Homework 2

Due: 11 October 2018, by end of the work day

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9. Page 294, 6.6, Problem 5
10. Page 295, 6.6, Problem 10
11. Page 295, 6.6, Problem 14
12. Page 295, 6.6, Problem 21
13. Page 298, 6.7, Compute the divergence for Problems 6 and 7
14. Page 299, 6.7, Problem 19
15. page 524, 10.8, Problems 1 and 3

~~9.~~ 6.6(5)

a) find the ~~derivative~~ gradient of $\phi = z\sin y - xz$ at point $(2, \pi/2, -1)$.

$$\vec{\nabla}\phi = \hat{i}\frac{\partial}{\partial x}(z\sin y - xz) + \hat{j}\frac{\partial}{\partial y}(z\sin y - xz) + \hat{k}\frac{\partial}{\partial z}(z\sin y - xz)$$

$$\boxed{\vec{\nabla}\phi = \hat{i}(-z) + \hat{j}(z\cos y) + \hat{k}(\sin y - x)}$$

at $(2, \frac{\pi}{2}, -1)$

$$\boxed{\vec{\nabla}\phi = \hat{i} + \hat{j}(-0) + \hat{k}(-1)}$$

b) from $(2, \pi/2, -1)$, in which direction is ϕ decreasing most rapidly?

$$\vec{\nabla}\phi \downarrow \Rightarrow \boxed{-\hat{i} + \hat{k}}$$

c) find derivative of ϕ in the direction $2\hat{i} + 3\hat{j}$

$$\vec{\nabla}\phi \cdot (2\hat{i} + 3\hat{j})$$

$$= \left(-z\hat{i} + z\cos y\,\hat{j} + (\sin y - x)\hat{k}\right)\cdot(2\hat{i} + 3\hat{j})$$

$$= -2z + 3z\cos y$$

$$= \boxed{-2z\left(1 - \frac{3}{2}\cos y\right)}$$

at $(2, \pi/2, -1) \Rightarrow 2$

10. 6.6(10)

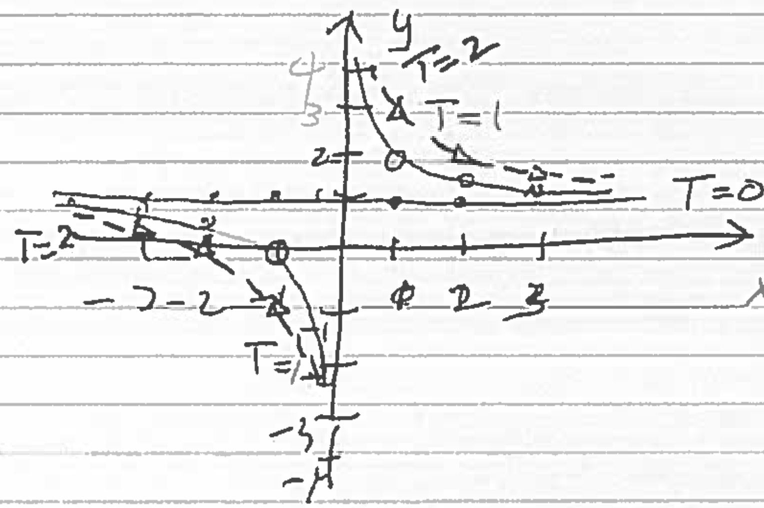
$T(x,y) = xy - x = x(y-1)$

a) Sketch a few isotherms for $T = 0, 1, 2, -1, -2$

$T=0$

x	y
1	1
0	—
−1	1
2	1
−2	1
+3	1
−3	1

$\Rightarrow$



$T=1$

x	y
1	2
−1	0
2	3/2
−2	1/2
3	4/3
−3	1/3

$\Rightarrow$

$T=2$

x	y
1	3
0	$\pm\infty$
−1	−1
2	2
−2	0
3	5/3
−3	1/3

b) find the dir^n in which T changes most rapidly from point (1,1) and the maximum rate of change.

$$T(x,y) = xy - x = x(y-1)$$

find $\vec\nabla T = \hat{i}\left[y-1\right] + \hat{j}\left[x\right]$

$$\Rightarrow \boxed{\vec\nabla T\big|_{(1,1)} = 0\hat{i} + \hat{j}} \Rightarrow \left|\vec\nabla T\right|_{(1,1)} = 1$$

c) find $\vec\nabla T|_{(1,1)} \cdot (3\hat{i} - 4\hat{j})$ from (1,1)

$$= 3(y-1) - 4x \quad \text{at } (1,1)$$

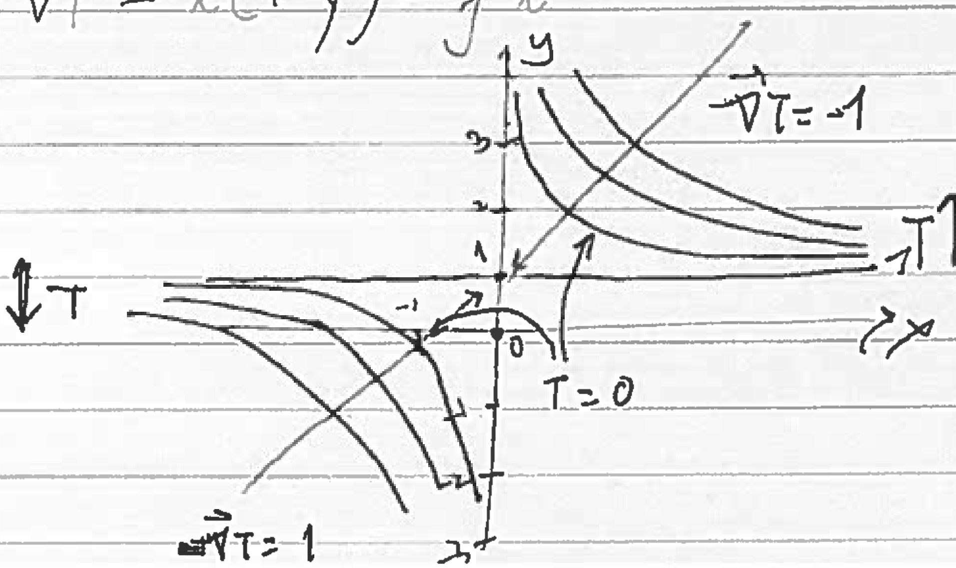
$$= 0 - 4$$

d) Heat flows in direction of $-\nabla T$. Sketch a few $-\vec\nabla T$ curves.

$$-\vec\nabla T = \hat{i}(1-y) - \hat{j}x$$

$-\vec\nabla T = -1$

x	y
0	1
1	2
2	3
$-\vec\nabla T = +1$	
0	0
-1	0
-2	-
-3	-2



11. 6.6(14)

Suppose a hill has equation, $z = 32 - x^2 - 4y^2$ = height, sketch a contour map. Use $z = 32, 19, 12, 7, 0$

(a) $z=0 \rightarrow 0 = 32 - x^2 - 4y^2 \Rightarrow$

x	y
0	$\pm\sqrt{8}$
± 2	$\pm\sqrt{7}$
± 4	± 2
$\pm\sqrt{32}$	0

(b) $z=32 \Rightarrow 32 = 32 - x^2 - 4y^2$

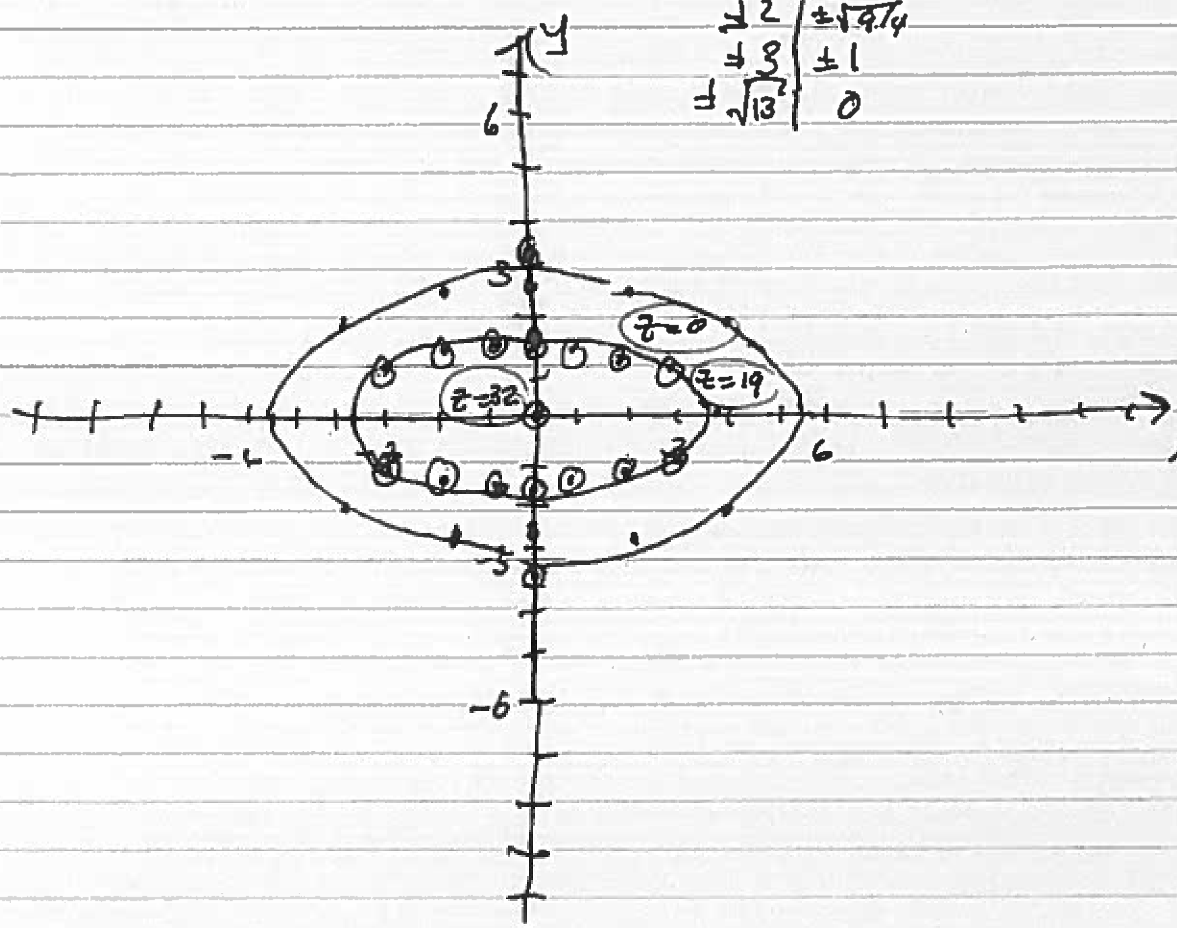
$0 = -x^2 - 4y^2 \Rightarrow x = y = 0$

$\boxed{x^2 = -4y^2}$ X

(c) $z=19 \Rightarrow 13 = x^2 + 4y^2$

$\Rightarrow$

x	y
0	$\pm\sqrt{13/4}$
± 1	$\pm\sqrt{3}$
± 2	$\pm\sqrt{9/4}$
± 3	± 1
$\pm\sqrt{13}$	0



(b) if you start at (3,2) in the dirⁿ (1,1) are you going uphill or downhill and how fast?

$\vec{\nabla} z = -2x\hat{\imath} - 8y\hat{\jmath}$ at $(3,2)$

$\rightarrow \boxed{\vec{\nabla} z = -6\hat{\imath} - 16\hat{\jmath}}$

$\vec{\nabla} z \cdot (1,1) = -6 - 16 = -22 \quad < 0$

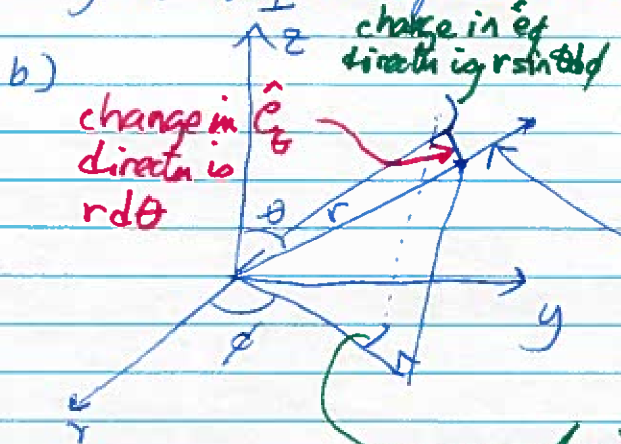
$\rightarrow$ downhill at 22 ft/ft

12. 6.4.21

Find $\vec{\nabla} f$ in spherical coordinates as we did for cylindrical coordinates. What is ds in the ϕ direction? See Chapter 5, Figure 4.5

a) following the text, find $d\vec{s}$ in spherical coordinates to find $\vec{\nabla}\Phi$ in spherical coordinates.

b)



$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$\Rightarrow \boxed{\vec{\nabla}\Phi = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{\partial}{r\,\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}}$$

13. 6.7.6 & 6.7.7

find divergence of

$\vec{V} = x^2 y \hat{i} + y^2 x \hat{j} + xyz\hat{k}$

and

$\vec{V} = \sinh z \hat{i} + 2y\hat{j} + x\cosh z \hat{k}$

a) $\vec{\nabla} \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(x^2 y \hat{i} + y^2 x \hat{j} + xyz\hat{k}\right)$

$= 2xy + x + xy$

$= 3xy + x \quad = x(1+3y)$

b) $\vec{\nabla} \cdot \left(\sinh z \hat{i} + 2y\hat{j} + x\cosh z \hat{k}\right)$

$= \frac{\partial}{\partial x}\sinh z + \frac{\partial}{\partial y} 2y + \frac{\partial}{\partial z} x\cosh z$

$= 0 + 2 + \frac{\partial}{\partial z} x\left(\frac{e^z + e^{-z}}{2}\right)$

$= 2 + \frac{x}{2}\left(e^z - e^{-z}\right)$

$= 2 + x\sinh z$

14

6.7.19

Evaluate $\vec{\nabla}\times\frac{\vec{r}}{|\vec{r}|}$

$$\frac{\vec{r}}{|\vec{r}|}=\frac{\hat{i}x+\hat{j}y+\hat{k}z}{\sqrt{x^2+y^2+z^2}}$$

and

$$\vec{\nabla}\times\frac{\vec{r}}{|\vec{r}|}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}}\end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)-\frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\right]$$
$$+\hat{j}\left[\frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)-\frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right]$$
$$+\hat{k}\left[\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)-\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

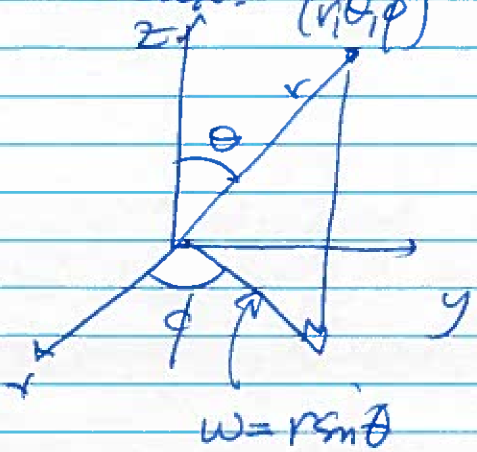
$$=\hat{i}\left[\frac{-\frac{1}{2}z\,2y}{(x^2+y^2+z^2)^{3/2}}-\frac{-\frac{1}{2}y\,2z}{(x^2+y^2+z^2)^{3/2}}\right]$$
$$+\hat{j}\left[\frac{-\frac{1}{2}x\,2z}{(x^2+y^2+z^2)^{3/2}}-\frac{-\frac{1}{2}z\,2x}{(x^2+y^2+z^2)^{3/2}}\right]$$
$$+\hat{k}\left[\frac{-\frac{1}{2}y\,2x}{(x^2+y^2+z^2)^{3/2}}-\frac{-\frac{1}{2}x\,2y}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$=\hat{i}(0)+\hat{j}(0)+\hat{k}(0)$$

$$=0$$

15. 10.8.1

find ds^2 in spherical coordinates by the method used to obtain (8.5) for cylindrical coordinates. Use your result to find scale factors, $d\vec{s}$, volume element, basis vectors, and unit basis vectors. Write g_{ij} matrix.



We see

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\longrightarrow \begin{cases} dx = dr\sin\theta\cos\phi + r\cos\theta\, d\theta\cos\phi - r\sin\theta\sin\phi\, d\phi \\ dy = dr\sin\theta\sin\phi + r\cos\theta\, d\theta\sin\phi + r\sin\theta\cos\phi\, d\phi \\ dz = dr\cos\theta - r\sin\theta\, d\theta \end{cases}$$

ⓐ find $ds^2 = dx^2 + dy^2 + dz^2$

$$= \Big([dr\sin\theta\cos\phi]^2 + [r\cos\theta\, d\theta\cos\phi - r\sin\theta\sin\phi\, d\phi]^2$$
$$+ 2(dr\sin\theta\cos\phi)(r\cos\theta\, d\theta\cos\phi - r\sin\theta\sin\phi\, d\phi)\Big)$$
$$+ \Big([dr\sin\theta\sin\phi]^2 + [r\cos\theta\, d\theta\sin\phi + r\sin\theta\cos\phi\, d\phi]^2$$
$$+ 2(dr\sin\theta\sin\phi)(r\cos\theta\, d\theta\sin\phi + r\sin\theta\cos\phi\, d\phi)\Big)$$
$$+ \Big([dr\cos\theta]^2 + [r\sin\theta\, d\theta]^2 - 2\,dr\cos\theta\sin\theta\, d\theta\Big)$$

$$= dr^2\left[\overbrace{\sin^2\theta\left(\cos^2\phi+\sin^2\phi\right)+\cos^2\theta}^{1}\right]$$

$$+\,d\theta^2\left[\overbrace{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}^{r^2}\right]$$

$$+\,d\phi^2\left[\overbrace{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi}^{r^2\sin^2\theta}\right]$$

$$+\,d\theta\,d\phi\left[\cancelto{0}{-2r^2\cos\theta\sin\theta\cos\phi\sin\phi + 2r^2\cos\theta\sin\theta\sin\phi\cos\phi}\right]$$

$$+\,dr\,d\theta\left[\cancelto{0}{\left[2r\cos\theta\sin\theta\cos^2\phi + 2r\sin\theta\sin^2\phi\cos\theta\right] - 2\cos\theta\sin\theta}\right]$$

$$+\,dr\,d\phi\left[\cancelto{0}{-2r\sin^2\theta\cos\phi\sin\phi + 2r\sin^2\theta\cos\phi\sin\phi}\right]$$

$$= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

(b) (a) $\longrightarrow$ $h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta$

(c) find $d\vec{s}$

$$d\vec{s} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$$

$$= \hat{i}\left[dr\sin\theta\cos\phi + r\cos\theta\,d\theta\cos\phi - r\sin\theta\sin\phi\,d\phi\right]$$

$$+ \hat{j}\left[dr\sin\theta\sin\phi + r\cos\theta\,d\theta\sin\phi + r\sin\theta\cos\phi\,d\phi\right]$$

$$+ \hat{k}\left[dr\cos\theta - r\sin\theta\,d\theta\right]$$

gather terms in $dr, d\theta, d\phi$

$$= dr\left[\hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta\right]$$

$$+ r\,d\theta\left[\hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta\right]$$

$$+ r\sin\theta\,d\phi\left[-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right]$$

$$\Rightarrow \begin{cases} \vec{a}_r = \sin\theta(\hat{i}\cos\phi + \hat{j}\sin\phi) + \cos\theta\,\hat{k} \\ \vec{a}_\theta = r(\hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \sin\theta\,\hat{k}) \\ \vec{a}_\phi = r\sin\theta\left[-\hat{i}\sin\phi + \hat{j}\cos\phi\right] \end{cases}$$

$$\Rightarrow \begin{cases} \hat{e}_r = \dfrac{\vec{a}_r}{|\vec{a}_r|} = \hat{i}\sin\theta(\cos\phi) + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta \\ \hat{e}_\theta = \dfrac{\vec{a}_\theta}{|\vec{a}_\theta|} = \hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta \\ \hat{e}_\phi = \dfrac{\vec{a}_\phi}{|\vec{a}_\phi|} = -\hat{i}\sin\phi + \hat{j}\cos\phi \end{cases}$$

(d) metric, g_{ij}

$$ds^2 = (dx_1 \; dx_2 \; dx_3)\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

$$\left[g_{ij} = \vec{a}_i \cdot \vec{a}_j \right]$$

$$\begin{cases} g_{11} = \vec{a}_1 \cdot \vec{a}_1 = \vec{a}_r \cdot \vec{a}_r = 1 \\ g_{12} = \vec{a}_r \cdot \vec{a}_\theta = \underbrace{\sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \cos\theta\sin\theta}_{=0} \\ g_{13} = \vec{a}_r \cdot \vec{a}_\phi = \underbrace{-\sin\theta\cos\phi\sin\phi + \sin\theta\cos\phi\sin\phi}_{=0} \end{cases}$$

$$\begin{cases} g_{21} = \vec{a}_\theta \cdot \vec{a}_r = 0 = g_{12} \\ g_{22} = \vec{a}_\theta \cdot \vec{a}_\theta = \underbrace{r^2(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta)}_{= r^2} \\ g_{23} = \vec{a}_\theta \cdot \vec{a}_\phi = \underbrace{r^2\sin\theta(-\cos\theta\cos\phi\sin\phi + \cos\theta\sin\phi\cos\phi)}_{=0} \end{cases}$$

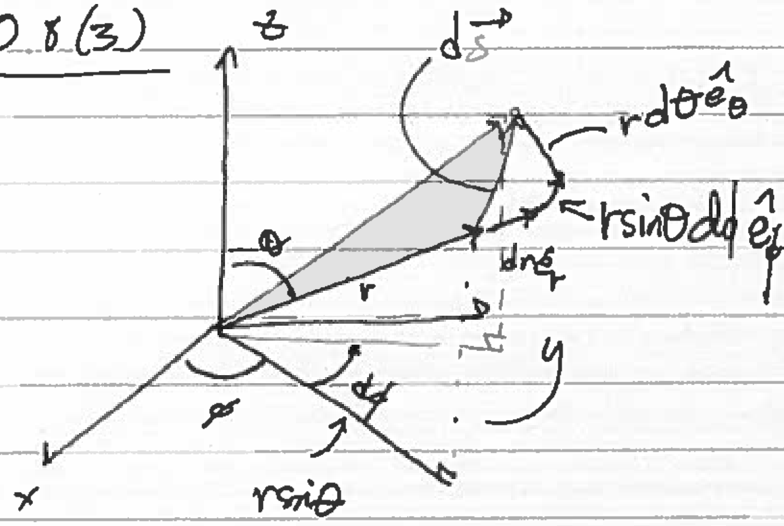
$$g_{31} = \vec{a}_\phi \cdot \vec{a}_r = g_{13} = 0$$

$$g_{32} = \vec{a}_\phi \cdot \vec{a}_\theta = g_{23} = 0$$

$$g_{33} = \vec{a}_\phi \cdot \vec{a}_\phi = r^2\sin^2\theta\left[\sin^2\phi + \cos^2\phi\right] = r^2\sin^2\theta$$

15. 10.8(3)

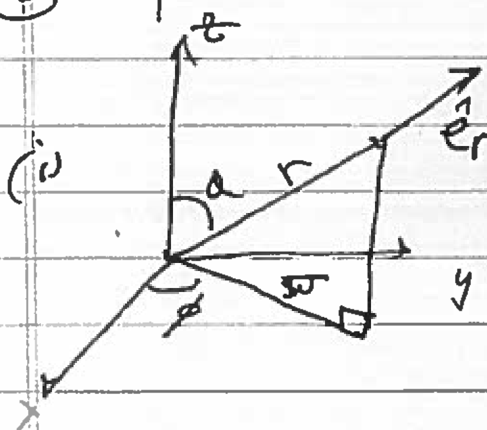
ⓐ



$$\boxed{\begin{aligned} d\vec{s} &= dr\,\hat{e}_r + r\sin\theta\,d\phi\,\hat{e}_\phi + r\,d\theta\,\hat{e}_\theta \\ &\rightarrow ds^2 = d\vec{s}\cdot d\vec{s} = dr^2 + r^2\sin^2\theta\,d\phi^2 + r^2 d\theta^2 \end{aligned}}$$

ⓑ Express in Cartesian coordinates

(i)



$\hat{e}_r$

① project r into xy plane, $\varpi = r\sin\theta$

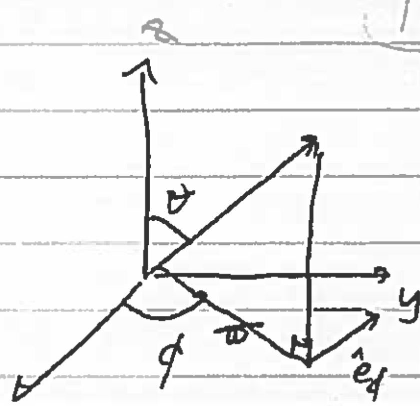
② $h_\varpi \hat{e}_\varpi = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j}$

$\hat{e}_r = \hat{e}_\varpi + \hat{z}$

$= r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$

$$\boxed{\hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}$$

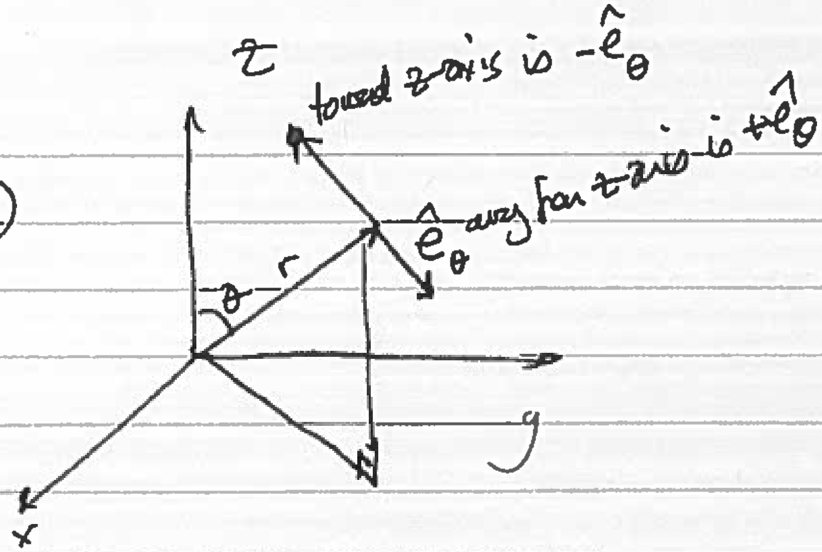
(ii)



$\hat{e}_\phi$ ① project r into xy plane.

② $h_\phi \hat{e}_\phi = -r\sin\theta\sin\phi\,\hat{i}$
$+ r\sin\theta\cos\phi\,\hat{j}$

(iii)



$$\Rightarrow h_\theta \hat{e}_\theta = r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}$$

$$\boxed{\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}}$$

(c) from a, b, we have $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ in terms of (r, θ, ϕ) & $\hat{i}, \hat{j}, \hat{k}$

and we can find scale factors,

$$\boxed{h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta}$$

(d) from ds^2, we see that the metric is

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

18. 10.8(3)

(a) $\vec{r} = r\hat{e}_r$

$$\frac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

$$\frac{d}{dt}\hat{e}_r = \frac{d}{dt}\left[\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right]$$

$$= \left[\cos\theta\dot{\theta}\cos\phi - \sin\theta\sin\phi\dot{\phi}, \cos\theta\dot{\theta}\sin\phi + \sin\theta\cos\phi\dot{\phi}, -\sin\theta\dot{\theta}\right]$$

$$= \dot{\theta}\left[\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta\right] + \dot{\phi}\left[-\sin\theta\sin\phi, \sin\theta\cos\phi, 0\right]$$

$$= \dot{\theta}\hat{e}_\theta + \dot{\phi}\hat{e}_\phi\sin\theta$$

$$\boxed{\frac{d\vec{r}}{dt} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\dot{\phi}\hat{e}_\phi\sin\theta}$$

$$d\vec{s} = dr\hat{e}_r + r\sin\theta d\phi\hat{e}_\phi + r d\theta\hat{e}_\theta$$

$$\Rightarrow \boxed{\frac{d\vec{s}}{dt} = \dot{r}\hat{e}_r + r\sin\theta\dot{\phi}\hat{e}_\phi + r\dot{\theta}\hat{e}_\theta}$$

(b) find $\ddot{\vec{r}}$

$$\frac{d}{dt}(\dot{\vec{r}}) = \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta$$
$$+ \dot{r}\dot{\phi}\sin\theta\hat{e}_\phi + r\ddot{\phi}\sin\theta\hat{e}_\phi + r\dot{\phi}\cos\theta\dot{\theta}\hat{e}_\phi$$
$$+ r\dot{\phi}\sin\theta\dot{\hat{e}}_\phi$$

recall: $\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta + \dot{\phi}\sin\theta\hat{e}_\phi$

and also

$$\dot{\hat{e}}_\phi = -\cos\phi\dot{\phi}\hat{i} - \sin\phi\dot{\phi}\hat{j} = -\dot{\phi}(\cos\phi\hat{i} + \sin\phi\hat{j})$$

$$\dot{\hat{e}}_\theta = -\sin\theta\dot{\theta}\cos\phi\hat{i} - \cos\theta\sin\phi\dot{\phi}\hat{i} - \sin\theta\dot{\theta}\sin\phi\hat{j}$$
$$+ \cos\theta\cos\phi\dot{\phi}\hat{j} - \cos\theta\dot{\theta}\hat{k}$$
$$= \dot{\theta}[-\sin\theta\cos\phi\hat{i} - \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}]$$
$$- \dot{\phi}\cos\theta[\sin\phi\hat{i} - \cos\phi\hat{j}]$$
$$= -\dot{\theta}\hat{e}_r + \dot{\phi}\cos\theta\hat{e}_\phi$$

hmm, what is $-\dot{\phi}[\cos\phi\hat{i} + \sin\phi\hat{j}]$?

take $\sin\theta\hat{e}_r + \cos\theta\hat{e}_\theta$

$$= (\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \sin\theta\cos\theta) + (\cos^2\theta\cos\phi, \cos^2\theta\sin\phi, -\cos\theta\sin\theta)$$

$$= (\cos\phi, \sin\phi, 0) \checkmark$$

$$\frac{d}{dt}\dot{\vec{r}} = \hat{e}_r\left[\ddot{r} + r\dot{\theta}\left(-\dot{\theta}\right) + r\sin\theta\,\dot{\phi}\left(-\dot{\phi}\sin\theta\right)\right]$$

$$+\hat{e}_\theta\left[\dot{r}\dot{\theta} + r\ddot{\theta} + \dot{r}\left(\dot{\theta}\right) + r\sin\theta\,\dot{\phi}\left(-\dot{\phi}\cos\theta\right)\right]$$

$$+\hat{e}_\phi\left[\dot{r}\dot{\phi}\sin\theta + r\ddot{\phi}\sin\theta + r\dot{\phi}\cos\theta\,\dot{\theta} + \dot{r}\left(\dot{\theta}\right) + r\dot{\theta}\left(\cos\theta\,\dot{\phi}\right)\right]$$