Homework 3

Due: 23 October 2018, by end of the work day

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16. page 299, 6.7, Problem 17
17. page 307, 6.8, Problem 16
18. page 323, 6.10, Problem 15
19. page 336, 6.12, Problem 3
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21. page 527, 10.9, Problem 1
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23. page 528, 10.9, Problem 10

HW #3

1B. 6.7(17) Verify b, c, d, g, h, i, j, k

(b) $\vec{\nabla} \times \vec{\nabla}\phi = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}, \frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}, \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right)$$

if ϕ & partials of ϕ are continuous

$$\Longrightarrow \frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial^2 \phi}{\partial z \partial y}, \dots$$

$$\rightarrow = (0, 0, 0)$$

(c) $\vec{\nabla}(\vec{\nabla} \cdot \vec{V}) = ?$

(1) $\vec{\nabla} \cdot \vec{V} = \left(\frac{\partial}{\partial x} V_x + \frac{\partial}{\partial y} V_y + \frac{\partial}{\partial z} V_z \right)$

$$\rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) = \hat{i}\left[\frac{\partial^2}{\partial x^2} V_x + \frac{\partial^2}{\partial x \partial y} V_y + \frac{\partial^2}{\partial x \partial z} V_z \right]$$

$$+ \hat{j}\left[\frac{\partial^2}{\partial y \partial x} V_x + \frac{\partial^2}{\partial y^2} V_y + \frac{\partial^2}{\partial y \partial z} V_z \right]$$

$$+ \hat{k}\left[\frac{\partial^2}{\partial z \partial x} V_x + \frac{\partial^2}{\partial z \partial y} V_y + \frac{\partial^2}{\partial z^2} V_z \right]$$

$$= \underbrace{\nabla^2 \vec{V}}_{\checkmark s} + \left(\frac{\partial}{\partial x}\left[\frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right], \frac{\partial}{\partial y}\left[\frac{\partial V_x}{\partial x} + \frac{\partial V_z}{\partial z} \right], \frac{\partial}{\partial z}\left[\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \right] \right)$$

(d) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = ?$

(i) $\vec{\nabla}\times\vec{V} = \left(\frac{\partial}{\partial y}V_z - \frac{\partial}{\partial z}V_y,\ \frac{\partial}{\partial z}V_x - \frac{\partial}{\partial x}V_z,\ \frac{\partial}{\partial x}V_y - \frac{\partial}{\partial y}V_x\right)$

$$\rightarrow \vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = \left[\frac{\partial^2}{\partial x\partial y}V_z - \frac{\partial^2}{\partial x\partial z}V_y + \frac{\partial^2}{\partial y\partial z}V_x - \frac{\partial^2}{\partial y\partial x}V_z + \frac{\partial^2}{\partial z\partial x}V_y - \frac{\partial^2}{\partial z\partial y}V_x\right]$$

if $\vec{V}$ & partial of $\vec{V}$ are continuous order of differentiation doesn't matter

$= 0$

(g) $$\vec{\nabla}\times(\phi\vec{V}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi V_x & \phi V_y & \phi V_z \end{vmatrix} =$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\phi V_z - \frac{\partial}{\partial z}\phi V_y\right] + \hat{j}\left[\frac{\partial}{\partial z}\phi V_x - \frac{\partial}{\partial x}\phi V_z\right] + \hat{k}\left[\frac{\partial}{\partial x}\phi V_y - \frac{\partial}{\partial y}\phi V_x\right]$$

$$= \hat{i}\left[\phi\frac{\partial V_z}{\partial y} + V_z\frac{\partial\phi}{\partial y} - \phi\frac{\partial V_y}{\partial z} - V_y\frac{\partial\phi}{\partial z}\right] + \hat{j}\left[\phi\frac{\partial V_x}{\partial z} + V_x\frac{\partial\phi}{\partial z} - \phi\frac{\partial V_z}{\partial x} - V_z\frac{\partial\phi}{\partial x}\right] + \hat{k}\left[\phi\frac{\partial V_y}{\partial x} + V_y\frac{\partial\phi}{\partial x} - \phi\frac{\partial V_x}{\partial y} - V_x\frac{\partial\phi}{\partial y}\right]$$

$$= \hat{i}\left[\phi\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + V_z\frac{\partial\phi}{\partial y} - V_y\frac{\partial\phi}{\partial z}\right] + \hat{j}\left[\phi\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + V_x\frac{\partial\phi}{\partial z} - V_z\frac{\partial\phi}{\partial x}\right] + \hat{k}\left[\phi\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) + V_y\frac{\partial\phi}{\partial x} - V_x\frac{\partial\phi}{\partial y}\right]$$

1^{st} column $\qquad\qquad -2^{nd}$ column

$$= \phi\, \vec{\nabla}\times\vec{V} - \vec{V}\times\vec{\nabla}\phi$$

(h) $\vec{\nabla}\cdot(\vec{u}\times\vec{v}) = ?$

(i) $\vec{u}\times\vec{v} = \left(u_y v_z - u_z v_y,\ u_z v_x - u_x v_z,\ u_x v_y - u_y v_x\right)$

$$\rightarrow \vec{\nabla}\cdot(\vec{u}\times\vec{v}) = \frac{\partial}{\partial x}\left(u_y v_z - u_z v_y\right) + \frac{\partial}{\partial y}\left(u_z v_x - u_x v_z\right) + \frac{\partial}{\partial z}\left(u_x v_y - u_y v_x\right)$$

$$= \left[u_y\frac{\partial v_z}{\partial x} + \frac{\partial u_y}{\partial x}v_z - u_z\frac{\partial v_y}{\partial x} - \frac{\partial u_z}{\partial x}v_y\right]$$

$$+ \left[u_z\frac{\partial v_x}{\partial y} + \frac{\partial u_z}{\partial y}v_x - u_x\frac{\partial v_z}{\partial y} - \frac{\partial u_x}{\partial y}v_z\right]$$

$$+ \left[u_x\frac{\partial v_y}{\partial z} + \frac{\partial u_x}{\partial z}v_y - u_y\frac{\partial v_x}{\partial z} - \frac{\partial u_y}{\partial z}v_x\right]$$

gather terms

$$= u_x\left[-\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right] + u_y\left[\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right] + u_z\left[-\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]$$

$$+ v_x\left[\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right] + v_y\left[-\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right] + v_z\left[\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right]$$

$$= \vec{u}\cdot\left(-\vec{\nabla}\times\vec{v}\right) + \vec{v}\cdot\left(\vec{\nabla}\times\vec{u}\right)$$

(ii) $\vec{\nabla} \times (\vec{u} \times \vec{v})$

$$(\text{ii})\ (\vec{u} \times \vec{v}) = (u_y v_z - u_z v_y,\ u_z v_x - u_x v_z,\ u_x v_y - u_y v_x)$$

$$\rightarrow \vec{\nabla} \times (\vec{u} \times \vec{v}) = \hat{i}\left[\frac{\partial}{\partial y}(u_x v_y - u_y v_x) - \frac{\partial}{\partial z}(u_z v_x - u_x v_z)\right]$$

$$+ \hat{j}\left[\frac{\partial}{\partial z}(u_y v_z - u_z v_y) - \frac{\partial}{\partial x}(u_x v_y - u_y v_x)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(u_z v_x - u_x v_z) - \frac{\partial}{\partial y}(u_y v_z - u_z v_y)\right]$$

$$= \hat{i}\left[\left(u_x \frac{\partial v_y}{\partial y} + \frac{\partial u_x}{\partial y} v_y - u_y \frac{\partial v_x}{\partial y} - \frac{\partial u_y}{\partial y} v_x\right) - \left(u_z \frac{\partial v_x}{\partial z} + \frac{\partial u_z}{\partial z} v_x - u_x \frac{\partial v_z}{\partial z} - \frac{\partial u_x}{\partial z} v_z\right)\right]$$

$$+ \hat{j}\left[\left(u_y \frac{\partial v_z}{\partial z} + \frac{\partial u_y}{\partial z} v_z - u_z \frac{\partial v_y}{\partial z} - \frac{\partial u_z}{\partial z} v_y\right) - \left(u_x \frac{\partial v_y}{\partial x} + \frac{\partial u_x}{\partial x} v_y - u_y \frac{\partial v_x}{\partial x} - \frac{\partial u_y}{\partial x} v_x\right)\right]$$

$$+ \hat{k}\left[\left(u_z \frac{\partial v_x}{\partial x} + \frac{\partial u_z}{\partial x} v_x - u_x \frac{\partial v_z}{\partial x} - \frac{\partial u_x}{\partial x} v_z\right) - \left(u_y \frac{\partial v_z}{\partial y} + \frac{\partial u_y}{\partial y} v_z - u_z \frac{\partial v_y}{\partial y} - \frac{\partial u_z}{\partial y} v_y\right)\right]$$

gather terms and add and subtract some terms

$$= \hat{i}\left[\left(v_y \frac{\partial}{\partial y} u_x + v_z \frac{\partial u_x}{\partial z} + v_x \frac{\partial}{\partial x} u_x\right) - v_x \frac{\partial}{\partial x} u_x\right.$$

$$\left. + \left(-u_y \frac{\partial}{\partial y} v_x - u_z \frac{\partial}{\partial z} v_x - u_x \frac{\partial}{\partial x} v_x\right) + u_x \frac{\partial}{\partial x} v_x\right.$$

similarly except for y, z terms

$$+\hat{j}\,[\quad] + \hat{k}\,[\quad]$$

$$= \hat{i}\left[(\vec{V}\cdot\vec{\nabla})\vec{u}_x - (\vec{u}\cdot\vec{\nabla})V_x - V_x\vec{\nabla}\cdot\vec{u} + u_x\vec{\nabla}\cdot\vec{V}\right]$$

$$+\hat{j}\,[\quad] + \hat{k}\,[\quad]$$

except for u_y, V_y except for u_z, V_z

Express as a vector formula

$$\boxed{(\vec{V}\cdot\vec{\nabla})\vec{u} - (\vec{u}\cdot\vec{\nabla})\vec{V} - \vec{V}(\vec{\nabla}\cdot\vec{u}) + \vec{u}(\vec{\nabla}\cdot\vec{V})}$$

(j) $\vec{\nabla}(\vec{u}\cdot\vec{V}) = \vec{\nabla}(u_xV_x + u_yV_y + u_zV_z)$

<u>look at x-compnt</u>

$$= \left[u_x\frac{\partial V_x}{\partial x} + \frac{\partial u_x}{\partial x}V_x + u_y\frac{\partial V_y}{\partial x} + \frac{\partial u_y}{\partial x}V_y + V_z\frac{\partial V_z}{\partial x} + \frac{\partial u_z}{\partial x}V_z\right]$$

$$= \Big[\underbrace{u_x\frac{\partial}{\partial x}V_x + u_y\frac{\partial}{\partial y}V_x + u_z\frac{\partial}{\partial z}V_x}_{(\vec{u}\cdot\vec{\nabla})V_x} + \underbrace{V_x\frac{\partial}{\partial x}u_x + V_y\frac{\partial}{\partial y}u_x + V_z\frac{\partial}{\partial z}u_x}_{(\vec{V}\cdot\vec{\nabla})u_x}\Big]$$

$$+\Big[+u_y\underbrace{\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)}_{\vec{\nabla}\times\vec{V}|_z} + V_y\underbrace{\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)}_{\vec{\nabla}\times\vec{u}|_z}\Big]$$

$$+\Big[u_z\underbrace{\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right)}_{-\vec{\nabla}\times\vec{V}|_y} + V_z\underbrace{\left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right)}_{-\vec{\nabla}\times\vec{u}|_y}\Big]$$

$$= (\vec{u}\cdot\vec{\nabla}) V_x + (\vec{V}\cdot\vec{\nabla}) u_x + \vec{u}\times(\vec{\nabla}\times\vec{V})\Big|_{x\text{-compnt}}$$

$$+ \vec{V}\times(\vec{\nabla}\times\vec{u})\Big|_{x\text{compnt}}$$

similarly for y, z compnts

$$\Rightarrow \quad = (\vec{u}\cdot\vec{\nabla})\vec{V} + (\vec{V}\cdot\vec{\nabla})\vec{u} + \vec{u}\times(\vec{\nabla}\times\vec{V}) + \vec{V}\times(\vec{\nabla}\times\vec{u})$$

(c) $\vec{\nabla}\cdot(\vec{\nabla}\phi\times\vec{\nabla}\psi) = ?$

(i) $$\vec{\nabla}\phi\times\vec{\nabla}\psi = \left(\frac{\partial}{\partial y}\phi\frac{\partial}{\partial z}\psi - \frac{\partial}{\partial z}\phi\frac{\partial}{\partial y}\psi,\ \frac{\partial}{\partial z}\phi\frac{\partial\psi}{\partial x} - \frac{\partial}{\partial x}\phi\frac{\partial\psi}{\partial z},\ \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial x}\right)$$

$$\Rightarrow \vec{\nabla}\cdot(\vec{\nabla}\phi\times\vec{\nabla}\phi) = \Big[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial y}\right)$$

$$+ \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial z}\right)$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial x}\right)\Big]$$

$$= 0$$

#7. 6.8(16) Given $\vec{F}_1 = 2x\hat{i} - 2yz\hat{j} - y^2\hat{k}$

ϕ

$\vec{F}_2 = y\hat{i} - x\hat{j}$

a) Are these conservative? If conservative find the potential.

(i) $$\vec{\nabla}\times\vec{F}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & -2yz & -y^2 \end{vmatrix} = (-2y+2y,\ 0-0,\ 0-0)$$

$= 0 \rightarrow$ conservative

$$\vec{F}_1 = \vec{\nabla}\phi = ? = \left(\frac{\partial}{\partial x}\phi, \frac{\partial}{\partial y}\phi, \frac{\partial}{\partial z}\phi\right) = (2x, -2yz, -y^2)$$

Integrate $\xrightarrow{\text{yields}}$

$\phi = x^2 + f(y,z)$ — $\frac{\partial\phi}{\partial x}$ term

$= -y^2 z + g(x,z)$ — $\frac{\partial\phi}{\partial y}$ term

$= -y^2 z + h(x,y)$ — $\frac{\partial\phi}{\partial z}$ term

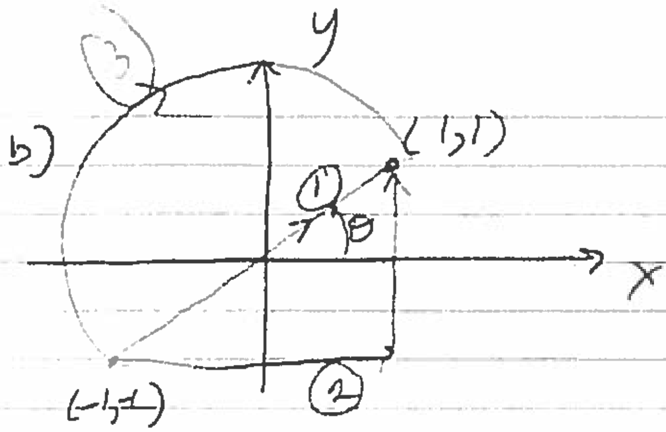
functions of (y,z), (x,z) & (x,y)

$$\Longrightarrow \boxed{\phi = -y^2 z + x^2 + C}$$

(ii) $$\vec{\nabla}\times\vec{F}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = (0-0,\ 0-0,\ \underbrace{-1-1}_{-2})$$

$\neq 0$

$\Longrightarrow$ not conservative



Integrate $\int \vec{F}_2 \cdot d\vec{r}$ along the 3 paths shown at left

$$② \int \vec{F}_2 \cdot d\vec{r} = \int_{(-1,-1)}^{(1,-1)} y\,dx - \int_{(1,-1)}^{(1,1)} x\,dy$$

$$= \int_{-1}^{1} (-1)\,dx - \int_{-1}^{1} (1)\,dy$$

$$\boxed{\int \vec{F}_2 \cdot d\vec{r} = -2 - (2) = -4}$$

$$① \int_{1} \vec{F}_2 \cdot d\vec{r} = \int_{1} (y, -x) \cdot d\vec{r}$$, $y = x$ is path and

$$dr^2 = dx^2 + dy^2$$

$$= dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right) = 2dx^2$$

$$= \int_{1} (y\,dx - x\,dy)$$

$$= \int_{1} (x\,dx - x\,[dx])$$

$$\boxed{\int \vec{F}_2 \cdot d\vec{r} = 0}$$

$$③ \int_{3} \vec{F}_2 \cdot d\vec{r} = \int_{3} (y, -x) \cdot d\vec{r}$$, $\overbrace{x^2 + y^2}^{r^2} = 2$ is path and

~~$dr^2 = dx^2 + dy^2$~~

$$= \int \left((r\sin\theta, -r\cos\theta) \cdot (\hat{x}\,dx + \hat{y}\,dy) \right)$$

let's use polar coordinates

$$dr^2 = r_0^2\, d\theta^2$$

$$\overset{3}{=} \int [r\sin\theta\,dx - r\cos\theta\,dy)$$

$r = $ constant

(1) $x = r\cos\theta$

$dx = -r\sin\theta\,d\theta$

$y = r\sin\theta$

$$= \int \left[r\sin\theta(-r\sin\theta)\,d\theta - r\cos\theta(r\cos\theta)\,d\theta \right]$$

$$= -\int r^2(\sin^2\theta + \cos^2\theta)\,d\theta$$

$$= -r^2(\theta_1 - \theta_0)$$

where $\cos\theta_0 = \cos(\theta_1 + \pi)$ $\quad \cos\theta_1 = \frac{1}{\sqrt{2}}$

$$= -\cos\theta_1$$

$$= -\frac{1}{\sqrt{2}}$$

$$\Rightarrow -r^2\left(\frac{\pi}{4} - \left[+\frac{\pi}{4} + \pi\right]\right) = -r^2\left[-\pi\right] = 2\pi$$

$$\boxed{\int_{③} \vec{F}_2 \cdot d\vec{r} = 2\pi}$$

~~10~~. 6.10 (15)

Given $\rho(x,y,z,t)$. If we follow a streamline then (x,y,z) are functions of time such that the flow velocity is

$$\vec{v} = \hat{\imath}\frac{dx}{dt} + \hat{\jmath}\frac{dy}{dt} + \hat{k}\frac{dz}{dt}$$

a) Show that

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + (\vec{v}\cdot\vec{\nabla})\rho$$

① $$\frac{d}{dt}\rho(x,y,z,t) = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}\frac{dx}{dt} + \frac{\partial\rho}{\partial y}\frac{dy}{dt} + \frac{\partial\rho}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}v_x + \frac{\partial\rho}{\partial y}v_y + \frac{\partial\rho}{\partial z}v_z$$

$$\boxed{\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + (\vec{v}\cdot\vec{\nabla})\rho}$$

b) Equation (10.9) $\vec{\nabla}\cdot\vec{v}\rho + \frac{\partial\rho}{\partial t} = 0$

$$\cancel{\Rightarrow \frac{d\rho}{dt} = -\vec{\nabla}\cdot\vec{v}\rho + (\vec{\nabla}\cdot\vec{v})\rho}$$

$$\cancel{= -(\vec{v}\cdot\vec{\nabla})\rho - \rho(\vec{\nabla}\cdot\vec{v}) + (\vec{\nabla}\cdot\vec{v})\rho}$$

$$\cancel{\frac{d\rho}{dt} + (\vec{v}\cdot\vec{\nabla})\rho = 0}$$

$$\frac{d\rho}{dt} = (\vec{V}\cdot\vec{\nabla})\rho \overbrace{- \vec{\nabla}\cdot(\rho\vec{V})}^{\text{substitute for } \frac{\partial\rho}{\partial t}}$$

$$= (\vec{V}\cdot\vec{\nabla})\rho - \rho(\vec{\nabla}\cdot\vec{V}) - (\vec{V}\cdot\vec{\nabla})\rho$$

$$= -\rho(\vec{\nabla}\cdot\vec{V})$$

$$\Longrightarrow \boxed{\frac{d\rho}{dt} + \rho(\vec{\nabla}\cdot\vec{V}) = 0}$$

19. 6.12.3

The force on a charge q moving w/ $\vec{v} \equiv \frac{d\vec{r}}{dt}$ in a magnetic field $\vec{B}$ is

$$\vec{F} = q(\vec{v} \times \vec{B}).$$

Write $\vec{B} = \vec{\nabla} \times \vec{A}$. Show that (if $\vec{r} = (x,y,z)$ for q at t)

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{A}$$

so that

$$\vec{F} = q\vec{v} \times (\vec{\nabla} \times \vec{A}) = q\left[\vec{\nabla}(\vec{v} \cdot \vec{A}) - \frac{d\vec{A}}{dt} + \frac{\partial \vec{A}}{\partial t}\right]$$

Solⁿ

a) $$\frac{d\vec{A}(x,y,z,t)}{dt} = \frac{\partial \vec{A}}{\partial t} + \left[\frac{\partial \vec{A}}{\partial x}\overbrace{\frac{dx}{dt}}^{v_x} + \frac{\partial \vec{A}}{\partial y}\overbrace{\frac{dy}{dt}}^{v_y} + \frac{\partial \vec{A}}{\partial z}\overbrace{\frac{dz}{dt}}^{v_z}\right]$$

$$= \frac{\partial \vec{A}}{\partial t} + \left[\frac{dx}{dt}\frac{\partial \vec{A}}{\partial x} + \frac{dy}{dt}\frac{\partial \vec{A}}{\partial y} + \frac{dz}{dt}\frac{\partial \vec{A}}{\partial z}\right]$$

$$= \frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{A}$$

b) $$\vec{F} = q(\vec{v} \times \vec{B}) = q(\vec{v} \times [\vec{\nabla} \times \vec{A}])$$

$$\boxed{\begin{array}{l}\text{expand } \vec{v} \times (\vec{\nabla} \times \vec{A}) \text{ using vector ID} \\ \vec{\nabla}(\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{A}\end{array}}$$

$$\rightarrow \vec{\nabla}(\vec{v} \cdot \vec{A}) = \vec{v} \times (\vec{\nabla} \times \vec{A}) + \vec{A} \times \cancelto{0}{(\vec{\nabla} \times \vec{v})} + (\vec{v} \cdot \vec{\nabla})\vec{A} + \cancelto{0}{(\vec{A} \cdot \vec{\nabla})\vec{v}}$$

following point charge q → $\vec{v}$ independent of $\vec{r}$

$$\rightarrow \quad \vec{\nabla}\times(\vec{v}\times\vec{A}) = \vec{\nabla}(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\vec{\nabla})\vec{A}$$

and so,

$$\vec{F} = q\left[\vec{v}\times(\vec{\nabla}\times\vec{A})\right]$$

$$= q\left[\vec{\nabla}(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\vec{\nabla})\vec{A}\right]$$

$$= q\left[\vec{\nabla}(\vec{v}\cdot\vec{A}) - \left\{\frac{d\vec{A}}{dt} - \frac{\partial\vec{A}}{\partial t}\right\}\right]$$

$$\boxed{\vec{F} = q\left[\vec{\nabla}(\vec{v}\cdot\vec{A}) + \frac{\partial\vec{A}}{\partial t} - \frac{d\vec{A}}{dt}\right]}$$

Pb. 6.12.5

Use Green's theorem to do Problem 8.2

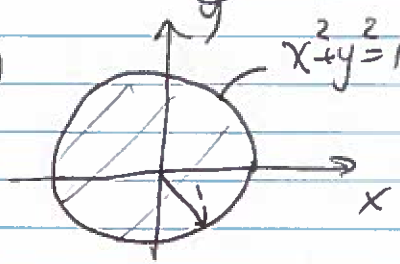
Evaluate $\oint (x+2y)dx - \oint 2x\,dy$ along each of the following paths;

a) circle $x^2+y^2=1$

b) square w/corners at $(1,1), (-1,1), (-1,-1), (1,-1)$

c) square w/corners at $(0,1), (-1,0), (0,-1), (1,0)$

a)



$$\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \oint_C (P\,dx + Q\,dy)$$

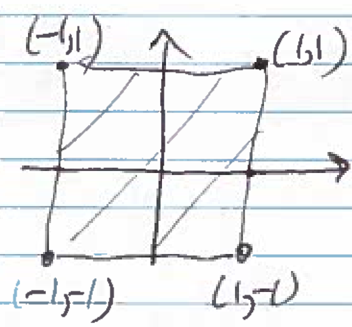
let $P = x+2y$, $Q = -2x$

$$\Rightarrow \frac{\partial P}{\partial y} = 2 \qquad \Rightarrow \frac{\partial Q}{\partial x} = -2$$

$$\Rightarrow \oint_{C,\ \text{circle}} \left[(x+2y)dx - 2y\,dy\right] = \int_A \left[-2-2\right]dx\,dy$$

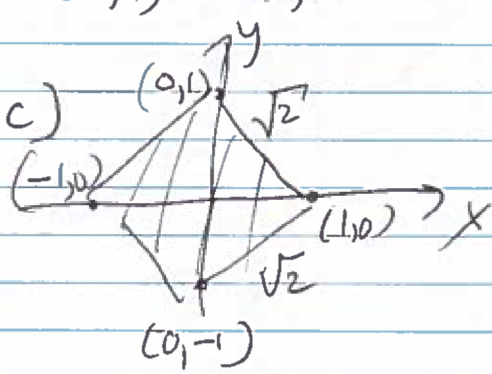
$$= -4\int_A dx\,dy = -4(\underbrace{\pi}_{\text{area of circle}})$$

b)



$$\Rightarrow \oint_{\text{square}} \left[(x+2y)dx - 2y\,dy\right] = -4(\underbrace{4}_{\text{area of square}})$$

c)



$$\Rightarrow \oint_{\text{square}} \left[(x+2y)dx - 2y\,dy\right] = -4(\underbrace{2}_{\text{area of square}})$$

28. 10.9(1)

Prove 9.4, $\vec{\nabla}\cdot\left(\frac{\hat{e}_3}{h_1 h_2}\right) = \vec{\nabla}\cdot\left(\frac{\hat{e}_2}{h_1 h_3}\right) = \vec{\nabla}\cdot\left(\frac{\hat{e}_3}{h_1 h_2}\right) = 0$

Sol^{n}

okay $\vec{\nabla} = \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial x_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial x_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial x_3}$ (9.2)

(i) find $\vec{\nabla} x_1 = \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial x_1} x_1 = \frac{\hat{e}_1}{h_1}$, $\frac{\partial x_1}{\partial x_2} = \frac{\partial x_1}{\partial x_3} = 0$

$\rightarrow \vec{\nabla} x_2 = \frac{\hat{e}_2}{h_2}$ and $\vec{\nabla} x_3 = \frac{\hat{e}_3}{h_3}$

(ii) let $\hat{e}_1, \hat{e}_2, \hat{e}_3$ form a right-handed ~~tetrad~~ triad so that

$$(\vec{\nabla} x_1) \times (\vec{\nabla} x_2) = \left(\frac{\hat{e}_1}{h_1}\right) \times \left(\frac{\hat{e}_2}{h_2}\right)$$

$$= \frac{1}{h_1 h_2}\hat{e}_3$$

(iii) take $\vec{\nabla}\cdot\left[\vec{\nabla} x_1 \times \vec{\nabla} x_2\right] = 0$ (ID B)

$$\cancel{= \vec{\nabla} x_1 \cdot \left(-\vec{\nabla} \times \vec{\nabla} x_2\right) + \vec{\nabla} x_2 \cdot \left(\vec{\nabla} \times \vec{\nabla} x_1\right)}$$

$$\cancel{= \frac{\hat{e}_1}{h_1 h_2} \cdot \left(-\vec{\nabla} \times \frac{\hat{e}_2}{h_2}\right) + \frac{\hat{e}_2}{h_2} \cdot \left(\vec{\nabla} \times \frac{\hat{e}_1}{h_1}\right)}$$

$$= \vec{\nabla}\cdot\left[\frac{\hat{e}_3}{h_1 h_2}\right]$$

$$= 0$$

Repeat for $\vec{\nabla} x_3 \times \vec{\nabla} x_1 = \dfrac{\hat{e}_2}{h_1 h_3}$

ad

$$\vec{\nabla} x_2 \times \vec{\nabla} x_3 = \frac{\hat{e}_1}{h_2 h_3}$$

~~29~~. 10.9(2)

Derive 9.11, $\vec{\nabla}\times\vec{V} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1V_1 & h_2V_2 & h_3V_3 \end{vmatrix}$

Sol^{n}

(i) recall $\vec{\nabla}x_1 = \frac{\hat{e}_1}{h_1}$ and then take

$$\vec{\nabla}\times(\vec{\nabla}x_1) = \vec{\nabla}\times\frac{\hat{e}_1}{h_1} \checkmark = 0 \quad (\text{ID } 6)$$

(ii) write $\vec{V} = \frac{\hat{e}_1}{h_1}(h_1V_1) + \frac{\hat{e}_2}{h_2}(h_2V_2) + \frac{\hat{e}_3}{h_3}(h_3V_3)$

(iii) take $\vec{\nabla}\times\vec{V} = \vec{\nabla}\times\left(\frac{\hat{e}_1}{h_1}h_1V_1\right) + \vec{\nabla}\times\left(\frac{\hat{e}_2}{h_2}h_2V_2\right)$
$+ \vec{\nabla}\times\left(\frac{\hat{e}_3}{h_3}h_3V_3\right)$

Use ID 9

0, from (i) above

$$= \left(h_1V_1\cancel{\vec{\nabla}\times\frac{\hat{e}_1}{h_1}} - \left(\frac{\hat{e}_1}{h_1}\times\vec{\nabla}\right)h_1V_1\right)$$
$$+ \left(h_2V_2\cancel{\vec{\nabla}\times\frac{\hat{e}_2}{h_2}} - \left(\frac{\hat{e}_2}{h_2}\times\vec{\nabla}\right)h_2V_2\right)$$
$$+ \left(h_3V_3\cancel{\vec{\nabla}\times\frac{\hat{e}_3}{h_3}} - \left(\frac{\hat{e}_3}{h_3}\times\vec{\nabla}\right)h_3V_3\right)$$

$$= -\left(0,\ -\frac{1}{h_1 h_3}\frac{\partial}{\partial x_3} h_1 V_1,\ \frac{1}{h_1}\frac{1}{h_2}\frac{\partial}{\partial x_2} h_1 V_1\right)$$

$$-\left(\frac{1}{h_2}\frac{1}{h_3}\frac{\partial}{\partial x_3} h_2 V_2,\ 0,\ -\frac{1}{h_2}\frac{1}{h_1}\frac{\partial}{\partial x_1} h_2 V_2\right)$$

$$-\left(-\frac{1}{h_3}\frac{1}{h_2}\frac{\partial}{\partial x_2} h_3 V_3,\ \frac{1}{h_3}\frac{1}{h_1}\frac{\partial}{\partial x_1} h_3 V_3,\ 0\right)$$

$$= \left(\frac{1}{h_2 h_3}\frac{\partial}{\partial x_2} h_3 V_3 - \frac{1}{h_2 h_3}\frac{\partial}{\partial x_3} h_2 V_2,\ \frac{1}{h_1 h_3}\frac{\partial}{\partial x_3} h_1 V_1 - \frac{1}{h_1 h_3}\frac{\partial}{\partial x_1} h_3 V_3,\right.$$

$$\left.\frac{1}{h_1 h_2}\frac{\partial}{\partial x_1} h_2 V_2 - \frac{1}{h_1 h_2}\frac{\partial}{\partial x_2} h_1 V_1\right)$$

$$= \frac{1}{h_1 h_2 h_3}\left(h_1 \frac{\partial}{\partial x_2} h_3 V_3 - h_1 \frac{\partial}{\partial x_3} h_2 V_2,\right.$$

$$h_2 \frac{\partial}{\partial x_3} h_1 V_1 - h_2 \frac{\partial}{\partial x_1} h_3 V_3,$$

$$\left. h_3 \frac{\partial}{\partial x_1} h_2 V_2 - h_3 \frac{\partial}{\partial x_2} h_1 V_1\right)$$

$$= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix} \checkmark$$

23. 10.9.10

find $\vec{\nabla} u$, $\vec{\nabla}\cdot\vec{\nabla}$, $\vec{\nabla}\cdot\vec{V}$, $\vec{\nabla}\times\vec{V}$ for parabolic cylindrical coordinates

$$\begin{cases} x = \frac{1}{2}(u^2 - v^2) \\ y = uv \\ z = z \end{cases}$$

$$\begin{cases} dx = u\,du - v\,dv \\ dy = u\,dv + du\,v \\ dz = dz \end{cases}$$

$$\rightarrow ds^2 = dx^2 + dy^2 + dz^2$$
$$= u^2 du^2 + v^2 dv^2 - 2uv\cancel{du}\,dv$$
$$+ u^2 dv^2 + du^2 v^2 + 2\cancel{uv}\,dv\,du$$
$$+ dz^2$$

$$\boxed{ds^2 = (u^2+v^2)du^2 + (u^2+v^2)dv^2 + dz^2}$$

$$\rightarrow h_u = \sqrt{u^2+v^2},\ h_v = \sqrt{u^2+v^2},\ h_z = 1$$

a) $\vec{\nabla} u = \frac{\hat{e}_u}{\sqrt{u^2+v^2}}\frac{\partial}{\partial u} + \frac{\hat{e}_v}{\sqrt{u^2+v^2}}\frac{\partial}{\partial v} + \hat{e}_z\frac{\partial}{\partial z}$

b) $\nabla^2 = \frac{1}{\sqrt{u^2+v^2}}\frac{1}{\sqrt{u^2+v^2}}\left[\frac{\partial}{\partial u}\frac{\cancel{\sqrt{u^2+v^2}}}{\cancel{\sqrt{u^2+v^2}}}\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\frac{\partial}{\partial v} + \frac{\partial}{\partial z}(u^2+v^2)\frac{\partial}{\partial z}\right]$

$$= \frac{1}{u^2+v^2}\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + (u^2+v^2)\frac{\partial^2}{\partial z^2}\right]$$

(c) $$\vec{\nabla}\cdot\vec{V} = \frac{1}{u^2+v^2}\left[\frac{\partial}{\partial u}\sqrt{u^2+v^2}\,V_u + \frac{\partial}{\partial v}\sqrt{u^2+v^2}\,V_v + \frac{\partial}{\partial z}(u^2+v^2)\,V_z\right]$$

$$= \frac{1}{u^2+v^2}\left[\frac{\partial}{\partial u}\sqrt{u^2+v^2}\,V_u + \frac{\partial}{\partial v}\sqrt{u^2+v^2}\,V_v + (u^2+v^2)\frac{\partial}{\partial z}V_z\right]$$

(d) $\vec{\nabla}\times\vec{V}$

$$= \frac{1}{(u^2+v^2)}\begin{vmatrix} \sqrt{u^2+v^2}\,\hat{e}_u & \sqrt{u^2+v^2}\,\hat{e}_v & \hat{e}_z \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial z} \\ \sqrt{u^2+v^2}\,V_u & \sqrt{u^2+v^2}\,V_v & V_z \end{vmatrix}$$

$$= \frac{\hat{e}_u}{\sqrt{u^2+v^2}}\left(\frac{\partial}{\partial v}V_z - \sqrt{u^2+v^2}\frac{\partial}{\partial z}V_v\right)$$

$$+ \frac{\hat{e}_v}{1}\left(\frac{\partial}{\partial z}V_u - \frac{1}{\sqrt{u^2+v^2}}\frac{\partial}{\partial u}V_z\right)$$

$$+ \frac{\hat{e}_z}{(u^2+v^2)}\left(\frac{\partial}{\partial u}\sqrt{u^2+v^2}\,V_v - \frac{\partial}{\partial v}\sqrt{u^2+v^2}\,V_u\right)$$