Homework 4

Due: 1 November 2018

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24. page 347, 7.3, Problem 3
25. page 350, 7.4, Problem 13
26. page 355, 7.5 Problem 4
27. page 355, 7.6 Problem 14
28. page 360, 7.7 Problem 6
29. page 363, 7.8 Problem 12
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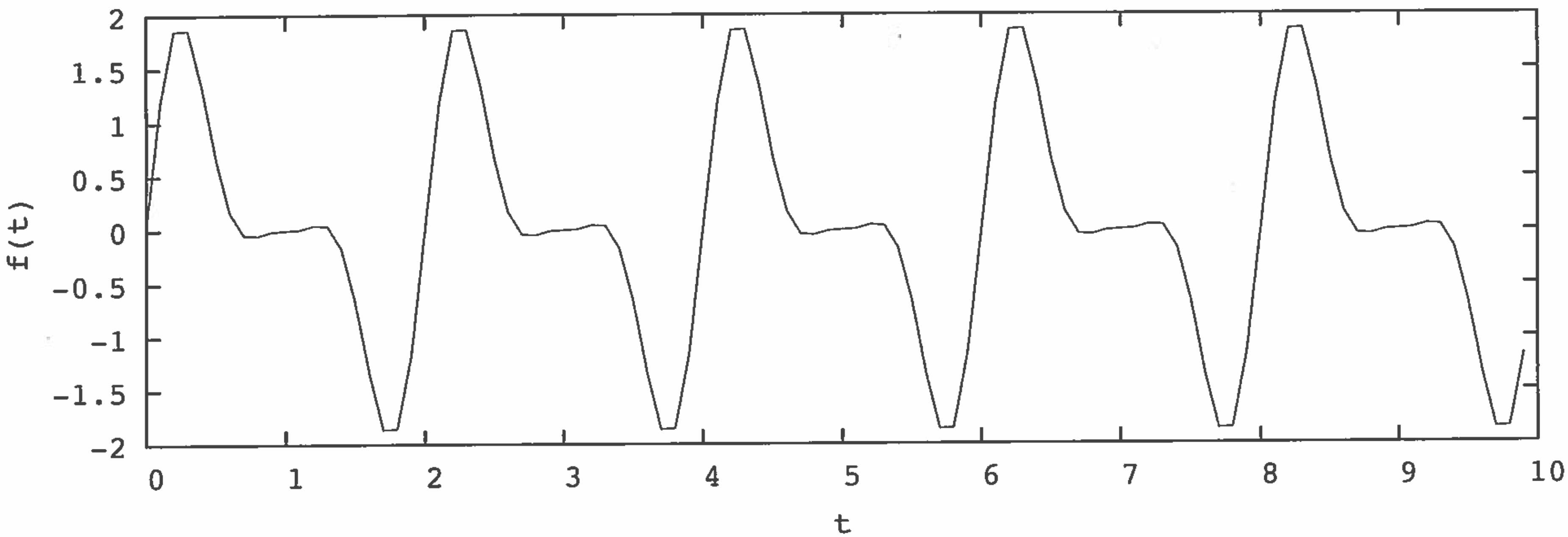
HW#4, 7.3.3, 7.4.13, 7.5.4, 7.6.14, 7.7.6, 7.8.12,
7.9.24

24. 7.3.3

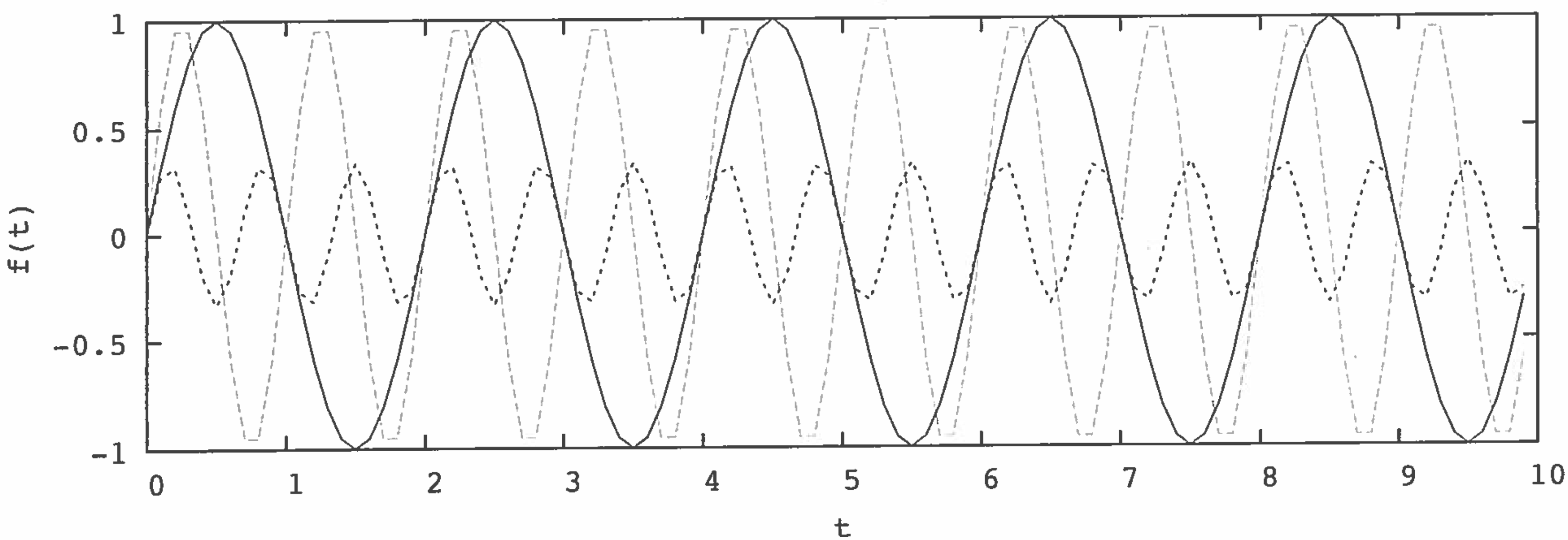
$\sin \pi t + \sin 2\pi t + \frac{1}{3} \sin 3\pi t$

a) plot each component

b) plot the sum



f(t)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
1
2
3
4
5
6
7
8
9
10
t



f(t)
1
0.5
0
-0.5
-1
0
1
2
3
4
5
6
7
8
9
10
t

25. 7.4.13

Show that $\int_a^b \sin^2 kx\,dx = \int_a^b \cos^2 kx\,dx = \frac{1}{2}(b-a)$

if $k(b-a)$ is an integral multiple of π or if kb and ka both integral multiples of $\pi/2$

Sol'n

n = integer

(a) $k(b-a) = n\pi \longrightarrow k = \dfrac{n\pi}{b-a}$

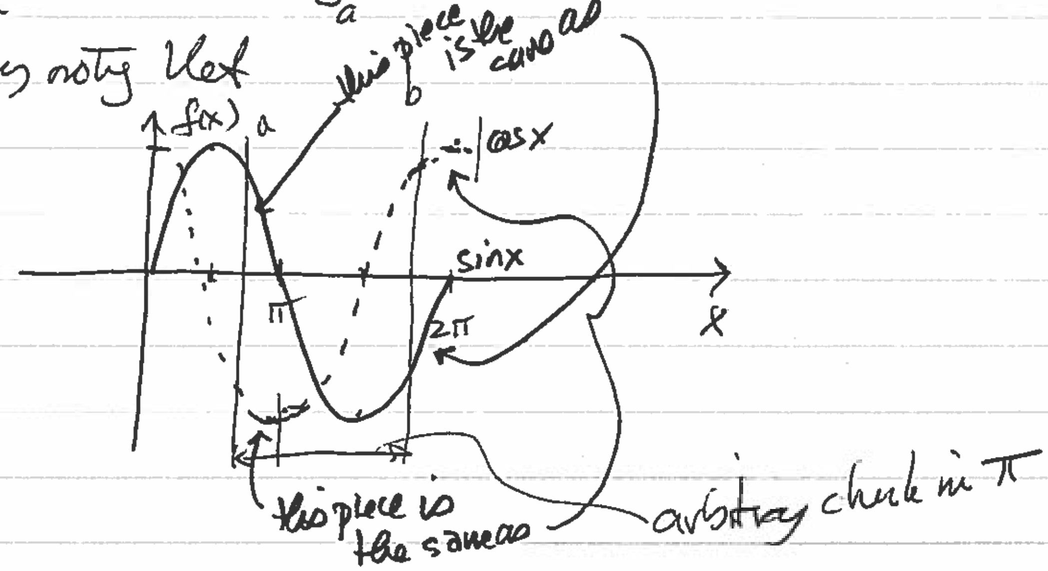
$\rightarrow \sin^2\left(\dfrac{n\pi x}{b-a}\right)$ & $\cos^2\left(\dfrac{n\pi x}{b-a}\right)$

each go over $a \rightarrow b$ $\rightarrow$ each goes over at least $\frac{1}{2}$ cycle.

(for even n sweep a full cycle). In this case,

$\int_a^b \sin^2 kx\,dx = \int_a^b \cos^2 kx\,dx$

by noting that



(b) $\int_a^b \cos^2 kx\,dx$ ← Use trig ID

$= \int_a^b \left[\cos 2kx + \sin^2(kx)\right]dx$

$= \left.\frac{\sin 2kx}{2k}\right|_a^b + \int_a^b \sin^2(kx)\,dx$

$= \frac{\sin 2kb}{2k} - \frac{\sin 2ka}{2k} + \int_a^b \sin^2(kx)\,dx$

if kb, ka are integral multiples of $\frac{\pi}{2} \Rightarrow$
$2kb, 2ka$ are integral multiples of π

$\Rightarrow \frac{\sin 2kb}{2k} - \frac{\sin 2ka}{2k} + \int_a^b \sin^2(kx)\,dx$

$= \int_a^b \sin^2(kx)\,dx$

and we have $\int_a^b \cos^2(kx)\,dx = \int_a^b \sin^2(kx)\,dx$

(c) okay, look at

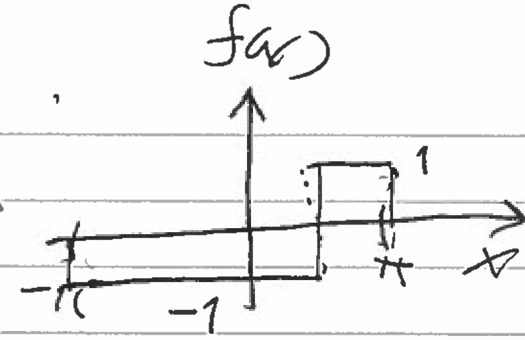
$$\int_a^b (\cos^2 kx + \sin^2 kx)\, dx = \int_a^b dx = (b-a)$$

and we have shown that $\int_a^b \cos^2 kx\, dx = \int_a^b \sin^2 kx\, dx$

$$\Rightarrow \int_a^b \cos^2 kx\, dx = \int_a^b \sin^2 kx\, dx = \frac{(b-a)}{2}$$

26. 7.5.4

$$f(x) = \begin{cases} -1 & -\pi < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

a) $\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} \frac{a_0}{2} dx$

$$-x\Big|_{-\pi}^{\frac{\pi}{2}} + x\Big|_{\frac{\pi}{2}}^{\pi} = \frac{a_0}{2} 2\pi = a_0\pi$$

$$-\left(\frac{\pi}{2} - -\pi\right) + \left(\pi - \frac{\pi}{2}\right) = -\pi = a_0\pi \rightarrow \boxed{a_0 = -1}$$

b) $\int_{-\pi}^{\pi} f(x) \cos n'x \, dx = \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos n'x \, dx$

$$-\int_{-\pi}^{\frac{\pi}{2}} \cos n'x \, dx + \int_{\frac{\pi}{2}}^{\pi} \cos n'x \, dx = a_n \pi \delta_{nn'}$$

$$-\frac{\sin n'x}{n'}\Big|_{-\pi}^{\frac{\pi}{2}} + \frac{\sin n'x}{n'}\Big|_{\frac{\pi}{2}}^{\pi} =$$

n' odd

$$-\left[\frac{\sin n'\frac{\pi}{2}}{n'}\right] + \left[\frac{-\sin n'\frac{\pi}{2}}{n'}\right] =$$

$$-\frac{2}{n'} \sin n'\frac{\pi}{2} = -2, 0, \frac{2}{3}, 0, -\frac{2}{5}, \ldots, n \text{ odd}$$

$$= -2 \sum_{\substack{n'=1 \\ n\ \text{odd}}}^{\infty} \frac{(-1)^{\frac{n'-1}{2}}}{n'} = \pi a_n \delta_{nn'}$$

$$\rightarrow \boxed{a_n = -\frac{2}{n\pi}(-1)^{\frac{n-1}{2}}, \ n\ \text{odd}}$$

c) $\int_{-\pi}^{\pi} f(x) \sin n'x \, dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin n'x \, dx$

$$-\int_{-\pi}^{\frac{\pi}{2}} \sin n'x \, dx + \int_{\frac{\pi}{2}}^{\pi} \sin n'x \, dx = \pi b_n \delta_{nn'}$$

$$\left[\frac{\cos n'x}{n'}\Big|_{-\pi}^{\pi/2}\right] + \left[-\frac{\cos n'x}{n'}\Big|_{\frac{\pi}{2}}^{\pi}\right] =$$

$$\left(\frac{\cos \frac{n'\pi}{2}}{n'} - \frac{\cos n'\pi}{n'}\right) - \left(\frac{\cos n'\pi}{n'} - \frac{\cos \frac{n'\pi}{2}}{n'}\right) =$$

$$\frac{2}{n'}\left(\cos \frac{n'\pi}{2} - \cos n'\pi\right) = +\frac{2}{1}, \frac{-1-1}{2}2, \frac{2}{3}, \cancel{1}\ 0, \ldots$$

$n'=1 \quad 2 \quad 3 \quad 4 \quad 5$

funny series
for b_n

$$\boxed{\rightarrow f(x) = -\frac{1}{2} + \frac{2}{\pi}\sum_{\substack{n=1, \\ \text{odd}}}^{\infty} (-1)^{\frac{n-1}{2}} \frac{\cos nx}{n} + \frac{2}{\pi}\left[\sin x - \frac{2\sin 2x}{2} + \frac{2}{3}\sin 3x + \cdots\right]}$$

we could have used Prob 3 to gener this solution.

Prob 3,

$$f_3(x) = \begin{cases} 0 & -\pi < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$

if we subtract 1 from $f_3(x) \rightarrow \begin{cases} -1 & -\pi < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$

so, if we then add $f_3(x)$ to the tasked $f_3(x)$

$$f_3(x) + \left[f_3(x) - 1\right] = \begin{cases} -1 & -\pi < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$

the $f(x)$ of 7.5.4 is found.

Linearity of fourier series $\rightarrow$ $f(x)$ of 7.5.4 can be constructed in this manner.

21. 7.6.14

Use: $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

$\Rightarrow f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right)$
$+ \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$

to find $\sum_{n, odd} \frac{1}{n^2} = \frac{\pi^2}{8}$

Use $\begin{cases} x = 0 \Rightarrow f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = 0 \\ x = \pi \Rightarrow f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(-1 - \frac{1}{3^2} - \frac{1}{5^2} + \cdots\right) = \pi \\ x = \frac{\pi}{2} \Rightarrow f(x) = \frac{\pi}{4} + \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right) = \frac{\pi}{2} \end{cases}$

a) $\Rightarrow \frac{\pi^2}{8} = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$ ✓

b) $\Rightarrow \frac{\pi^2}{8} = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$ ✓

c) $\Rightarrow \frac{\pi}{4} = \left(1 - \frac{1}{3} + \frac{1}{5} + \cdots\right)$

28 7.7.6

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ -1 & 0 < x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x < \pi \end{cases}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Sol^n

$$\int_{-\pi}^{\pi} f(x) e^{-in'x} dx = \sum_{n=-\infty}^{\infty} c_n \underbrace{\int_{-\pi}^{\pi} e^{inx} e^{-in'x} dx}$$

$$-\int_0^{\pi/2} e^{-in'x} dx + \int_{\frac{\pi}{2}}^{\pi} e^{-in'x} dx = 2\pi c_n \delta_{nn'}$$

$$\left.\frac{e^{-in'x}}{in'}\right|_0^{\frac{\pi}{2}} - \left.\frac{e^{-in'x}}{in'}\right|_{\frac{\pi}{2}}^{\pi} =$$

$$\left(\frac{e^{-in'\frac{\pi}{2}} - e^{i0}}{in'}\right) - \left(\frac{e^{-in'\pi} - e^{-in'\frac{\pi}{2}}}{in'}\right) =$$

$$\underbrace{2\frac{e^{-in'\pi/2}}{in'}} - \underbrace{\frac{1+e^{-in'\pi}}{in'}}_{\frac{2}{in'} \text{ if } n' \text{ even}} =$$

$$-\frac{2i}{in'}(-1)^{\frac{n'-1}{2}} \quad - \frac{2}{in'}(-1)^{\frac{n'-2}{2}}$$

n' odd $\quad$ n' even

→ funny form for series ad

$$f(x) = \frac{1}{2\pi}\sum_{\substack{n=-\infty,\\ \text{even}}}^{\infty} \left[\frac{2}{in}\left(1-(-1)^{\frac{n-2}{2}}\right) e^{inx}\right]$$

$$+\frac{1}{2\pi}\sum_{\substack{n=-\infty,\\ \text{odd}}}^{\infty} \left[\frac{2}{n}(-1)^{\frac{n-1}{2}} e^{inx}\right]$$

Write out $f(x)$

$$f(x) = -\frac{2}{1}(-1)e^{-ix} - \frac{2}{3}(-1)^{\frac{4}{2}} e^{-3ix} - \frac{2}{5}(-1)^{-\frac{6}{2}} e^{-5xi} - \cdots$$

$n=1 \qquad n=-3 \qquad n=-5$

$$+\frac{2}{1}e^{ix} + \frac{2}{3}(-1)^{\frac{2}{2}} e^{3ix} + \frac{2}{5}(-1)^{\frac{4}{2}} e^{i5x} + \cdots$$

$n=1 \qquad n=3 \qquad n=5$

$$-\frac{2}{i2}\cancel{(1-1)}^{0} e^{-2ix} - \frac{2}{i4}(1+1)e^{-4ix} - \frac{2}{i6}\cancel{(1-1)}^{0} e^{-6ix} - \cdots$$

$n=-2 \qquad n=-4 \qquad n=-6$

$$+\frac{2}{i2}\cancel{(1-1)}^{0} e^{2ix} + \frac{2}{i4}(1+1)e^{4ix} + \frac{2}{i6}\cancel{(1-1)}^{0} e^{6ix} + \cdots$$

$n=2 \qquad n=4 \qquad n=6$

$$= 2\left[e^{ix}\left(\frac{1}{1}\right) + \frac{1}{1}e^{-ix} + e^{3ix}\left(-\frac{1}{3}\right) + e^{-3ix}\left(-\frac{1}{3}\right) + e^{i5x}\left(\frac{1}{5}\right) + e^{-5ix}\left(\frac{1}{5}\right) + \cdots\right]$$

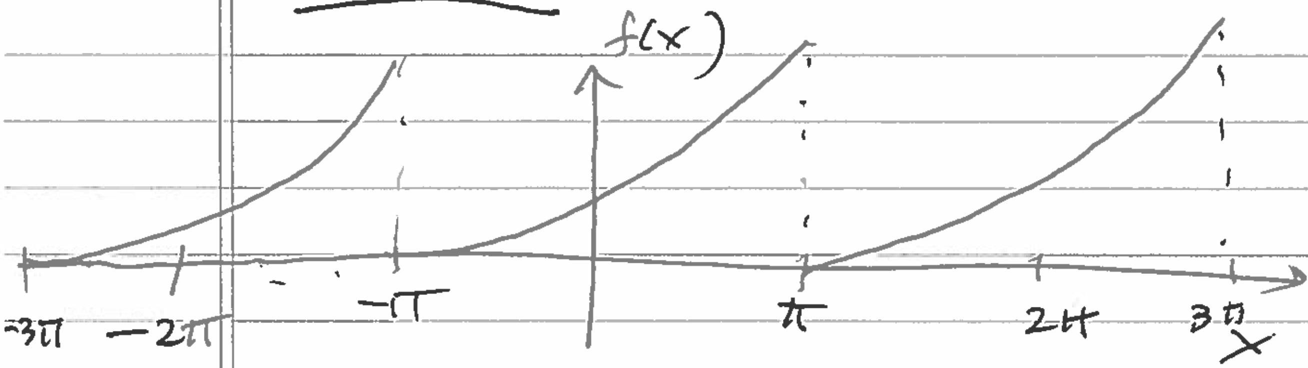
$$+2\left[\frac{2}{4i}\left(e^{4ix} - e^{-4ix}\right) + \cdots\right]$$

$$= 2\left[2\cos x - \frac{2}{3}\cos 3x + \frac{2}{5}\cos 5x + \cdots\right]$$

$$+2\left[\sin 4x + \frac{1}{2}\sin 6x + \cdots\right]$$

7.8 12

$f(x) = e^x, \quad -\pi < x < \pi$



a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx) + \sum_{n=1}^{\infty} (b_n \sin nx)$

(i) $\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} e^x\,dx = e^x \Big|_{-\pi}^{\pi} = e^{\pi} - e^{-\pi}$

$= 2\sinh\pi$

(ii) $\int_{-\pi}^{\pi} f(x)\cos n'x\,dx = \int_{-\infty}^{\infty} e^x \frac{e^{in'x} + e^{-in'x}}{2}\,dx$

$= \frac{e^{x+in'x}}{2(1+in')}\Big|_{-\pi}^{\pi} + \frac{e^{x-in'x}}{2(1-in')}\Big|_{-\pi}^{\pi}$

$= \frac{e^{\pi}(-1)^{n'}}{2(1+in')} - \frac{e^{-\pi}(-1)^{n'}}{2(1+in')}$

$+ \frac{e^{\pi}(-1)^{n'}}{2(1-in')} - \frac{e^{-\pi}(-1)^{n'}}{2(1-in')}$

$= \frac{(-1)^{n'}}{2}\left[\left(\frac{e^{\pi} - e^{-\pi}}{1+in'}\right) + \left(\frac{e^{\pi} - e^{-\pi}}{1-in'}\right)\right]$

$= (-1)^{n'} \sinh\pi \left[\frac{1-in' + 1+in'}{1+n'^2}\right] = (-1)^{n'} \frac{2\sinh\pi}{1+n'^2}$

(iii) $\int_{-\pi}^{\pi} f(x)\sin n'x\,dx = \int_{-\pi}^{\pi} e^{x}\left(\frac{e^{in'x}-e^{-in'x}}{2i}\right)dx$

$= \frac{e^{x+in'x}}{2i(1+in')}\Big|_{-\pi}^{\pi} - \frac{e^{x-in'x}}{2i(1-in')}\Big|_{-\pi}^{\pi}$

$= \frac{e^{\pi}(-1)^{n'}}{2i(1+in')} - \frac{e^{-\pi}(-1)^{n'}}{2i(1+in')} - \frac{e^{\pi}(-1)^{n'}}{2i(1-in')} + \frac{e^{-\pi}(-1)^{n'}}{2i(1-in')}$

$= \frac{(-1)^{n'}}{2i}(e^{\pi}-e^{-\pi})\left[\frac{1}{\cancel{2i}(1+in')} - \frac{1}{1-in'}\right]$

$= \frac{(-1)^{n'}}{i}\sinh\pi\left[\frac{1-in'-1-in'}{1+n'^2}\right]$

$= -\frac{(-1)^{n'}}{1+n'^2}2n'\sinh\pi$

$\Rightarrow a_0 = \frac{2}{\pi}\sinh\pi$

$a_n = \frac{2}{\pi}\sinh\pi\frac{(-1)^n}{1+n^2}$

$b_n = -\frac{2}{\pi}\sinh\pi\frac{(-1)^n}{1+n^2}$

$\Rightarrow f(x) = \frac{\sinh\pi}{\pi}\left[\sum_{n=1}^{\infty}\frac{2(-1)^n}{1+n^2}\cos nx + \sum_{n=1}^{\infty}\frac{2(-1)^n}{1+n^2}\sin nx + 1\right]$

b) $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$

$$\int_{-\pi}^{\pi} f(x) e^{-in'x} dx = 2\pi c_n \delta_{nn'}$$

$$\int_{-\pi}^{\pi} e^{+x-in'x} dx =$$

$$\rightarrow \frac{e^{x-in'x}}{1-in'}\Big|_{-\pi}^{\pi} = \frac{e^{\pi}(-1)^{n'}}{1-in'} - \frac{e^{-\pi}(-1)^{n'}}{1-in'}$$

$$= \frac{(-1)^{n'}}{1-in'}\left(e^{\pi} - e^{-\pi}\right)$$

$$= \frac{2}{1-in'}(-1)^{n'} \sinh\pi$$

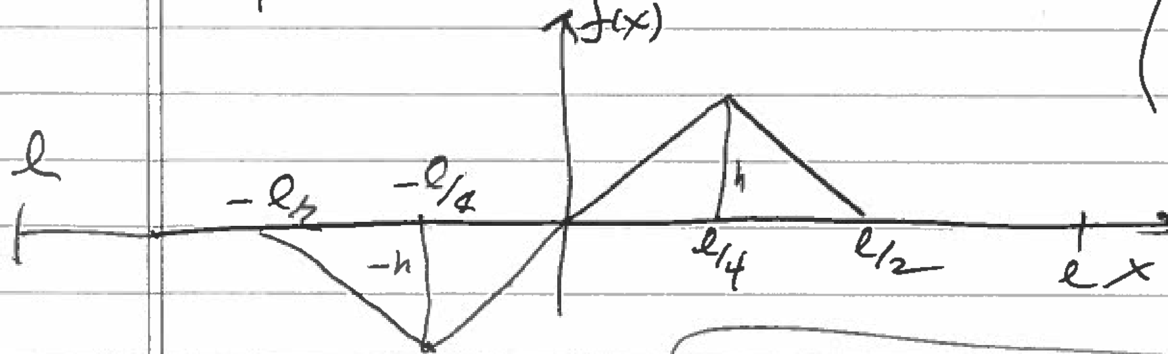
$$\Rightarrow \boxed{f(x) = \sum_{n=-\infty}^{\infty} \frac{\cancel{2}\,1}{1-in'}(-1)^{n'} \frac{\sinh\pi}{\pi} e^{inx}}$$

$$= \frac{\sinh\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n'}}{1-in'} e^{inx}$$

30 · 7.9.24

Expand as a sine series

$$f(x)=\begin{cases} x\frac{4h}{l}, & 0<x<\frac{l}{4} \\ 2h-x\frac{4h}{l}, & \frac{l}{4}<x<\frac{l}{2} \\ 0, & \frac{l}{2}<x \end{cases}$$



$$\Rightarrow \text{Period} = 2l \rightarrow \boxed{\text{frequency} = \frac{2\pi n}{2l} = \frac{\pi}{l} n}$$

$$\Rightarrow b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi}{l}x\right)dx$$

$$= \frac{2}{l}\left[\int_0^{l/4} x\frac{4h}{l}\sin\left(\frac{n\pi}{l}x\right)dx + \int_{l/4}^{l/2}\left(2h - x\frac{4h}{l}\right)\sin\left(\frac{n\pi}{l}x\right)dx\right.$$

$$= \frac{2}{l}\left[\frac{4h}{l}\frac{l}{n\pi}\frac{l}{n\pi}\int_0^{\frac{n\pi}{4}}\xi\sin\xi\, d\xi + \int_{l/4}^{l/2} 2h\sin\left(\frac{n\pi}{l}x\right)dx - \int_{\frac{n\pi}{4}}^{\frac{n\pi}{2}}\frac{4h}{l}\frac{l^2}{n^2\pi^2}\xi\sin\xi\, d\xi\right.$$

$$= \frac{2}{l}\left[\frac{4h}{l}\frac{l^2}{n^2\pi^2}\left(\sin\xi - \xi\cos\xi\right)\Big|_0^{\frac{n\pi}{4}} - \frac{4h}{l}\frac{l^2}{n^2\pi^2}\left(\sin\xi - \xi\cos\xi\right)\Big|_{\frac{n\pi}{4}}^{\frac{n\pi}{2}}\right.$$

$$\left. + \frac{2h}{\frac{n\pi}{l}}\left(-\cos\left(\frac{n\pi}{l}x\right)\right)\Big|_{l/4}^{l/2}\right]$$

$$b_n = \frac{2}{l}\left[\frac{4hl}{n^2\pi^2}\left\{\left(\sin\frac{n\pi}{4} - \frac{n\pi}{4}\cos\frac{n\pi}{4}\right) - \left(\sin\frac{n\pi}{2} - \sin\frac{n\pi}{4} - \frac{n\pi}{2}\cos\frac{n\pi}{2} + \frac{n\pi}{4}\cos\frac{n\pi}{4}\right)\right\}\right.$$

$$\left.\cancel{+}\ \frac{2hl}{n\pi}\left(\cos\frac{n\pi}{2} - \cos\frac{n\pi}{4}\right)\right]$$

$$b_n = \frac{2}{l}\left[\frac{4hl}{n^2\pi^2}\left\{2\sin\frac{n\pi}{4} - \sin\frac{n\pi}{2} - \frac{n\pi}{2}\cos\frac{n\pi}{4} + \frac{n\pi}{2}\cos\frac{n\pi}{2}\right\}\right.$$

$$\left. - \frac{2hl}{n\pi}\left(\cos\frac{n\pi}{2} - \cos\frac{n\pi}{4}\right)\right]$$

$$b_n = \left[\frac{8h}{n^2\pi^2}\left(2\sin\frac{n\pi}{4} - \sin\frac{n\pi}{2} - \frac{n\pi}{2}\cos\frac{n\pi}{4} + \frac{n\pi}{2}\cos\frac{n\pi}{2}\right)\right.$$

$$\left. - \frac{4h}{n\pi}\left(\cos\frac{n\pi}{2} - \cos\frac{n\pi}{4}\right)\right]$$

$$= \frac{8h}{(n\pi)^2}\left[2\sin\frac{n\pi}{4} - \sin\frac{n\pi}{2}\right] + \frac{8h}{(n\pi)}\left[-\cancel{\frac{1}{2}\cos\frac{n\pi}{4}} + \cancel{\frac{1}{2}\cos\frac{n\pi}{2}} - \cancel{\frac{1}{2}\cos\frac{n\pi}{2}} + \cancel{\frac{1}{2}\cos\frac{n\pi}{4}}\right.$$

$$= \frac{8h}{(n\pi)^2}\left[2\sin\left(\frac{n\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\right]$$

first few terms

$$b_1 = \frac{8h}{\pi^2}\left(2\sin\frac{\pi}{4} - 1\right) = \frac{16h}{\pi^2}\left(\sin\frac{\pi}{4} - \frac{1}{2}\right)$$

$$b_2 = \frac{8h}{4\pi^2}(2) = \frac{\cancel{4}4h}{\pi^2}$$

$$b_3 = \frac{8h}{9\pi^2}\left(3\sin\frac{3\pi}{4} + 1\right) = \frac{+8h}{3\pi^2}\sin\frac{\pi}{4} + 1$$

$$b_4 = \frac{8h}{16\pi^2}(0) = 0$$

$$b_5 = \frac{8h}{25\pi^2}\left(2\sin\frac{5\pi}{4} - 1\right) = \frac{-16}{25\pi^2}\sin\frac{\pi}{4} - \frac{8h}{25\pi^2}$$

$$b_6 = \frac{8h}{36\pi^2}(-2) = \frac{-4h}{9\pi^2}$$

$$b_7 = \frac{8h}{49\pi^2}\left(2\sin\frac{7\pi}{4} + 1\right) = \frac{-16h}{49\pi^2}\sin\frac{\pi}{4} + \frac{8h}{49\pi^2}$$

$$b_8 = \frac{8h}{64\pi^2}(0) = 0$$

and, in general,

$$f(x) = \sum_{n=1}^{\infty} \frac{16h}{\pi^2}\left(\frac{\sin\left(\frac{n\pi}{4}\right)}{n^2} - \frac{\sin\left(\frac{n\pi}{2}\right)}{2n^2}\right)\sin\left(\frac{n\pi}{\ell}x\right)$$