Homework 5

Due: 15 November 2018

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31. page 377, 7.11.5
32. page 385, 7.12.18
33. page 386, 7.12.22
34. page 387, 7.13.9
35. page 633, 13.3.11
36. page 662, 13.9.4
37. A stretched string extends from $x = -\infty$ to $+\infty$. Between $x = 1$ and 2, it is displaced forming a waveform with shape $-\sin(\pi x)$. The string is then released from rest. Find the subsequent motion using d'Alembert's solution for the wave equation. Sketch the motion at several subsequent times. Suppose that the string is not displaced initially but is instead struck with a mallet, giving it an initial velocity impulse of $\cos(\pi x)$ between $x = 1$ and 2. Find the motion of the string and sketch it at several subsequent times.
38. Verify properties 1 through 7 of Fourier transforms.

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(7) 7.11.5

Sum $1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$

$$=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} \text{ or } \sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{1}{n^2}$$

Prob 9.6 $f(x)=\begin{cases}-1, & -\pi<x<0\\ 1, & 0<x<\pi\end{cases}$

$$\rightarrow f(x)=\frac{4}{\pi}\left[\sin\frac{\pi x}{l}+\frac{1}{3}\sin\frac{3\pi x}{l}+\frac{1}{5}\sin\frac{5\pi x}{l}+\cdots\right]$$

$$=\frac{4}{\pi}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\left(\frac{1}{n}\right)\sin\left(\frac{n\pi x}{l}\right)$$

Sol^n Parseval's thm

$$\overline{|f(x)|^2}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)^2dx=\left(\frac{a_0}{2}\right)^2+\frac{1}{2}\sum_{n=1}^{\infty}\left[a_n^2+b_n^2\right]$$

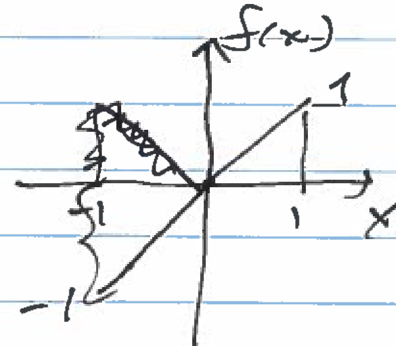
$$=\frac{1}{2\pi}\int_{-\pi}^{0}(+1)^2dx+\frac{1}{2\pi}\int_{0}^{\pi}(-1)^2dx=\frac{1}{2}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\left(\frac{4}{n\pi}\right)^2$$

$$\Rightarrow \frac{2\pi}{2\pi}=1=\frac{8}{\pi^2}\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{8}}$$

(32) 7.12.18

$$f(x) = \begin{cases} x & |x|<1 \\ 0 & |x|>1 \end{cases}$$



$f(x) = -f(-x)$

$\rightarrow f(x)$ is odd

(a) $\Rightarrow g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \alpha x \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x \sin \alpha x \, dx$$

let $u = x$, $dV = \sin \alpha x \, dx$

$du = dx \quad V = -\frac{\cos x\alpha}{\alpha}$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \left[-\frac{x \cos \alpha x}{\alpha} \Big|_0^1 + \frac{1}{\alpha} \int_0^\infty \cos \alpha x \, dx \right]$$

$$= + \sqrt{\frac{2}{\pi}} \left[-\frac{\cos \alpha}{\alpha} - 0 + \frac{\sin \alpha x}{\alpha^2} \Big|_0^1 \right]$$

$$\boxed{g_s(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left(\frac{\sin \alpha}{\alpha} - \cos \alpha \right)}$$

$$\rightarrow f(x) = \int_0^\infty \sqrt{\frac{2}{\pi}} \left\{ \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left(\frac{\sin \alpha}{\alpha} - \cos \alpha \right) \right\} \sin \alpha x \, d\alpha$$

(b) $g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$

$= \frac{1}{2\pi}\int_{-1}^{1} x e^{-i\alpha x}\, dx$

$= \frac{1}{2\pi}\left[\frac{e^{-i\alpha x}}{(-i\alpha)^2}\left(-i\alpha x - 1\right)\right]_{-1}^{1}$

$= -\frac{1}{2\pi\alpha^2}\left[e^{-i\alpha}\left(-i\alpha - 1\right) - e^{i\alpha}\left(i\alpha - 1\right)\right]$

$= -\frac{1}{2\pi\alpha^2}\left[-i\alpha\left(e^{-i\alpha} + e^{i\alpha}\right) + \left(e^{i\alpha} - e^{-i\alpha}\right)\right]$

$= -\frac{1}{\pi\alpha^2}\left[-\alpha i \cos\alpha + i \sin\alpha\right]$

$g(\alpha) = \frac{i}{\pi\alpha}\left(\cos\alpha - \frac{\sin\alpha}{\alpha}\right)$

(33). 7.12.22

Show that $\int_0^\infty j_1(\alpha) \sin \alpha x \, d\alpha = \begin{cases} \frac{\pi}{2} x & -1 < x < 1 \\ 0 & |x| > 1 \end{cases}$

where $j_1(\alpha) = \left(\cos\alpha - \frac{\sin\alpha}{\alpha}\right)$

from 7.12.18

$$f(x) = \int_0^\infty \frac{2}{\pi\alpha}\left[\frac{\sin\alpha}{\alpha} - \cos\alpha\right] \sin \alpha x \, d\alpha$$

$$= \begin{cases} x, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

note: spherical Bessel functions

$$j_n = x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right)$$

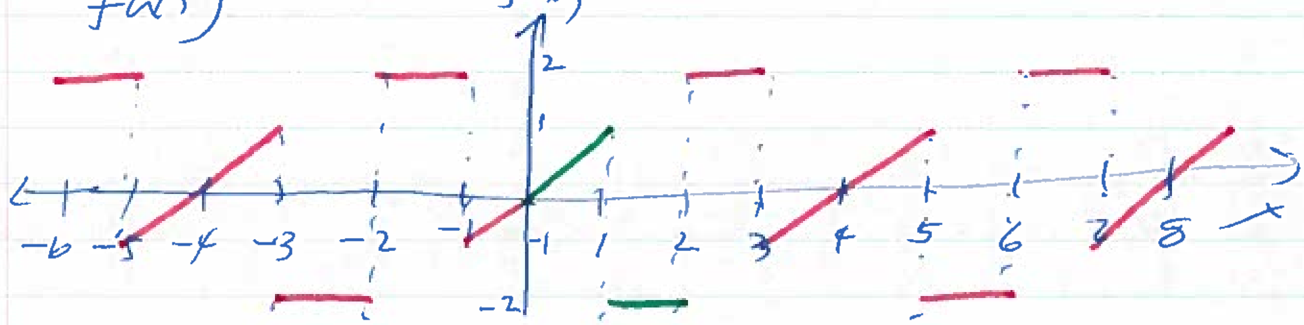
$$\rightarrow \boxed{\begin{aligned} j_1 &= x\left(-\frac{1}{x}\frac{d}{dx}\right)\left(\frac{\sin x}{x}\right) \\ &= -\left[\frac{\cos x}{x} - \frac{\sin x}{x^2}\right] \\ &= \left[\frac{\sin x}{x} - \cos x\right]\frac{1}{x} \end{aligned}}$$

← agrees w/ 7.12.18, but not 7.12.22

34 2.13.9

Given $f(x)=\begin{cases} x, & 0<x<1 \\ -2, & 1<x<2 \end{cases}$

ⓐ Sketch at least 3 periods of above sine series (an odd $f(x)$)



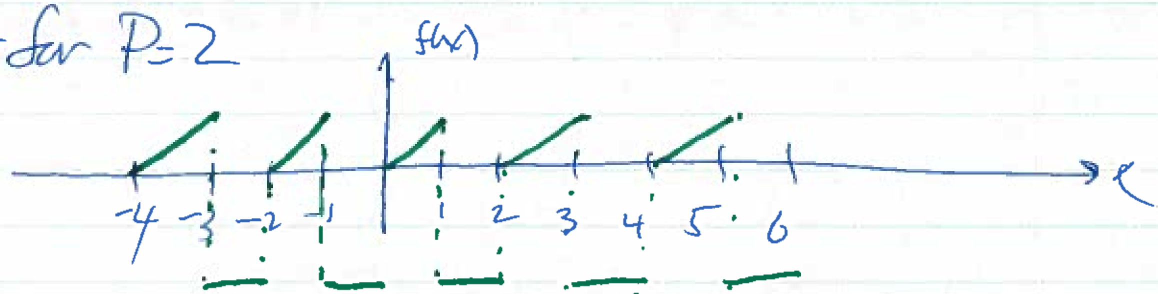
ⓑ at $x=1$, converges to $\frac{1+(-2)}{2}=-\frac{1}{2}=f(1)$

at $x=2$, converges to $f(2)=0$

at $x=0$, converges to $f(0)=0$

at $x=-1$, converges to $f(-1)=\frac{1}{2}$

ⓒ Plot for $P=2$



ⓓ find $\sum_{n=-\infty}^{\infty} C_n e^{-in\pi x} = f(x)$

$$f(x)=\sum_{n=\infty}^{\infty} C_n e^{-in\pi x}$$

$$\int^{1}\left(xe^{-in\pi x}dx\right)-\int^{2} 2e^{-in\pi x}dx = 2\pi C_n$$

$$\frac{x e^{-in\pi x}}{-in\pi}\Big|_0^1 + \frac{1}{in\pi}\int_0^1 e^{-in\pi x}\,dx - 2\,\frac{e^{-in\pi x}}{-in\pi}\Big|_1^2 = 2\pi C_n$$

$$-\frac{e^{-in\pi}}{in\pi} - \frac{1}{(in\pi)^2}\,e^{-in\pi x}\Big|_0^1 + \frac{2}{in\pi}\left(e^{-2in\pi} - e^{-in\pi}\right) = 2\pi C_n$$

$$-\frac{(-1)^n}{in\pi} - \left(\frac{e^{-in\pi}-1}{(in\pi)^2}\right) + \frac{2}{in\pi}\left(1-\cos n\pi\right) = 2\pi C_n$$

$$-\frac{\cos n\pi}{in\pi} - \frac{\cos n\pi - 1}{(in\pi)^2} + \frac{2}{in\pi} - \frac{2\cos n\pi}{in\pi} = 2\pi C_n$$

$$\rightarrow C_n = \begin{cases} \frac{1}{2\pi}\,\frac{-1}{in\pi}, & n \text{ even} \\ \frac{1}{2\pi}\left(\frac{1}{in\pi} + \frac{2}{(in\pi)^2} + \frac{2}{in\pi} + \frac{2}{in\pi}\right), & n \text{ odd} \end{cases}$$

oops didn't have to do this

$$= \frac{1}{2\pi}\begin{cases} \frac{-1}{in\pi}, & n \text{ even} \\ \frac{5}{in\pi} + \frac{2}{(in\pi)^2}, & n \text{ odd} \end{cases}$$

(e) find $\sum_{n=-\infty}^{\infty} C_n^* C_n$ from Parseval's theorem - find

$$|f(x)|^2 = \int_0^2 f(x)^2\,dx$$

$$= \int_0^1 x^2\,dx + \int_1^2 4\,dx$$

learn to read & integrate

$$= \frac{x^3}{3}\Big|_0^1 + 4x\Big|_1^2 = \frac{1}{3} + \{8-4\} = \frac{13}{3}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} C_n^* C_n = \frac{13}{6}$$

35) 13.3.15

Solve the "particle in a box" problem to find $\Psi(x,t)$ if

$$\Psi(x,0) = 1 \text{ on } [0,\pi].$$

What is E_n. Plot $|\Psi_n(x,t)|^2$

(i) $$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

Let $\Psi(x,t) = \psi(x)\,T(t)$

$$\Rightarrow -\frac{\hbar^2}{2m}T(t)\frac{\partial^2}{\partial x^2}\psi(x) = i\hbar\psi(x)\frac{\partial T(t)}{\partial t}$$

divide by $\Psi(x,t)$

$$-\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{\partial^2\psi(x)}{\partial x^2} = i\hbar\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = E$$

(ii) Solve for $T(t)$

$$i\hbar\frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = E$$

$$\Rightarrow \frac{1}{T(t)}\frac{\partial T(t)}{\partial t} = -i\frac{E}{\hbar}$$

$$\rightarrow \boxed{T(t) = T(0)e^{-i\frac{E}{\hbar}t}}$$

(iii) Solve for $\psi(x)$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) = E\psi(x)$$

$$\Rightarrow \psi(x) = a\cos\sqrt{\frac{2mE}{\hbar^2}}\,x + b\sin\sqrt{\frac{2mE}{\hbar^2}}\,x$$

@ at $x=0, \cancel{\pi}\,l$, $\psi(x)=0$

$$x=0 \Rightarrow a=0$$

$$x=\cancel{\pi}\,l \Rightarrow \sqrt{\frac{2mE}{\hbar^2}}\,\cancel{\pi}\,l = n\pi \rightarrow E = \left(\frac{n\pi}{l}\right)^2 \frac{\hbar^2}{2m}$$

darn it

$$\boxed{E_n = \frac{\hbar^2 n^2}{2ml^2}}$$

$$\Rightarrow \boxed{\Psi(x,t) = \sum_{n=1}^{\infty} b_n \sin nx\, e^{-i\frac{\hbar n^2}{2ml^2}t}}$$

(iv) at $t=0$, $\psi(x) = 1$

$$\Rightarrow 1 = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\int_0^{\pi} \sin n'x\,dx = \sum_{n=1}^{\infty} \int_0^{\pi} b_n \sin nx \sin n'x\,dx$$

$$\left.\frac{-\cos nx}{n}\right|_0^{\pi} = \delta_{nn'}\frac{\pi}{2} b_n$$

$$\Rightarrow b_n = \frac{2}{\pi}\left[\frac{-\cos n\pi}{n} + \frac{1}{n}\right]$$

$$= \frac{4}{\pi n}, \text{ n odd}$$

$$\Rightarrow \boxed{\Psi(x,t) = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \left(\frac{4}{n\pi}\right) \sin nx \, e^{-i \frac{\hbar n^2}{2m\ell^2} t}}$$

36 ~~3~~ P. 13.9.4

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}$$

find the sine transform

(a) $$\int_0^\infty \frac{\partial^2 T}{\partial x^2} \sin kx \, dx = \frac{1}{\alpha^2}\int_0^\infty \frac{\partial T}{\partial t} \sin kx \, dx$$

$$\Rightarrow -k^2 C_s(k,t) = \frac{1}{\alpha^2}\frac{\partial}{\partial t} C_s(k,t)$$

and $$\boxed{C_s(k,t) = C_s(k,0)\, e^{-\alpha^2 k^2 t}}$$

(b) find $T(x,t)$,

$$T(x,t) = \int_0^\infty C_s(k,t) \sin kx \, dk$$

$$= \int_0^\infty \underbrace{C_s(k,0)}_{} e^{-\alpha^2 k^2 t} \sin kx \, dk$$

what is Fourier transform of T at $t=0$

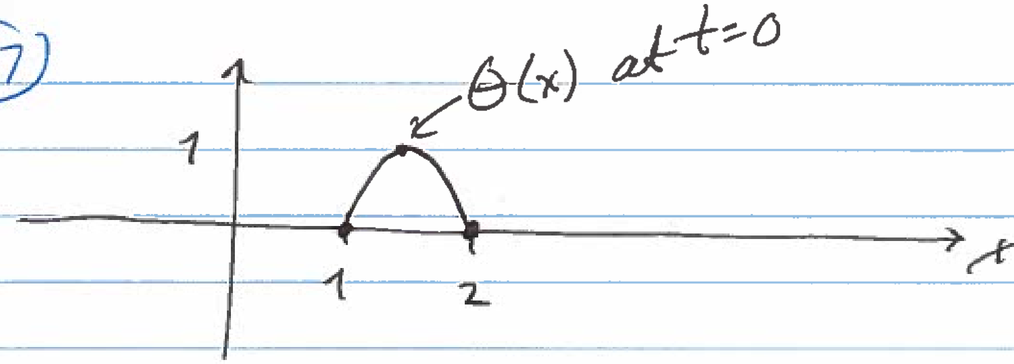
(c) at $t=0$, $$C_s(k,0) = \int_0^\infty 100° \sin kx \, dx$$

$$= -\frac{100 \cos kx}{k}\Big|_0^1$$

$$= -\frac{100}{k}\left[\cos k - 1\right]$$

$$\Rightarrow \boxed{T(x,t) = \int_0^\infty \frac{100}{k}(1-\cos k)\, e^{-\alpha^2 k^2 t} \sin kx \, dk}$$

(37)



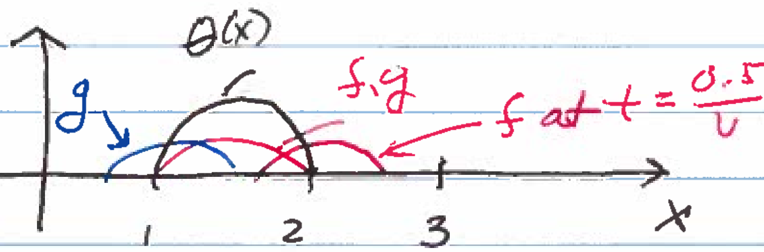
a) at $t=0$, $f(u) = g(v) = \frac{1}{2}\theta(x)$

$\rightarrow f(x,0) = g(x,0) = \frac{1}{2}\sin \pi x$, 0 for $x>2$ & $x<1$

$\Rightarrow f(x,t) = \frac{1}{2}\sin\pi(x-vt)$, $\longrightarrow$ propagate

$g(x,t) = \frac{1}{2}\sin\pi(x+vt)$, $\longleftarrow$ propagate

$\Rightarrow$



b) at $t=0$, $f(x,0) + g(x,0) = 0$ but

$\frac{\partial f}{\partial t} + \frac{\partial g}{\partial t} \neq 0$, $\frac{\partial f}{\partial t} + \frac{\partial g}{\partial t} = \dot{\theta}(x)$

$\rightarrow \frac{\partial f(u)}{\partial u} = \frac{1}{2}\cos\pi x$, $\frac{\partial g(v)}{\partial v} = +\frac{1}{2}\cos\pi x$, and $\frac{\partial u}{\partial t}$

and

$f + g = 0$ which follows from above

at $t=0$ $\frac{\partial f}{\partial u}, \frac{\partial g}{\partial u} \longrightarrow f = \frac{\sin\pi x}{2\pi}$, $g = -\frac{\sin\pi x}{2\pi}$

b) at $t=0$, $f(x,0)+g(x,0)=0$ but

$$\dot{f}(x,0)+\dot{g}(x,0) \neq 0, = \dot{\theta}(x)$$

$$\dot{f}(x,0)+\dot{g}(x,0) = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial t}$$

$$= -v\frac{\partial f}{\partial u} + v\frac{\partial g}{\partial v}$$

$$= \dot{\theta}(x) \text{ at } t=0$$

$$\frac{\cos\pi x}{v} = -\frac{\partial f}{\partial u} + \frac{\partial g}{\partial v} \quad \text{at } t=0$$

$$\Rightarrow \begin{cases} \dfrac{\partial f}{\partial u} = \dfrac{-1}{2}\dfrac{\cos\pi x}{v} \\ \dfrac{\partial g}{\partial v} = \dfrac{1}{2}\dfrac{\cos\pi x}{v} \end{cases}$$

$$\Rightarrow \boxed{f = -\frac{\sin\pi x}{2v\pi}, \quad g = +\frac{\sin\pi x}{2v\pi}}$$

35

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Property 1: Linearity

$$FT(a_1 f_1 + a_2 f_2) = \int_{-\infty}^{\infty} (a_1 f_1 + a_2 f_2) e^{-i\omega t} dt$$

$$= a_1 \int_{-\infty}^{\infty} f_1 e^{-i\omega t} dt + a_2 \int_{-\infty}^{\infty} f_2 e^{-i\omega t} dt$$

$$= a_1 FT(f_1) + a_2 FT(f_2)$$

Property 2: Symmetry

a) $F(\omega) = FT(f(t)) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

$$\rightarrow f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

b) let $t \rightarrow -t \Rightarrow f(-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega$

interchange ω and t

$$\Rightarrow f(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) e^{-i\omega t} dt$$

Property 3: Time translation

show $FT(f[t-t_0]) = e^{-i\omega t_0} FT(f(t)) = e^{-i\omega t_0} F(\omega)$

Pf. $FT(f[t-t_0]) = F(\omega) = \int_{-\infty}^{\infty} f(t-t_0) e^{-i\omega t} dt$

let $z = t - t_0 \rightarrow dz = dt$

$$\rightarrow F(\omega) = \int_{-\infty}^{\infty} f(z) e^{-i\omega(z+t_0)} dz$$

$$= e^{-i\omega t_0} FT(f(z))$$

Property 4: Frequency Translation

what is $FT(e^{i\omega_0 t} f(t))$?

$$FT(e^{i\omega_0 t} f(t)) = \int_{-\infty}^{\infty} \left[e^{i\omega_0 t} f(t)\right] e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-i(\omega-\omega_0)t}\, dt$$

let $\Omega = \omega - \omega_0$ ~~and~~ find

$$\Rightarrow FT(e^{i\omega_0 t} f(t)) = \int_{-\infty}^{\infty} f(t)\, e^{-i\Omega t}\, dt$$

$$= FT(f(t))$$

$$= F(\Omega)\ \checkmark$$

Property 5: Change of time scale

show $FT(f(t)) = \int_{-\infty}^{\infty} f(at)\, e^{-i\omega t}\, dt = \frac{1}{a} F\left(\frac{\omega}{a}\right)$

Suppose we change time scale by factor a, $t \to at$

let $z = at \to t = (z/a) \to dt = dz/a$

$$\cancel{\Rightarrow \int_{-\infty}^{\infty} f(at)\, e^{-i\omega at}\, d(at)}$$

$$\cancel{\int_{-\infty}^{\infty} f(z)}$$

$$\Rightarrow F(\omega) = \int_{-\infty}^{\infty} f(z)\, e^{\overbrace{-i\omega\left(\frac{z}{a}\right)}^{-i\left(\frac{\omega}{a}\right)z}}\, \frac{dz}{a} = \frac{1}{a} F\left(\frac{\omega}{a}\right)\ \checkmark$$

Property 6: FT of $\frac{df}{dt}$

$$FT\left(\frac{df}{dt}\right) = \int_{-\infty}^{\infty}\left(\frac{df}{dt}\right)e^{-i\omega t}dt = F(\omega)$$

do by parts

$$du = \frac{df}{dt}dt, \quad V = e^{-i\omega t}$$

$$u = f, \quad dV = -i\omega e^{-i\omega t}dt$$

$$\Rightarrow \quad = fe^{-i\omega t}\Big|_{-\infty}^{\infty} + i\omega\int_{-\infty}^{\infty} fe^{-i\omega t}dt$$

$$= 0 + i\omega FT(f(t)) \checkmark$$

if f doesn't have $\int_{-\infty}^{\infty} f dt \to \infty$

Property 7: Differentiation w/to frequency ω

show $\frac{d}{d\omega}\left[FT(f(t))\right] = \frac{d}{d\omega}\left[F(\omega)\right] = FT\left[-itf(t)\right]$

Pf

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$\frac{dF(\omega)}{d\omega} = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$= \int_{-\infty}^{\infty} f(t)\, dt \frac{d}{d\omega}\left[e^{-i\omega t}\right]$$

$$= +\int_{-\infty}^{\infty} f(t)\, dt \left(-i\omega e^{-i\omega t}\right)$$

$$= -i\omega \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$= -i\omega\, FT(f(t))$$

$$= -i\omega F(\omega) \checkmark$$