Homework 6

Due: 18 November 2016

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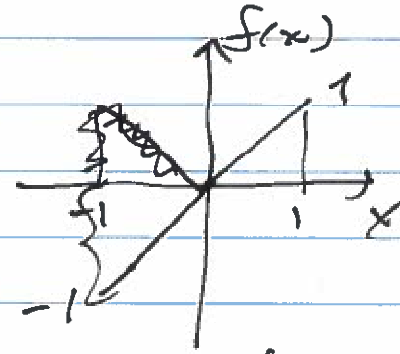
36. page 385, 7.12.18
37. page 385, 7.12.22
38. page 632, 13.3.3
39. page 662, 13.9.4
40. A stretched string extends from $x = -\infty$ to $+\infty$. Between $x = 1$ and 2, it is displaced forming a waveform with shape $-\sin(\pi x)$. The string is then released from rest. Find the subsequent motion using d'Alembert's solution for the wave equation. Sketch the motion at several subsequent times. Suppose that the string is not displaced initially but is instead struck with a mallet, giving it an initial velocity impulse of $\cos(\pi x)$ between $x = 1$ and 2. Find the motion of the string and sketch it at several subsequent times.
41. Verify properties 1 through 7 of Fourier transforms given in class.

HW 6

36. 2.12.18

$$f(x) = \begin{cases} x & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$



$f(x) = -f(-x)$
$\rightarrow f(x)$ is odd

(a) $\Rightarrow g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \alpha x \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x \sin \alpha x \, dx$$

Let $u = x$, $dV = \sin \alpha x \, dx$
$du = dx$ $\quad V = -\frac{\cos x\alpha}{\alpha}$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \left[-\frac{x \cos \alpha x}{\alpha} \Big|_0^1 + \frac{1}{\alpha} \int_0^\infty \cos \alpha x \, dx \right]$$

$$= + \sqrt{\frac{2}{\pi}} \left[-\frac{\cos \alpha}{\alpha} - 0 + \frac{\sin \alpha x}{\alpha^2} \Big|_0^1 \right]$$

$$\boxed{g_s(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left(\frac{\sin \alpha}{\alpha} - \cos \alpha \right)}$$

$$\rightarrow f(x) = \int_0^\infty \sqrt{\frac{2}{\pi}} \left\{ \sqrt{\frac{2}{\pi}} \frac{1}{\alpha} \left(\frac{\sin \alpha}{\alpha} - \cos \alpha \right) \right\} \sin \alpha x \, d\alpha$$

ⓑ $g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$

$= \frac{1}{2\pi}\int_{-1}^{1} x\, e^{-i\alpha x}\, dx$

$= \frac{1}{2\pi}\left[\frac{e^{-i\alpha x}}{(-i\alpha)^2}\left(-i\alpha x - 1\right)\right]_{-1}^{1}$

$= -\frac{1}{2\pi\alpha^2}\left[e^{-i\alpha}\left(-i\alpha - 1\right) - e^{i\alpha}\left(i\alpha - 1\right)\right]$

$= -\frac{1}{2\pi\alpha^2}\left[-i\alpha\left(e^{-i\alpha} + e^{i\alpha}\right) + \left(e^{i\alpha} - e^{-i\alpha}\right)\right]$

$= -\frac{1}{\pi\alpha^2}\left[-\alpha i \cos\alpha + i \sin\alpha\right]$

$g(\alpha) = \frac{i}{\pi\alpha}\left(\cos\alpha - \frac{\sin\alpha}{\alpha}\right)$

37. 7.12.22

Show that $\int_0^\infty j_1(\alpha) \sin \alpha x \, d\alpha = \begin{cases} \frac{\pi}{2} x & -1 < x < 1 \\ 0 & |x| > 1 \end{cases}$

where $j_1(\alpha) = \left(\cos\alpha - \frac{\sin\alpha}{\alpha} \right)$

from 7.12.18

$$f(x) = \int_0^\infty \frac{2}{\pi\alpha} \left[\frac{\sin\alpha}{\alpha} - \cos\alpha \right] \sin \alpha x \, d\alpha$$

$$= \begin{cases} x , & |x| < 1 \\ 0 , & |x| > 1 \end{cases}.$$

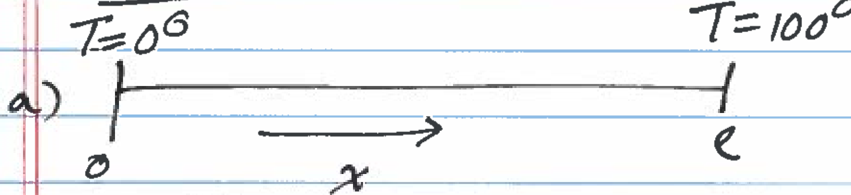
note: spherical Bessel functions,

$$j_n = x^n \left(-\frac{1}{x}\frac{d}{dx} \right)^n \left(\frac{\sin x}{x} \right)$$

$$\rightarrow \boxed{\begin{aligned} j_1 &= x \left(-\frac{1}{x}\frac{d}{dx} \right) \left(\frac{\sin x}{x} \right) \\ &= -\left[\frac{\cos x}{x} - \frac{\sin x}{x^2} \right] \\ &= \left[\frac{\sin x}{x} - \cos x \right] \frac{1}{x} \end{aligned}}$$

← agrees w/ 7.12.18, but not 7.12.22

38 13.3.3

a) 

in steady-state

(i) $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}$

Let: $T(x,t) = u(x)\tau(t)$

$\Rightarrow \frac{1}{u}\frac{\partial^2 u}{\partial x^2} = \frac{1}{\tau\alpha^2}\frac{\partial \tau}{\partial t} = -k^2 \Rightarrow \boxed{\tau(t) = \tau_0 e^{-k^2\alpha^2 t}}$

and

$\frac{\partial^2 u}{\partial x^2} = -k^2 u \rightarrow \boxed{u(x) = a\cos kx + b\sin kx}$

(ii) in steady state, $k^2 = 0 \rightarrow \frac{\partial \tau}{\partial t} = 0$

$\rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \rightarrow \boxed{u(x) = c_0 + c_1 x}$

and so, at $t=0$ $\boxed{u(x) = 100(x/\ell)}$

b) at $t>0$, $T(0,t) = 100°$, $T(\ell,t) = 0°$.

look at,

$\cancel{T(x,t) = (a\cos kx + b\sin kx)e^{-k^2\alpha^2 t}}$

~~(i) $x=0 \rightarrow T=100°$ & at $x=\ell \rightarrow T=0°$~~

$T(x,t) = \underbrace{c_0 + c_1 x}_{k^2=0} + \underbrace{(a\cos kx + b\sin kx)e^{-k^2\alpha^2 t}}_{k^2\neq 0}$

as $t \to \infty$, the $e^{-k^2\alpha^2 t} \to 0$ and we want

$T(x,\infty) \to \left(100^\circ - \frac{100^\circ}{\ell} x\right)$

$\Rightarrow T(x,t) = \left(100^\circ - \frac{100^\circ}{\ell} x\right) + \left(a\cos kx + b\sin kx\right) e^{-k^2\alpha^2 t}$

note: the stuff in ( ) must not alter BCs.

$\to (\ ) \to 0$ at $x=0$ and $x=\ell$

$\Rightarrow a=0,\ k=\left(\frac{n\pi}{\ell}\right)$

so that $\boxed{T(x,t) = \left(100^\circ - \frac{100^\circ}{\ell} x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\left(\frac{n\pi\alpha}{\ell}\right)^2 t}}$

at $t=0$, $T(x,0) = 100^\circ \frac{x}{\ell}$

$\Rightarrow 100^\circ \frac{x}{\ell} = 100^\circ - \frac{100^\circ}{\ell} x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$

$\frac{200^\circ}{\ell} x - 100^\circ = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)$

$\int_0^\ell \left[\frac{200}{\ell} x - 100\right] \sin\left(\frac{n'\pi x}{\ell}\right) dx = \sum_{n=1}^{\infty} b_n \int_0^\ell \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{n'\pi x}{\ell}\right) dx$

$\left\{ \frac{200}{\ell}\left(\frac{\ell}{n'\pi}\right)^2 \left[\sin\left(\frac{n'\pi x}{\ell}\right) - \left(\frac{n'\pi x}{\ell}\right)\cos\left(\frac{n'\pi x}{\ell}\right)\right]_0^\ell + 100\frac{\ell}{n'\pi}\cos\left(\frac{n'\pi x}{\ell}\right)\Big|_0^\ell \right\} = b_n \frac{\ell}{2}\delta_{nn'}$

$$\frac{200\cancel{\ell}}{n'^2\pi^2}\left[0-n'\pi\cos(n'\pi)\right]+\frac{100\cancel{\ell}}{n'\pi}\left[\cos n'\pi-1\right]=b_n\frac{\cancel{\ell}\delta_{nn'}}{2}$$

$$-\frac{200}{n'\pi}\left[(-1)^{n'}\right]+\frac{100}{n'\pi}\begin{bmatrix}-2, & n \text{ odd}\\ 0, & n \text{ even}\end{bmatrix}=\frac{b_n}{2}$$

$$\Rightarrow \tfrac{1}{2}b_1=\frac{200}{\pi}-\frac{200}{\pi}=0$$

$$\tfrac{1}{2}b_2=-\frac{100}{\pi}$$

$$\tfrac{1}{2}b_3=\frac{200}{3\pi}-\frac{200}{3\pi}=0$$

$$\tfrac{1}{2}b_4=-\frac{50}{\pi}$$

$$\vdots$$

$$\tfrac{1}{2}b_n=-\frac{200}{n}\ ,\ n \text{ even}$$

$$\Rightarrow \boxed{T(x,t)=100^\circ-100^\circ\left(\frac{x}{\ell}\right)-\sum_{n=1}^{\infty}\frac{400}{n\pi}\sin\left(\frac{n\pi x}{\ell}\right)e^{-\left(\frac{\alpha n\pi}{\ell}\right)^2 t}}$$

$$=100^\circ\left(1-\frac{x}{\ell}\right)-\frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{\ell}\right)e^{-\left(\frac{\alpha n\pi}{\ell}\right)^2 t}$$

39. 13.9.4

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}$$

find the sine transform

(a) $$\int_0^\infty \frac{\partial^2 T}{\partial x^2} \sin kx\, dx = \frac{1}{\alpha^2}\int_0^\infty \frac{\partial T}{\partial t} \sin kx\, dx$$

$$\Rightarrow -k^2 C_s(k,t) = \frac{1}{\alpha^2}\frac{\partial}{\partial t} C_s(k,t)$$

and $$\boxed{C_s(k,t) = C_s(k,0)\, e^{-\alpha^2 k^2 t}}$$

(b) find $T(x,t)$,

$$T(x,t) = \int_0^\infty C_s(k,t) \sin kx\, dk$$

$$= \int_0^\infty \underbrace{C_s(k,0)}\, e^{-\alpha^2 k^2 t} \sin kx\, dk$$

which is Fourier transform of T at t = 0

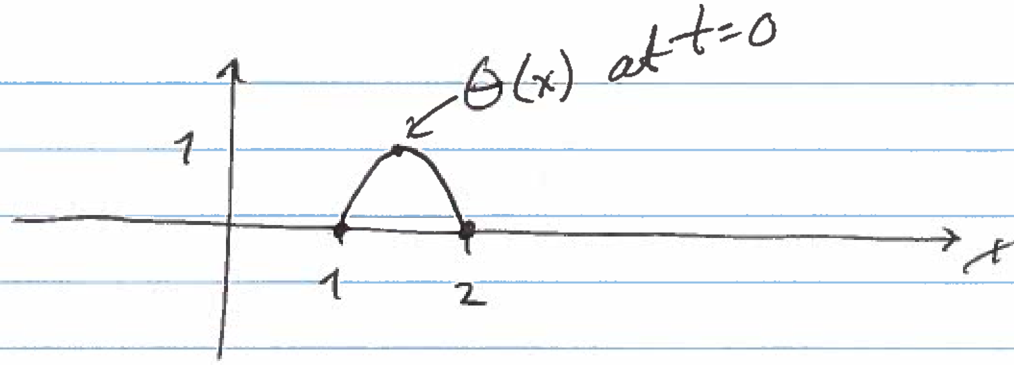
(c) at $t=0$, $$C_s(k,0) = \int_0^\infty 100° \sin kx\, dx$$

$$= \left.\frac{-100\cos kx}{k}\right|_0^{\infty}$$

$$= -\frac{100}{k}\left[\cos k - 1\right]$$

$$\Rightarrow \boxed{T(x,t) = \int_0^\infty \frac{100}{k}(1-\cos k)\, e^{-\alpha^2 k^2 t} \sin kx\, dk}$$

(40)



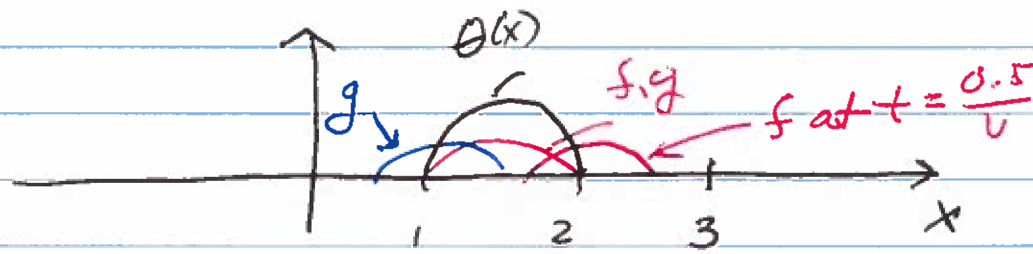
a) at $t=0$, $f(u) = g(v) = \frac{1}{2}\theta(x)$

$\rightarrow f(x,0) = g(x,0) = \frac{1}{2}\sin \pi x$, 0 for $x>2$ & $x<1$

$\Rightarrow f(x,t) = \frac{1}{2}\sin\pi(x-vt)$, $\longrightarrow$ propagates

$g(x,t) = \frac{1}{2}\sin\pi(x+vt)$, $\longleftarrow$ propagates

$\Rightarrow$



b) at $t=0$, $f(x,0) + g(x,0) = 0$ but

$\frac{\partial f}{\partial t} + \frac{\partial g}{\partial t} \neq 0$, $\frac{\partial f}{\partial t} + \frac{\partial g}{\partial t} = \dot{\theta}(x)$

$\rightarrow \frac{\partial f(u)}{\partial u} = \frac{1}{2}\cos\pi x$, $\frac{\partial g(u)}{\partial u} = +\frac{1}{2}\cos\pi x$, and $\frac{\partial u}{\partial t}$

and

$f + g = 0$ which follows from above

at $t=0$ $\frac{\partial f}{\partial u}, \frac{\partial g}{\partial u} \longrightarrow f = \frac{\sin\pi x}{2\pi}$, $g = -\frac{\sin\pi x}{2\pi}$

b) at $t=0$, $f(x,0)+g(x,0)=0$ but

$\dot{f}(x,0)+\dot{g}(x,0)\neq 0$, $=\dot{\theta}(x)$

$$\dot{f}(x,0)+\dot{g}(x,0)=\frac{\partial f}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial t}$$

$$=-v\frac{\partial f}{\partial u}+v\frac{\partial g}{\partial v}$$

$$=\dot{\theta}(x) \text{ at } t=0$$

$$\frac{\cos\pi x}{v}=-\frac{\partial f}{\partial u}+\frac{\partial g}{\partial v} \quad \text{at } t=0$$

$$\Rightarrow \begin{cases} \dfrac{\partial f}{\partial u}=\dfrac{-1}{2}\dfrac{\cos\pi x}{v} \\ \dfrac{\partial g}{\partial v}=\dfrac{1}{2}\dfrac{\cos\pi x}{v} \end{cases}$$

$$\Rightarrow \boxed{f=-\frac{\sin\pi x}{2v\pi}, \quad g=+\frac{\sin\pi x}{2v\pi}}$$

41. <u>Property 1: Linearity</u>

$$FT(a_1 f_1 + a_2 f_2) = \int_{-\infty}^{\infty} (a_1 f_1 + a_2 f_2) e^{-i\omega t} dt$$

$$= a_1 \int_{-\infty}^{\infty} f_1 e^{-i\omega t} dt + a_2 \int_{-\infty}^{\infty} f_2 e^{-i\omega t} dt$$

$$= a_1 FT(f_1) + a_2 FT(f_2) \checkmark$$

<u>Property 2: Symmetry</u>

a) $F(\omega) = FT(f(t)) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

$$\rightarrow f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

b) let $t \rightarrow -t \Longrightarrow f(-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega$

interchange ω and t

$$\Longrightarrow f(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t) e^{-i\omega t} dt$$

$\checkmark$

<u>Property 3: Time translation</u>

show $FT(f[t-t_0]) = e^{-i\omega t_0} FT(f(t)) = e^{-i\omega t_0} F(\omega)$

<u>Pf</u>, $FT(f[t-t_0]) = F(\omega) = \int_{-\infty}^{\infty} f(t-t_0) e^{-i\omega t} dt$

let $z = t - t_0 \rightarrow dz = dt$

$$\rightarrow F(\omega) = \int_{-\infty}^{\infty} f(z) e^{-i\omega(z+t_0)} dz$$

$$= e^{-i\omega t_0} FT(f(z)) \checkmark$$

Property 4: Frequency Translation

what is $FT(e^{i\omega_0 t} f(t))$?

$$FT(e^{i\omega_0 t} f(t)) = \int_{-\infty}^{\infty} \left[e^{i\omega_0 t} f(t)\right] e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-i(\omega-\omega_0)t}\, dt$$

let $\Omega = \omega - \omega_0$ ~~and~~ find

$$\Rightarrow FT(e^{i\omega_0 t} f(t)) = \int_{-\infty} f(t)\, e^{-i\Omega t}\, dt$$

$$= FT(f(t))$$

$$= F(\Omega)$$

Property 5: Change of time scale

show $FT(f(t)) = \int_{-\infty}^{\infty} f(at)\, e^{-i\omega t}\, dt = \frac{1}{a} F\left(\frac{\omega}{a}\right)$

Suppose we change the scale by factor a, $t \to at$

let $z = at \to t = (z/a) \to dt = dz/a$

$$\Rightarrow \cancel{\int_{-\infty}^{\infty} f(at)\, e^{-i\omega at}\, d(at)}$$

$$\cancel{\int_{-\infty}^{\infty} f(z)}$$

$$\Rightarrow F(\omega) = \int_{-\infty}^{\infty} f(z)\, e^{\overbrace{-i\omega\left(\frac{z}{a}\right)}^{-i\left(\frac{\omega}{a}\right)z}}\, \frac{dz}{a} = \frac{1}{a} F\left(\frac{\omega}{a}\right)$$

Property 6: FT of $\frac{df}{dt}$

$$FT\left(\frac{df}{dt}\right) = \int_{-\infty}^{\infty} \left(\frac{df}{dt}\right) e^{-i\omega t} dt = F(\omega)$$

do by parts

$$du = \frac{df}{dt} dt, \quad V = e^{-i\omega t}$$

$$u = f \quad , \quad dV = -i\omega e^{-i\omega t} dt$$

$$\Rightarrow \quad = f e^{-i\omega t} \Big|_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} f e^{-i\omega t} dt$$

$$= 0 + i\omega \, FT(f(t))$$

if f doesn't have $\int_{-\infty}^{\infty} f dt \to \infty$

Property 7: Differentiation w/to frequency ω

show $\frac{d}{d\omega}\left[FT(f(t))\right] = \frac{d}{d\omega}\left[F(\omega)\right] = FT\left[-itf(t)\right]$

Pf $\quad F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

$$\frac{dF(\omega)}{d\omega} = \frac{d}{d\omega}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$= \int_{-\infty}^{\infty} f(t)\, dt \frac{d}{d\omega}\left[e^{-i\omega t}\right]$$

$$= + \int_{-\infty}^{\infty} f(t)\, dt \left(-i\omega e^{-i\omega t}\right)$$

$$= -i\omega \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$= -i\omega\, FT(f(t))$$

$$= -i\omega F(\omega)$$