Physics 421M
Course CRN: 15532
Class: 09:30-10:30, MWF, 00 REMOTE
Text: Mathematical Methods in the Physical Sciences, 3rd Ed., Mary L. Boas

https://pages.uoregon.edu/imamura/421/physics.421.html

Instructor: James Imamura

Office: 457 Willamette Hall
E-mail: imamura@uoregon.edu
Phone: 541-346-5212
Office Hours: by appointment

Grading:

Homework: 50 pts

Tests will be take-home exams

Test 1: 50 pts

Test 2: 50 pts

Final: 70 pts

Total: 220 pts

Tests:

Test 1: Wednesday, October 28, 2020

Test 2: Wednesday, November 25, 2020

Final: Thursday, December 10, 2020 @ 10:15

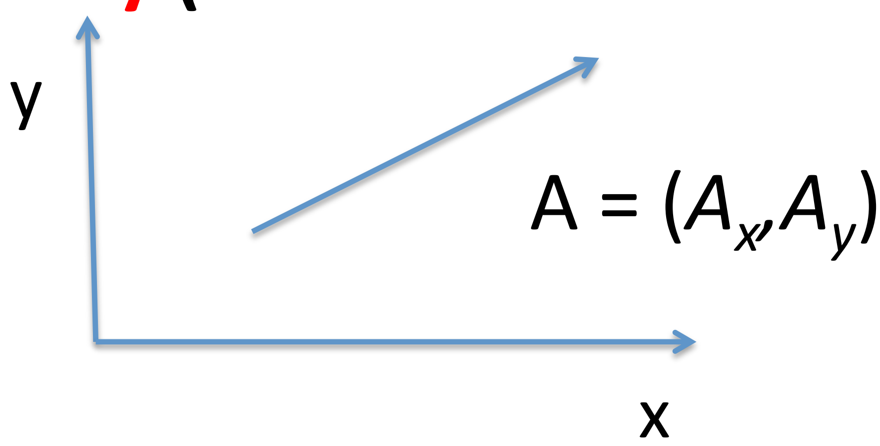
Material:

- Vector Algebra. **Chapters 3.4,3.5,6.1-6.3: vector analysis and vector operations, addition, subtraction, multiplication (scalar, dot product, cross product), multiplication involving 3 or more vectors (Triple Scalar Product, Triple Vector Product, Laplace's Identity, and applications); vector functions. rotations (Chapter 3.7) and vectors as first rank tensors (and scalars as zero rank tenors) (Chapter 10.2). For additional enrichment and enjoyment on other transformations needed for Cartesian vectors, see 10.6. 10.6 not to be tested.**
- **Vector Calculus**. Chapter 6.4-6.12, parts of Chapter 4 (Partial Differentiation), Chapter 5 (Multipole Integrals, Applications of Integration), and Chapter 10 curvlinear coordinate systems and some Tensor Analysis.
- Chapter 7 (**Fourier Series and Transforms**), Chapter 8 (**Ordinary Differential Equations**), Chapter 12 (Series solutions of differential equation, **Legendre, Bessel, Hermite, and Laguerre functions**), Chapter 13 (**Partial Differential Equations**)

VECTOR CALCULUS

Physical quantities, positions, velocities, accelerations, … , may be represented **geometrically**, by directed line segments,

algebraically (in some coordinate system),



Vector formulas, ** such as **F = m**a**, **p** = m**v**, ***l*** **= rxp, …,** do not depend upon coordinate system**,** consequently, we like express relations in vector forms.

We briefly review some notation, and simple operations, addition, subtraction, and multiplication of vectors before moving on to vector calculus.

Vector Length:

$$|A|^2=A_x{}^2+A_y{}^2$$

In n-dimensions

$$|A|^2 = \sum_{i=1}^{n} A_i A_i$$

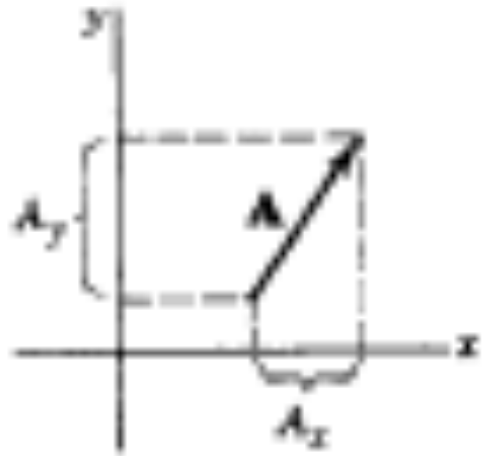


Figure 4.1

Vector Addition:

A+B

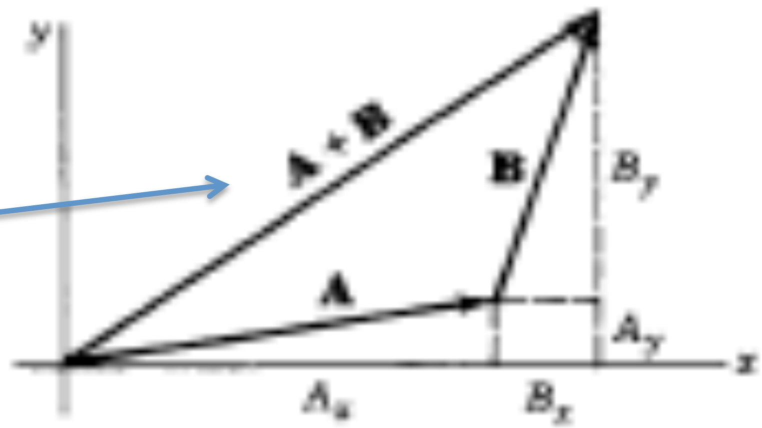


Figure 4.3

$$A+B=(A_x+B_x, A_y+B_y, A_z+B_z)$$

Commets:

- **A+B=B+A,** commutative
- **A+B+C =(A+B)+C=A+(B+C),** associative

Vector Subtraction:

A-B

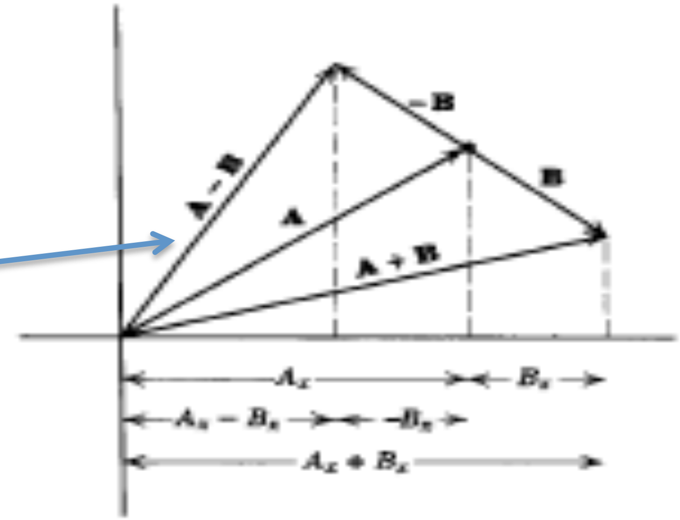


Figure 4.6

A-B=(A_x-B_x,A_y-B_y,A_z-B_z)

Comments:

- **A-B=A+(-B)=(-B)+A**
- **-A-B-C=(-A-B)-C=-A+(-B-C)**

Vector Multiplication by a constant:

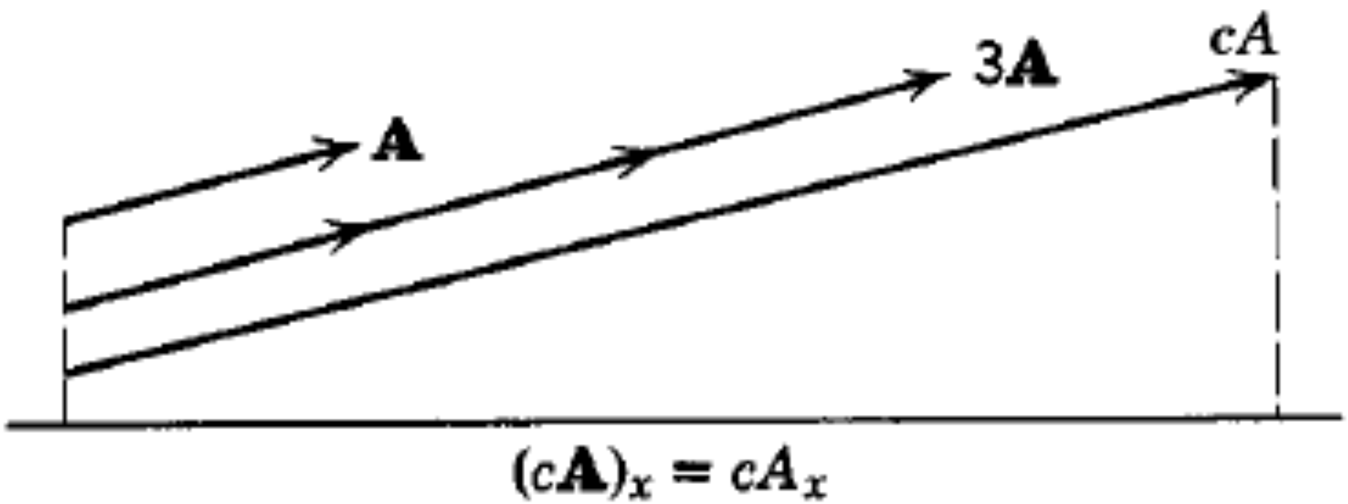
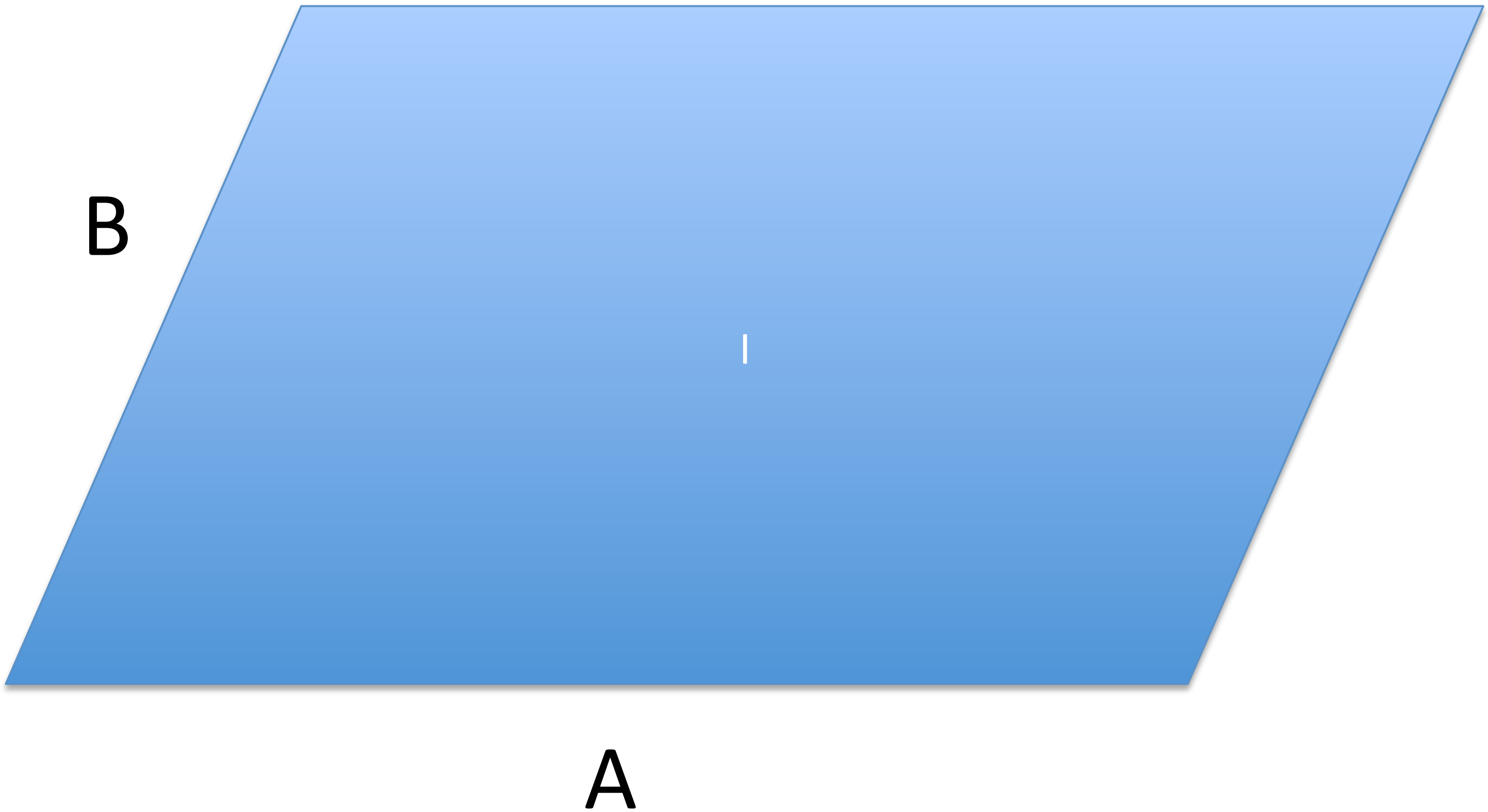


Figure 4.5

Problem 3.4: **Show that the diagonals of a parallelogram bisect each other.**



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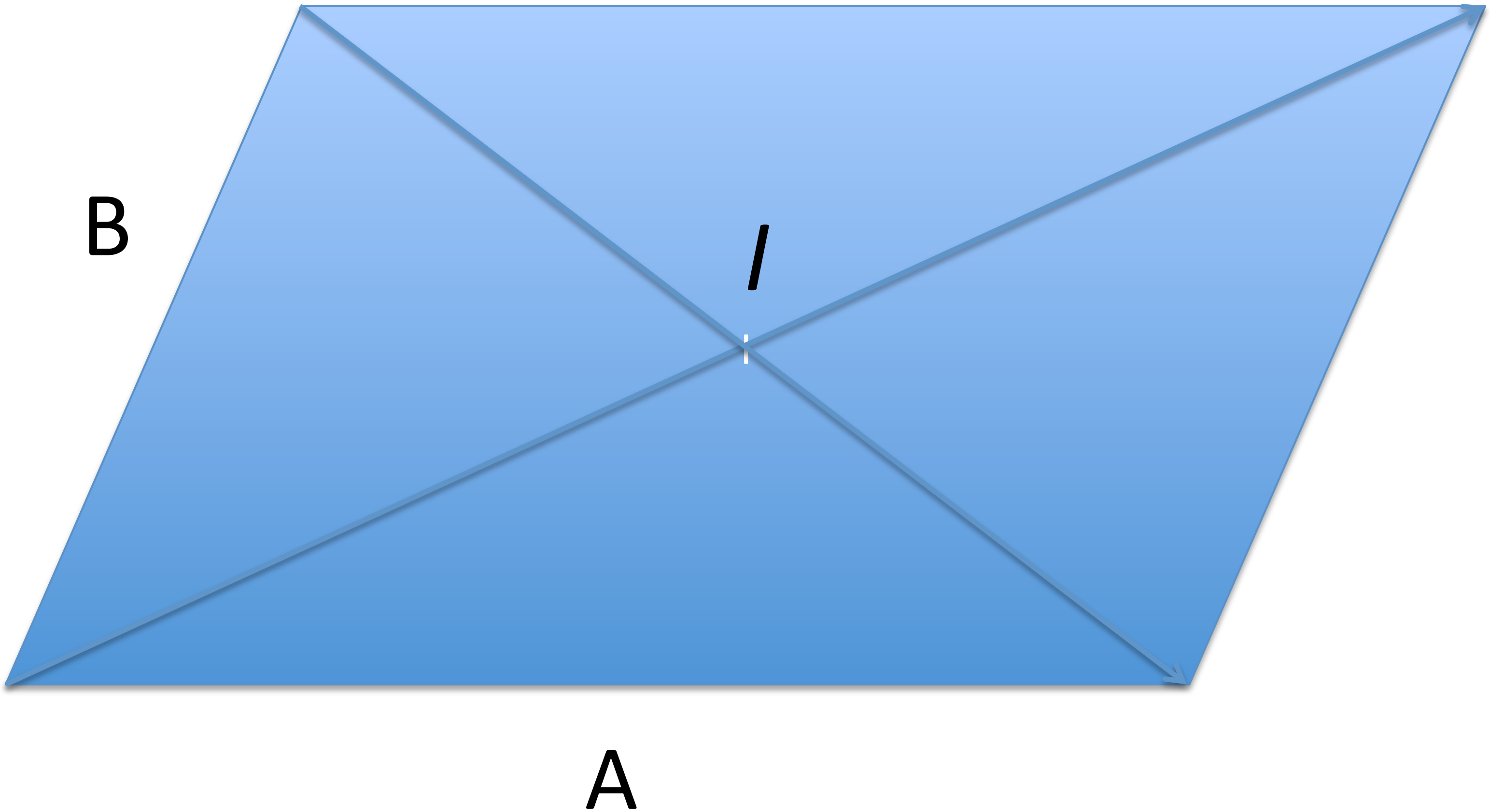
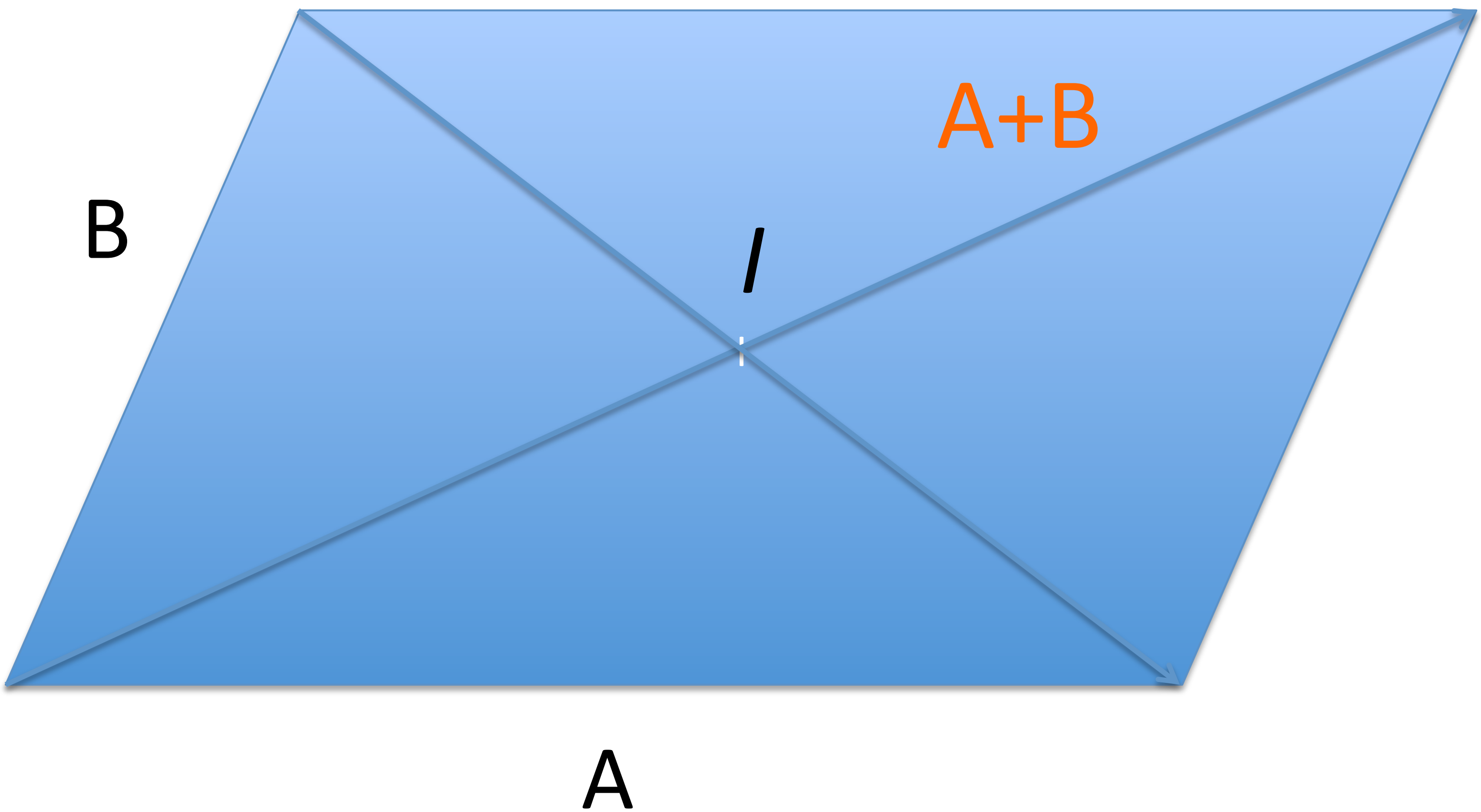
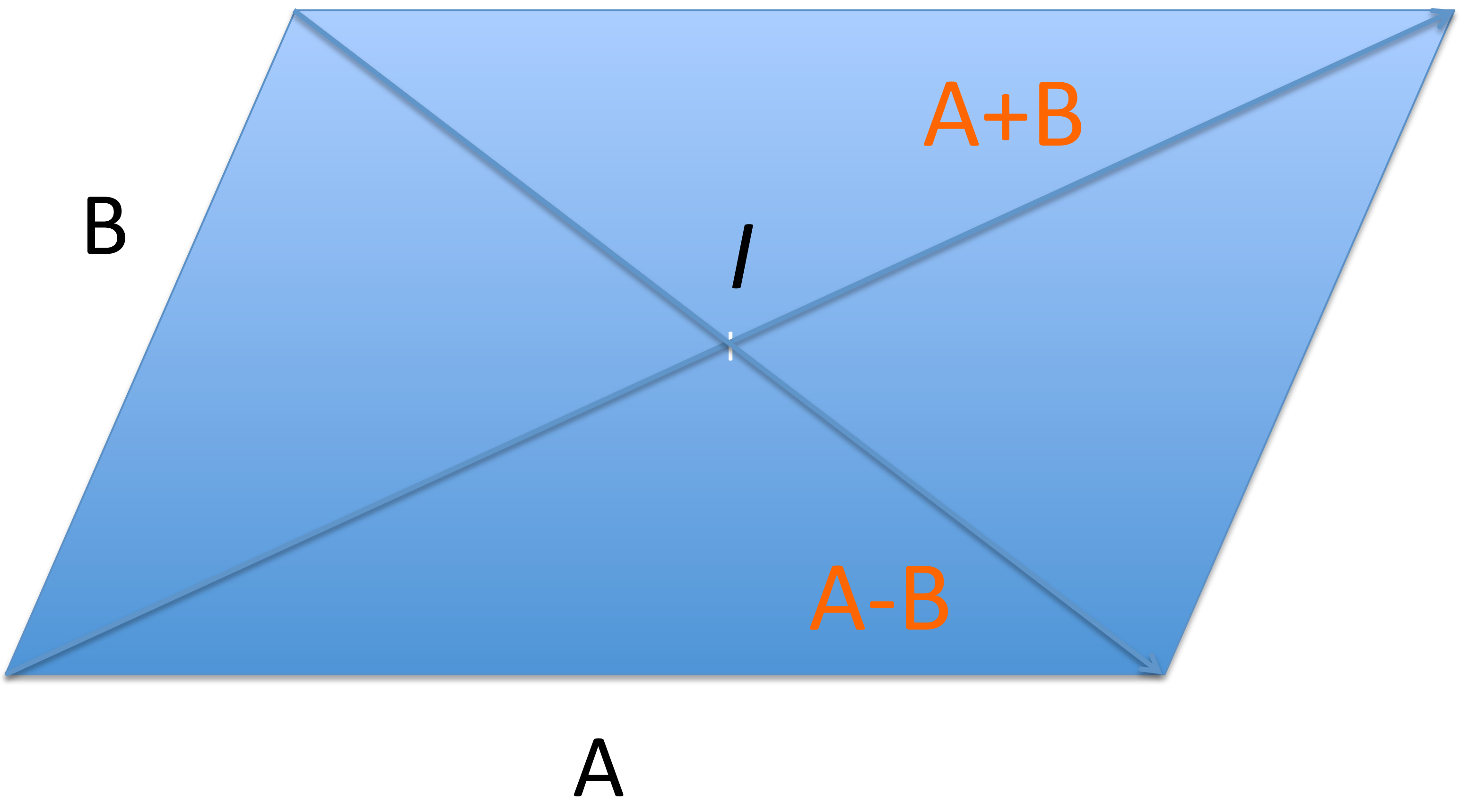


Problem 3.4: **Show that the diagonals of a parallelogram bisect each other.**



Problem 3.4: **Show that the diagonals of a parallelogram bisect each other.**



Write two vectors, one approaching from the left and one from the right,

$$I_L = B+\alpha(A-B) \text{ and } I_R = (A+B)-\beta(A+B)$$

where $\alpha > 0$ and $\beta > 0$. Note that $I_L = I_R$ and solve for α and β. This leads to $\alpha = \beta = ½$. Could also do this problem algebraically.

Vector Multiplication:

Consider two vector multiplication forms:

- Scalar product (dot product)
- Cross product (outer product)

Scalar Product

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta. \tag{4.2}$$

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z. \tag{4.10}$$

Comments:

- **A•B=B•A**, commutative
- **A•(B+C)=A•B+A•C**, distributive

Scalar Product

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta. \tag{4.2}$$

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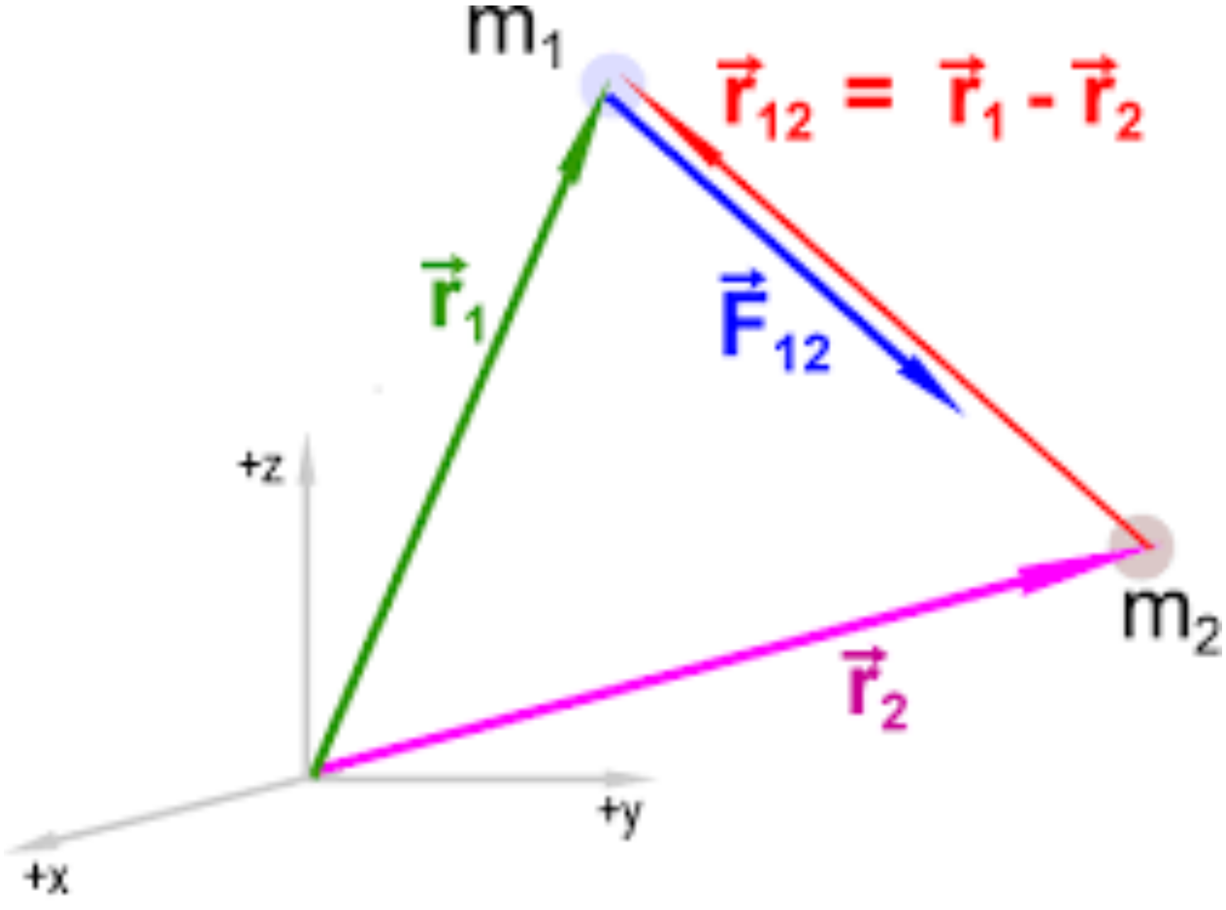
Comments:

- **A•B=B•A**, commutative
- **A•(B+C)=A•B+A•C**, distributive

Gravitational interaction:



Gravitational interaction:

$$|\mathbf{r_1}-\mathbf{r_2}|^2 = (\mathbf{r_1}-\mathbf{r_2}) \bullet (\mathbf{r_1}-\mathbf{r_2})$$

Law of Cosines

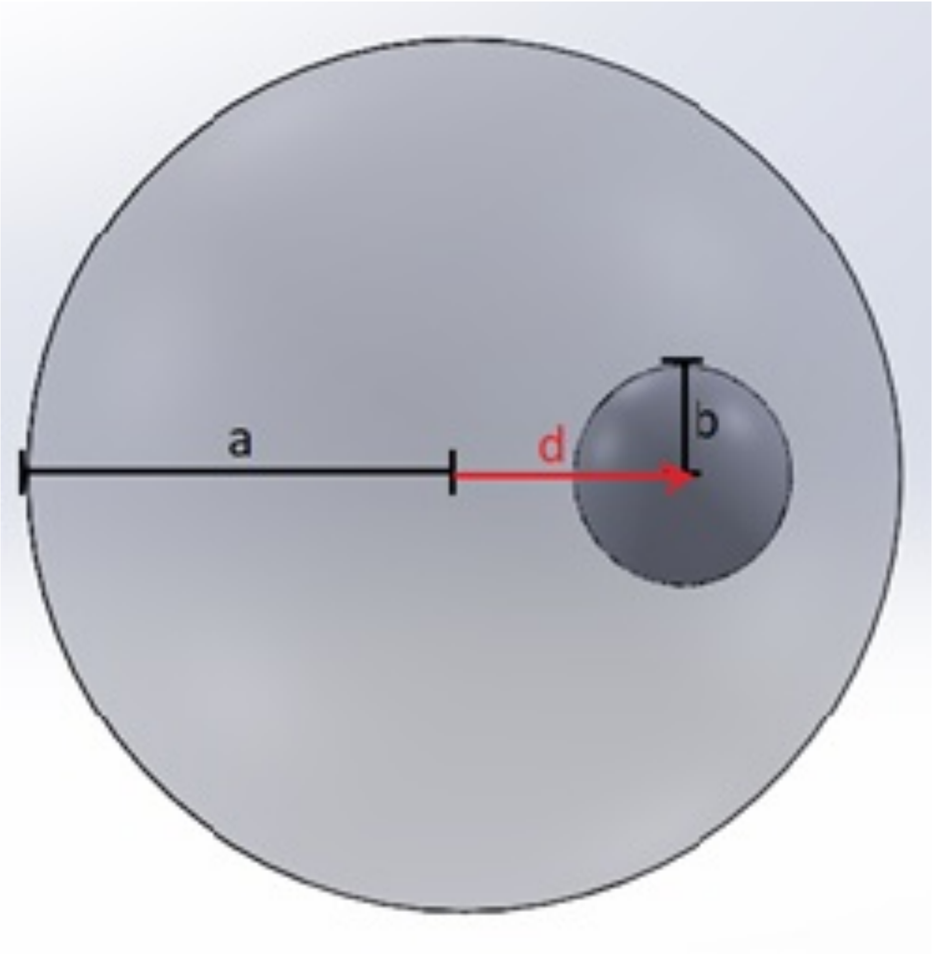
$$= r_2^2+r_1^2-2r_1r_2 \cos\theta$$

and the expression for the field Is

$$F = -Gm_1m_2(r_1-r_2)/(r_2^2+r_1^2-2r_1r_2 \cos\theta)^{3/2}$$

Electrostatic interaction:

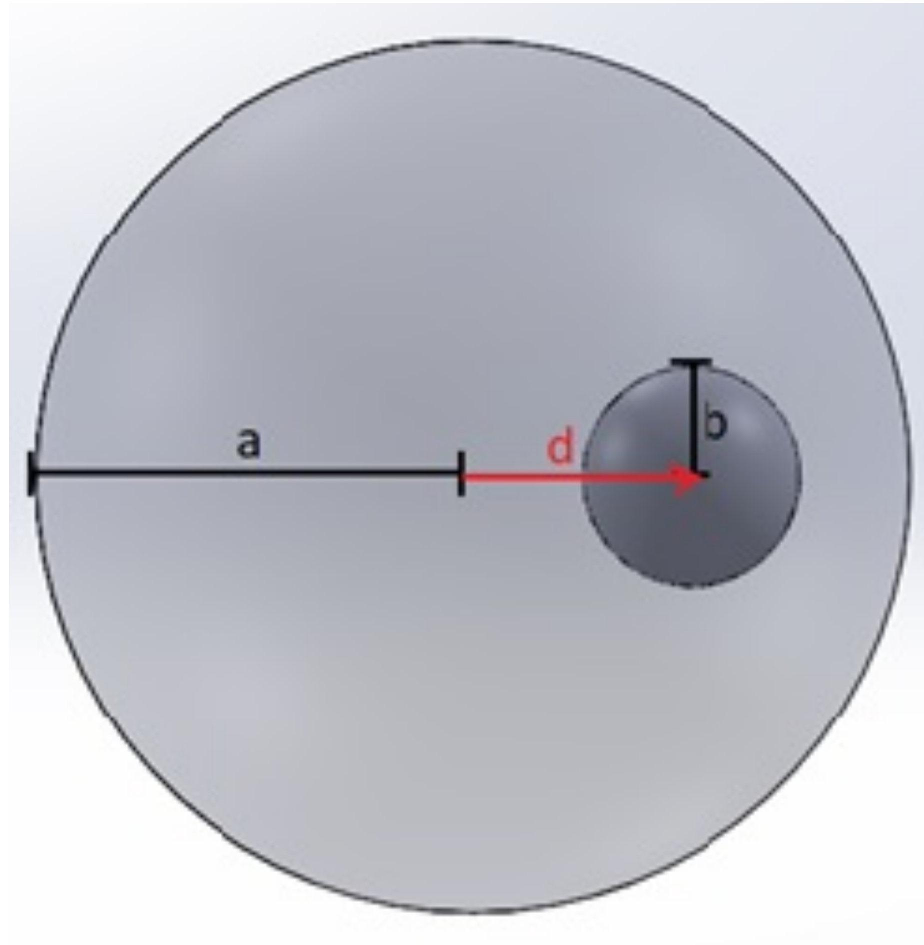
In a uniformly charged sphere of radius a, a spherical cavity of radius b is excised. What is the electric field in the cavity?



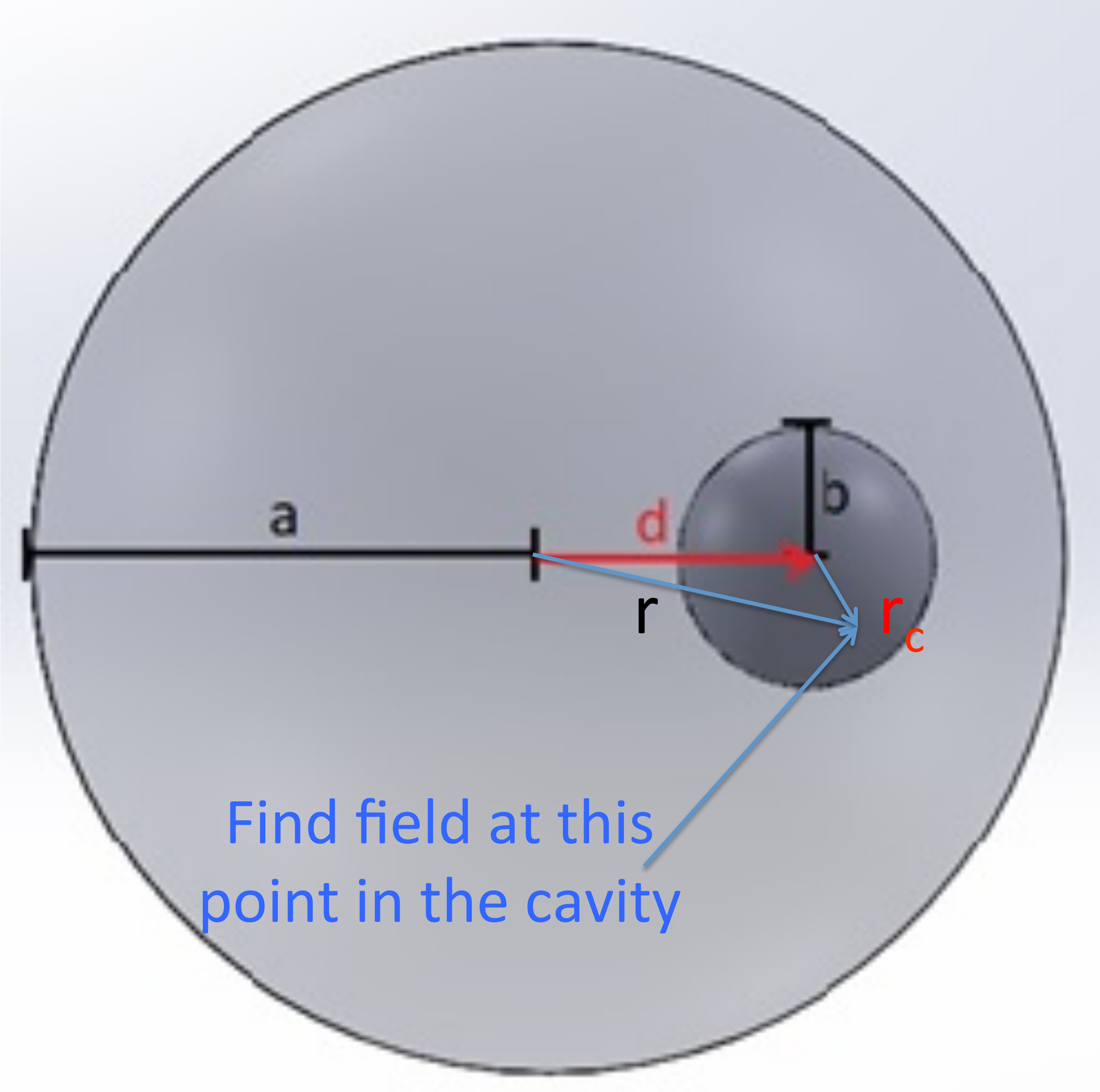
What is the electric field in the cavity?

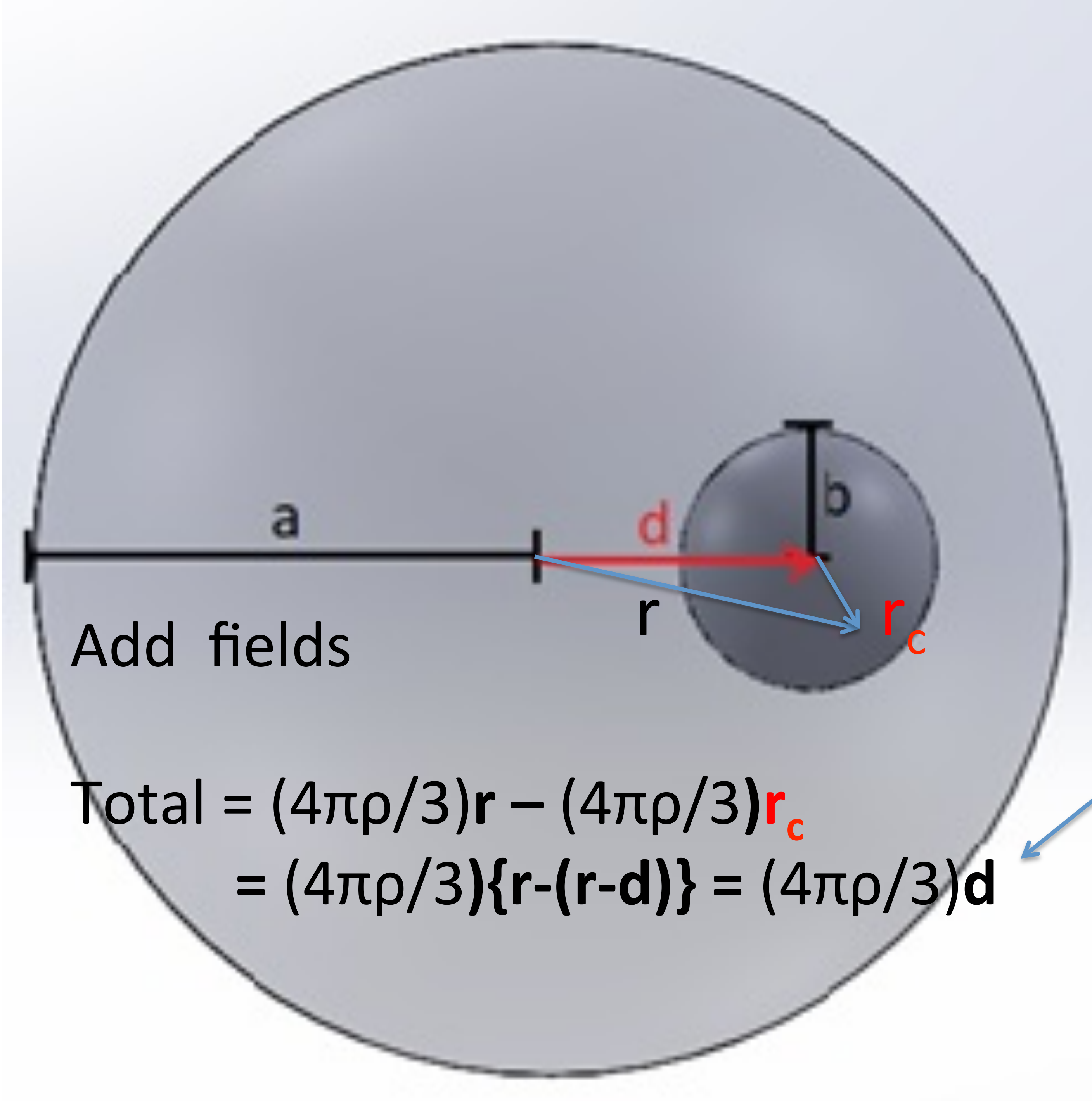
We know (can show) that the electric field in an uniformly charged sphere is radial and given by

$$E = Q(r)\mathbf{r}/r^3$$
$$= (4\pi r^3 \rho/3)\,\mathbf{r}/r^3$$
$$= (4\pi\rho/3)\mathbf{r}$$



Field is radial and increases with r


a
d
b
r
r c
Find field at this
point in the cavity



Add fields

$$\text{Total} = (4\pi\rho/3)\mathbf{r} - (4\pi\rho/3)\mathbf{r}_c$$
$$= (4\pi\rho/3)\{\mathbf{r}-(\mathbf{r}-\mathbf{d})\} = (4\pi\rho/3)\mathbf{d}$$

Vector Multiplication:

Cross Product

The magnitude of $\mathbf{A} \times \mathbf{B}$ is

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}|\,|\mathbf{B}| \sin\theta, \tag{4.14}$$

where θ is the positive angle ($\leq 180°$) between $\mathbf{A}$ and $\mathbf{B}$. The direction of $\mathbf{A} \times \mathbf{B}$ is perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$ and in the sense $\mathbf{C}$ of advance of a right-handed screw rotated from $\mathbf{A}$ to $\mathbf{B}$ as in Figure 4.12.

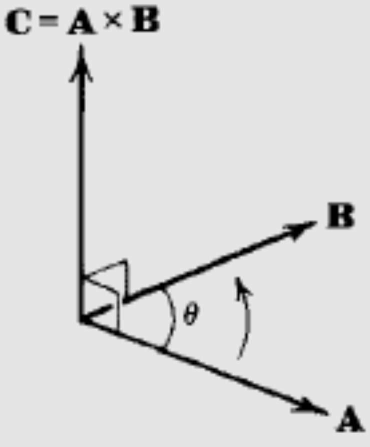


Figure 4.12

Cross Product

$$\begin{aligned} &\mathbf{A}\times\mathbf{B}=0 \quad \text{if } \mathbf{A} \text{ and } \mathbf{B} \text{ are parallel or antiparallel,} \\ &\mathbf{A}\times\mathbf{A}=0 \quad \text{for any } \mathbf{A}. \end{aligned} \tag{4.15}$$

$$\begin{aligned} &\mathbf{i}\times\mathbf{j}=\mathbf{k} \quad &&\mathbf{j}\times\mathbf{k}=\mathbf{i} \quad &&\mathbf{k}\times\mathbf{i}=\mathbf{j}. \\ &\mathbf{j}\times\mathbf{i}=-\mathbf{k} \quad &&\mathbf{k}\times\mathbf{j}=-\mathbf{i} \quad &&\mathbf{i}\times\mathbf{k}=-\mathbf{j}. \end{aligned} \tag{4.17}$$

Comments:

- **AxB=-BxA**, not commutative
- **Ax(B+C)=AxB+AxC**, distributive

Cross Product

cross product in component form (Cartesian coordinates—consider other coordinate systems later),

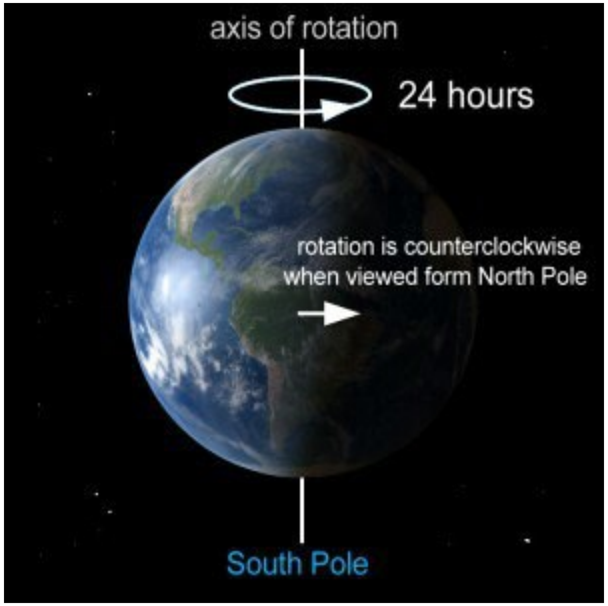
$$
\begin{aligned}
(4.19) \qquad \mathbf{A} \times \mathbf{B} &= (\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z) \times (\mathbf{i}B_x + \mathbf{j}B_y + \mathbf{k}B_z) \\
&= \mathbf{i}(A_yB_z - A_zB_y) + \mathbf{j}(A_zB_x - A_xB_z) + \mathbf{k}(A_xB_y - A_yB_x) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.
\end{aligned}
$$

Cross Product: Physical Examples

- Angular momentum, $\mathbf{J} = \mathbf{r} \times \mathbf{p}$, $\mathbf{r}$ = position vector, $\mathbf{p}$ = linear momentum
- Torque, $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, $\boldsymbol{\tau}$ = torque, $\mathbf{r}$ = position vector, $\mathbf{F}$ = force
- Lorentz Force, $\mathbf{F} = \mathbf{v} \times \mathbf{B}$, $\mathbf{v}$ = velocity, $\mathbf{B}$ = magnetic field
- …

Consider a rotating object:

The Earth spins on its axis with rotation period P= 23h56.091s.



What is the speed of an object fixed to the Earth at latitude $\lambda = (\frac{1}{2}\pi - \theta)$? θ is the polar angle.

The object moves a distance $D = 2\pi r \sin\theta$ per rotation of the Earth.

Its speed is then $|v| = D/P = 2\pi r \sin\theta \times (|\Omega|/2\pi)$
$= r|\Omega|\sin\theta$ ➔ $\mathbf{v = \Omega \times r}$

Kepler's laws of planetary motion

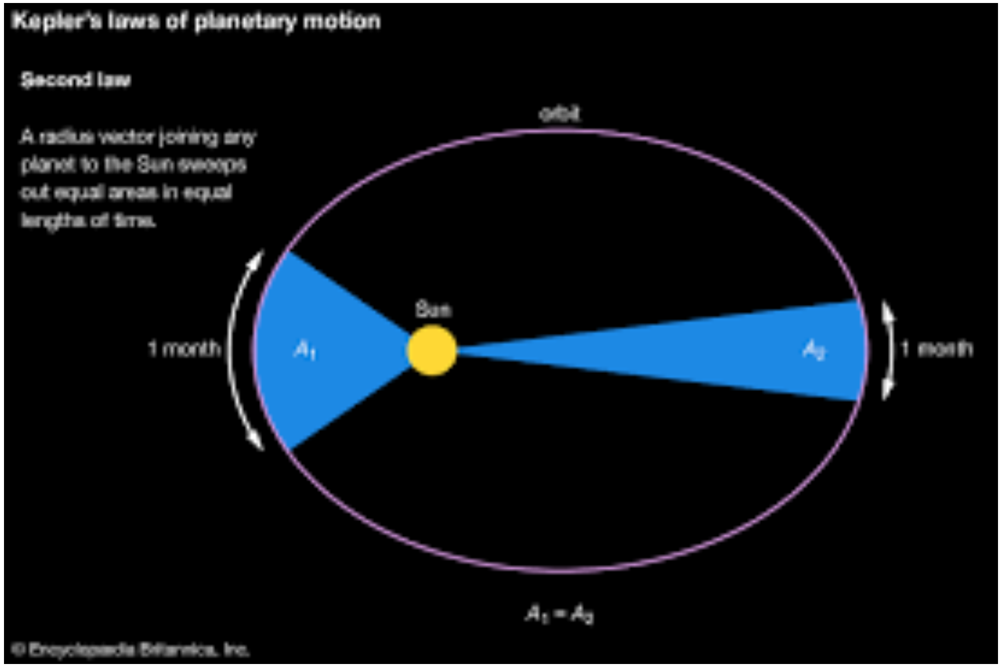
From Wikipedia, the free encyclopedia

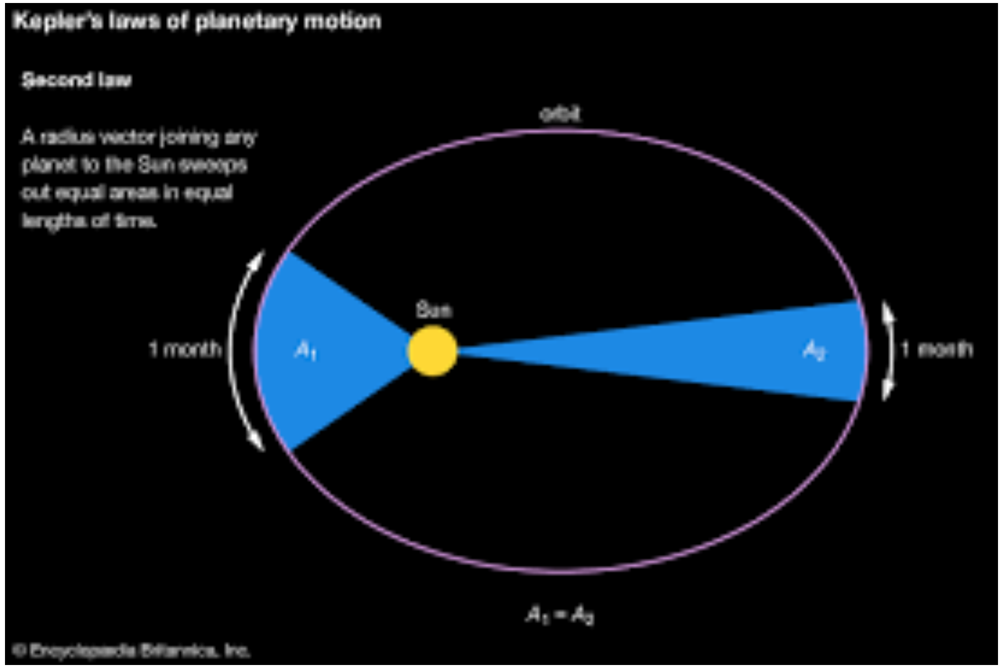
For a more precise historical approach, see in particular the articles Astronomia nova and Epitome Astronomiae Copernicanae.

In astronomy, **Kepler's laws of planet motion** are three scientific laws describing the motion of planets around the Sun, published by Johannes Kepler between 1609 and 1619. These modified the heliocentric theory of Nicolaus Copernicus, replacing its circular orbits and epicycles with elliptical trajectories, and explaining how planetary velocities vary. The laws state that:

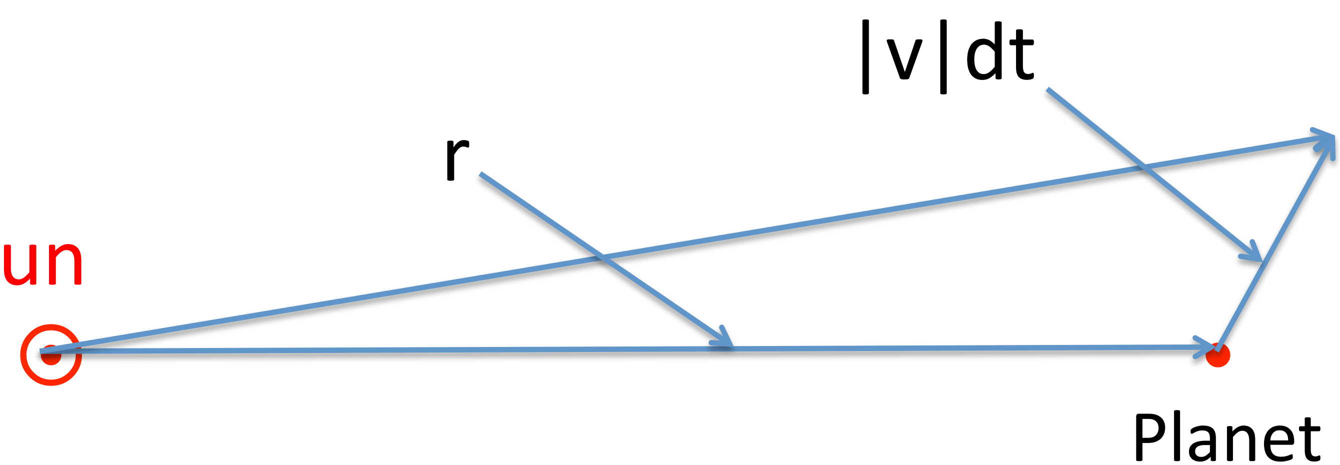
1. The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.

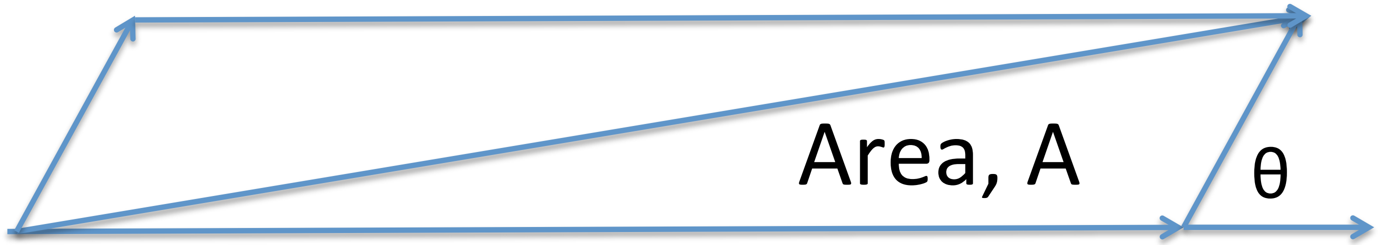
Geometrically, the 2^{nd} Law says that





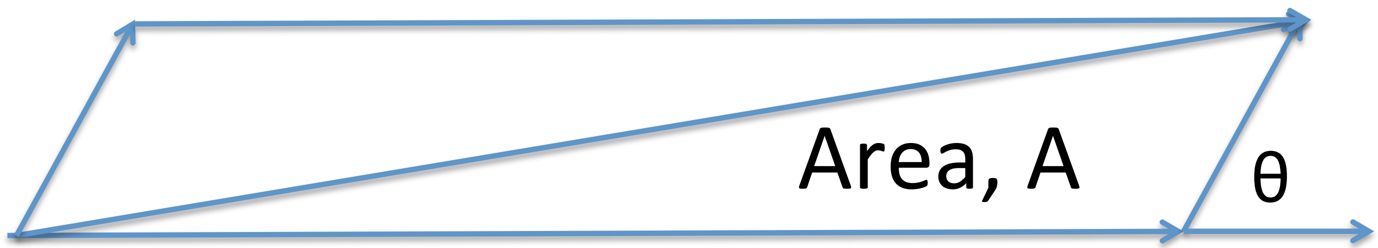
we construct





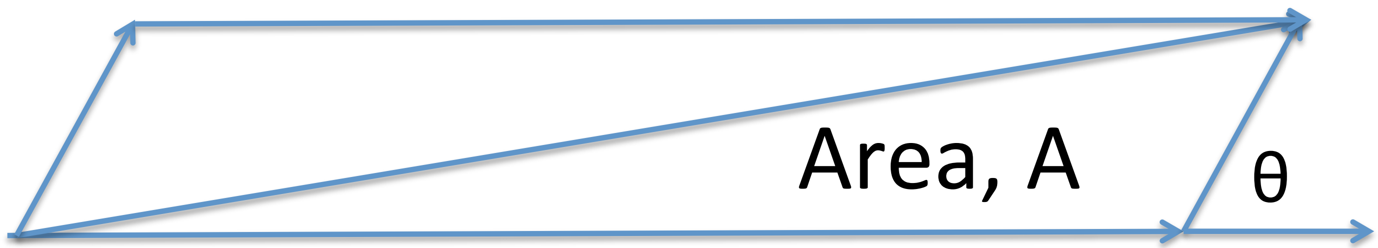
Form the parallelogram and find the area A, First find the area of the rectangle,

$(r+|v|dt\cos\theta)\ |v|dt\sin\theta$



Form the parallelogram and find the area A,
Next, subtract the triangles to find 2A,

$2A = (r+|v|dt\cos\theta)\ |v|dt\sin\theta - |v|^2dt^2\ (\cos\theta \sin\theta)$

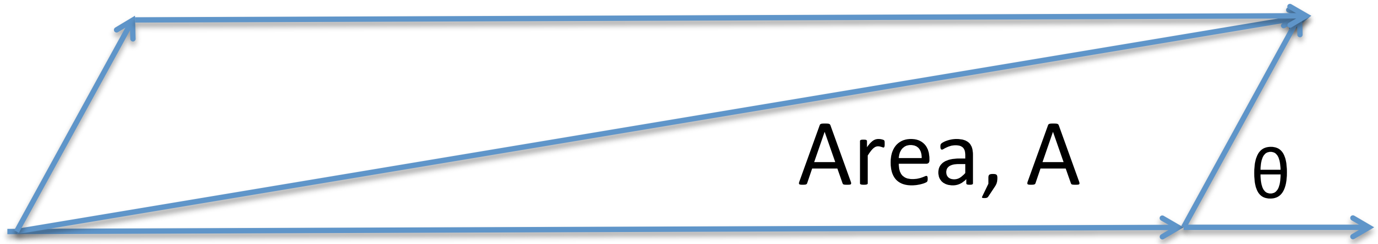


Form the parallelogram and find the area A,
Next, subtract the triangles to find 2A,

$2A = (r+|v|dt\ \cos\theta)\ |v|dt\ \sin\theta\ -\ |v|dt\ (\cos\theta\ \sin\theta)$

This yields,

$2A = r|v|dt\ \sin\theta \rightarrow 2A/dt = r|v|\sin\theta$

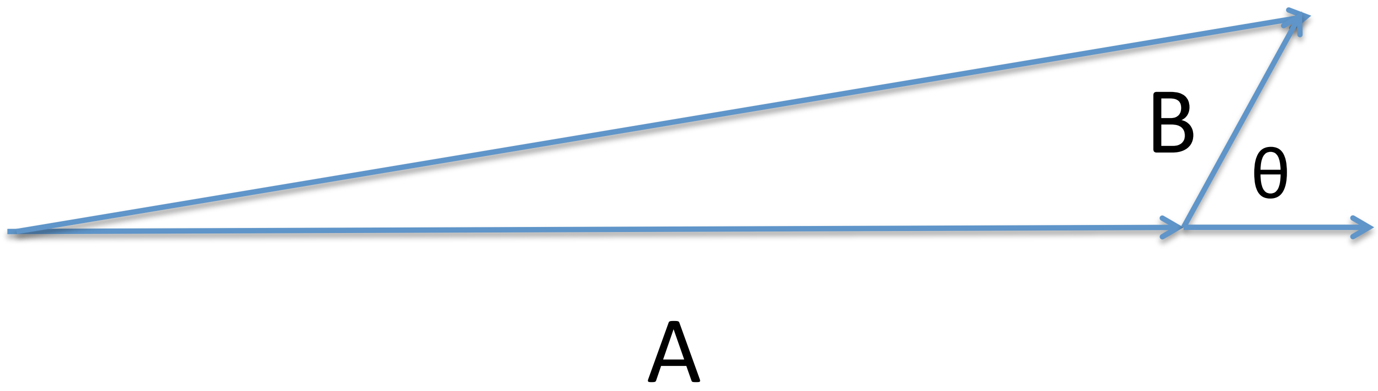


If the planet has mass m, then

$$2mA/dt = r|m\mathbf{v}|\sin\theta = |\mathbf{r} \times m\mathbf{v}| = |\mathbf{r} \times \mathbf{p}| = |\mathbf{J}|$$

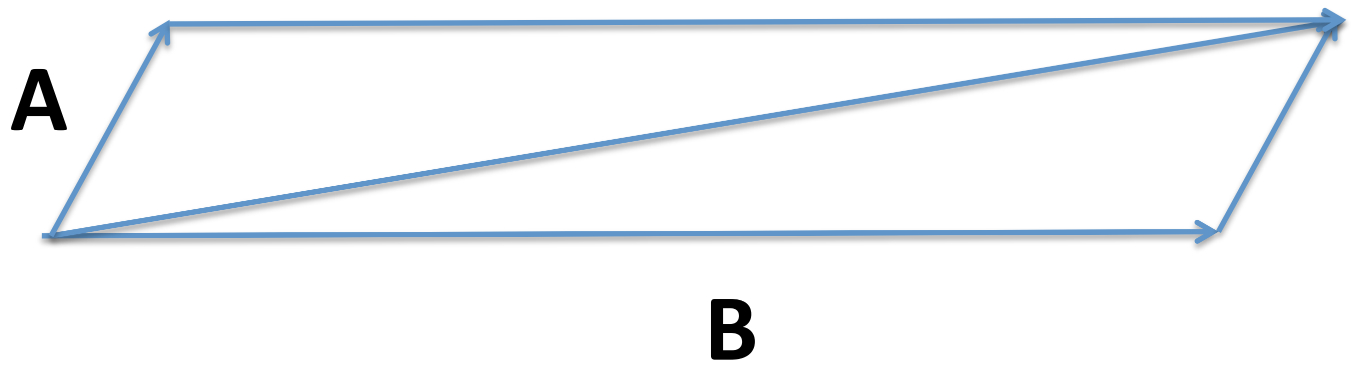
Kepler's 2nd says that that the area, A, swept out for any time interval dt is constant so that,

$$|\mathbf{J}| = \text{constant}$$



In general, the area of the triangle, sides **A** and **B** where **A** and **B** form angle θ, is given by

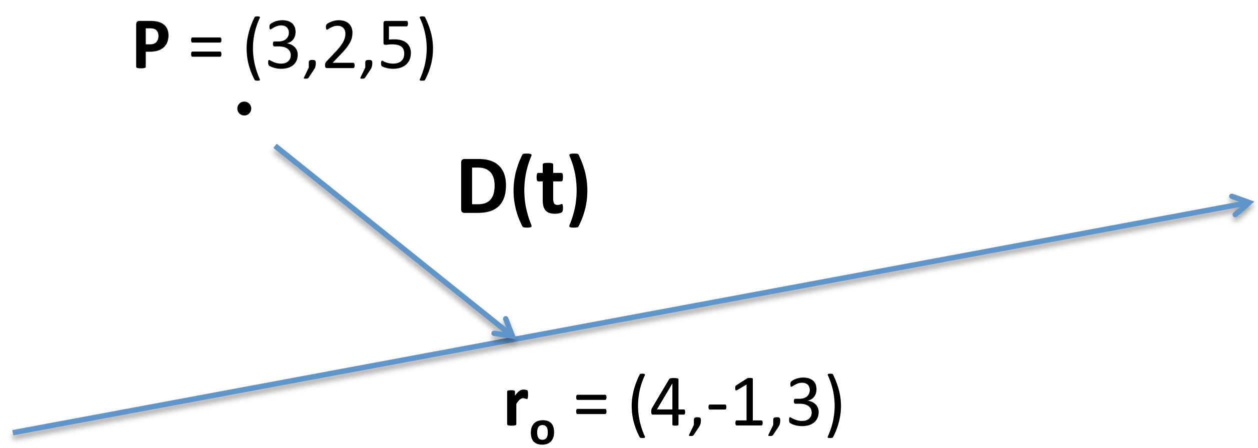
$$\text{Area} = \tfrac{1}{2}\,|\mathbf{A} \times \mathbf{B}|$$



In general, the area of a parallelogram, sides **A** and B, has area given by

$$\text{Area} = |\mathbf{A} \times \mathbf{B}|$$

Find the distance from point **P** = (3,2,5) to the line paasing through $\mathbf{r}_o$ = (4,-1,3) parallel to vector **A** = (1,0,-2)



Equation of line is $\mathbf{r}(t)-\mathbf{r}_o = \mathbf{A}\,t$, where t is a parameter

Find the distance from point **P** = (3,2,5) to the line paasing through $\mathbf{r_o}$ = (4,-1,3) parallel to vector **A** = (1,0,-2)

The vector between **P** and **r(t)** is **D = P – r** and we want the minimum value for this length, that is, we want to find |**D|** when **D** and **A** are perpendicular, that is, when

$$\mathbf{D} \bullet \mathbf{A} = \mathbf{0}$$

Find the distance from point **P** = (3,2,5) to the line passing through $\mathbf{r_o}$ = (4,-1,3) parallel to vector **A** = (1,0,-2)

$$\mathbf{D \bullet A = (P - r) \bullet A = (P - r_o} - A\, \mathbf{t) \bullet A = 0}$$

when

$$\mathbf{(P - r_o) \bullet A - A \bullet A\, t = 0}$$

so

$$t = (-5)/(5) = -1$$

and

$$|\mathbf{D_{min}}| = 3$$

Triple Vector Products

Given three vectors, **A, B**, and C, form products:

(A•B)C, A•(BxC), Ax(BxC) , ...

Other forms that are not valid

Triple Vector Products

(A•B) C = C(A•B), **commutative**

Ax(BxC) = -(BxC)xA, **not commutative**

A•(BxC) = (BxC)•A, **commutative**

Switching order only affects **Ax(BxC).**

What are some identities associated with these forms?

Triple Vector Products

Is **Ax(BxC) = (AxB)xC?** No, not commutative. imagine A,B,C are in the x,x,y directions

- **(BxC)** is then in z-direction and **Ax(BxC)** in –y direction
- **(AxB)** is then 0 and **(AxB)xC** = 0

identity associated with this form

$$A\times(B\times C) = B(A\bullet C)-C(A\bullet B)$$

(BAC-CAB rule). Easily shown (but tedious) using Cartesian coordinates.

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\bullet\mathbf{C})-\mathbf{C}(\mathbf{A}\bullet\mathbf{B})$

$\mathbf{(B\times C)} = (B_x, B_y, B_z) \times (C_x, C_y, C_z) = (B_y C_z - B_z C_y, B_z C_x - B_x C_z, B_x C_y - B_y C_x) = (f_x, f_y, f_z)$

$\mathbf{A\times(B\times C)} = (A_y f_z - A_z f_y, A_z f_x - A_x f_z, A_x f_y - A_y f_x)$

$$\mathbf{A x (B x C) = B(A \bullet C) - C(A \bullet B)}$$

$\mathbf{(BxC)} = (B_x, B_y, B_z) x (C_x, C_y, C_z) = (B_y C_z - B_z C_y, B_z C_x - B_x C_z, B_x C_y - B_y C_x) = (f_x, f_y, f_z)$

$\mathbf{Ax(BxC)} = (A_y f_z - A_z f_y, A_z f_x - A_x f_z, A_x f_y - A_y f_x)$

$= (A_y[B_x C_y - B_y C_x] - A_z[B_z C_x - B_x C_z], \ldots, \ldots)$

$$\mathbf{Ax(BxC) = B(A \bullet C) - C(A \bullet B)}$$

$$\mathbf{(BxC)} = (B_x, B_y, B_z) x (C_x, C_y, C_z) = (B_y C_z - B_z - C_y, B_z C_x - B_x C_z, B_x C_y - B_y C_x) = (f_x, f_y, f_z)$$

$$\mathbf{Ax(BxC)} = (A_y f_z - A_z f_y, A_z f_x - A_x f_z, A_x f_y - A_y f_x)$$

$$= (A_y [B_x C_y - B_y C_x] - A_z [B_z C_x - B_x C_z], \ldots, \ldots)$$

$$= (B_x [A_x C_x + A_y C_y + A_z C_z] - C_x [A_x Bx + A_Y B_Y + A_Z B_Z] - A_X B_X C_X + A_X B_X C_X, \ldots, \ldots)$$

$A\times(B\times C) = B(A\bullet C)-C(A\bullet B)$

$$\mathbf{A\times(B\times C)} = (A_y f_z - A_z f_y, A_z f_x - A_x f_z, A_x f_y - A_y f_x)$$
$$= (A_y[B_x C_y - B_y C_x] - A_z[B_z C_x - B_x C_z], \ldots, \ldots)$$
$$= (B_x[A_x C_x + A_y C_y + A_z C_z] - C_x[A_x B_x + A_Y B_Y$$
$$+ A_Z B_Z] - A_X B_X C_X + A_X B_X C_X, \ldots, \ldots)$$

Generalize to

$B(A\bullet C) - C(A\bullet B)$

identity associated with this form

$$(A \times B) \times C = -C \times (A \times B) = -A(C \bullet B) + B(C \bullet A)$$

identity associated with this form

$A \bullet (B \times C) = [ABC]$

$A \bullet (B \times C) = (A \times B) \bullet C$, can be shown expanding in Cartesian coordinates. Exchange dot and cross in this form.

identity associated with this form

A•(BxC)

$$\mathbf{A\bullet(B\times C)} = (A_x,A_y,A_z)(f_x,f_y,f_z)$$

$$= A_x(B_yC_z-B_zC_y)+A_y(B_zC_x-B_xC_z)$$

$$+A_z(B_xC_y-B_yC_x)$$

$$\mathbf{A\bullet(B\times C)} = C_x(A_yB_z-A_zB_y)+C_y(A_zB_x-A_xB_z)$$

$$+C_z(A_xB_y-A_yB_x)$$

$$= \mathbf{(A\times B)\bullet C = C\bullet(A\times B)}$$

identity associated with this for

$\mathbf{A} \bullet (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \bullet (\mathbf{A} \times \mathbf{B})$

$\mathbf{B} \bullet (\mathbf{C} \times \mathbf{A}) = \mathbf{A} \bullet (\mathbf{B} \times \mathbf{C})$

$\mathbf{C} \bullet (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \bullet (\mathbf{C} \times \mathbf{A})$

$\mathbf{A} \bullet (\mathbf{C} \times \mathbf{B}) = \mathbf{B} \bullet (\mathbf{A} \times \mathbf{C})$

$\mathbf{C} \bullet (\mathbf{B} \times \mathbf{A}) = \mathbf{A} \bullet (\mathbf{C} \times \mathbf{B})$

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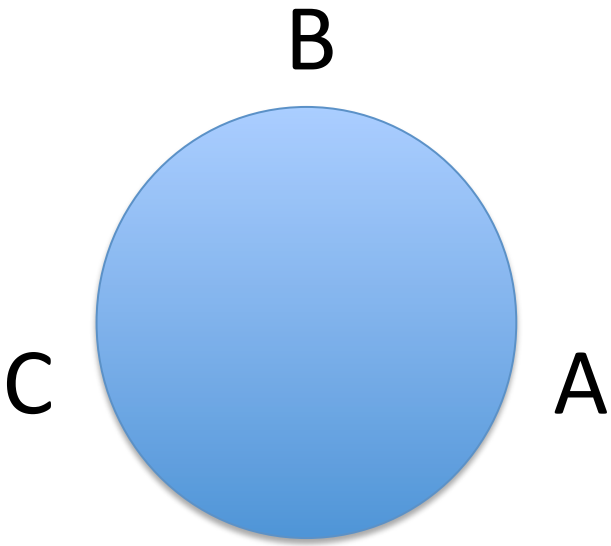

Same magnitudes and signs for CW or CCW permutations

Consider vectors, **A,B,C; A,B,C** are not all parallel to the same plane**,**

$$[\mathbf{ABC}] \neq 0$$

Show that any vector can be written as a linear combination of **A,B, & C.**

Let some vector V, be expressed as

$$\mathbf{V} = a\mathbf{A} + b\mathbf{B} + c\mathbf{C}$$

Cross **V** into **B,**

$$\mathbf{VxB} = a\mathbf{AxB} + c\mathbf{CxB}$$

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$$\mathbf{V} = a\mathbf{A} + b\mathbf{B} + c\mathbf{C}$$

Cross **V** into **B,**

$$\mathbf{V}\times\mathbf{B} = a\mathbf{A}\times\mathbf{B} + c\mathbf{C}\times\mathbf{B}$$

Dot **C** into the result,

$$(\mathbf{V}\times\mathbf{B})\cdot\mathbf{C} = a\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

Solve for a

$$a = (V \times B) \cdot C \,/\, C \cdot (A \times B)$$

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Put this into a form (prettier form)

$$a = V \cdot (B \times C) \,/ \{-C \cdot (B \times A)\}$$

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Change to "[]" notation

$$a = [VBC] \,/\, [ABC]$$

Similarly for b and c leads to

a = [VBC] / [ABC]

b = [AVC] / [ABC]

c = [ABV] / [ABC]