Divergence, $\vec{\nabla}\cdot$

to be shown,

$$\vec{\nabla}\cdot = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial x_1} h_2 h_3 + \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial x_2} h_1 h_3 + \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial x_3} h_1 h_2$$

But first we will consider Cartesian coordinates where $h_1 = h_2 = h_3 = 1$

$(h_x = h_y = h_z = 1)$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

Consider differentiation of vectors, $\vec{\nabla}\cdot$, $\vec{\nabla}\times$, ...

a) Divergence, $\vec{\nabla}\cdot$

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}A_x + \hat{j}A_y + \hat{k}A_z\right)$$

$$= \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

Examples

Let: $\vec{A}(r) = \hat{r}f(r)$ Central force field

where $\vec{r} = (\hat{i}x + \hat{j}y + \hat{k}z)$ and $\hat{r} = \dfrac{\vec{r}}{|\vec{r}|}$

(i) $\vec{\nabla}\cdot\vec{r} =$, $f(r) = r$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(x, y, z) = 1 + 1 + 1 = 3$$

(ii) Let: $f(r) = ar^n$

$$\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\left[\hat{r}ar^n\right] = \vec{\nabla}\cdot\left[\vec{r}ar^{n-1}\right]$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\left(axr^{n-1}, ayr^{n-1}, azr^{n-1}\right)$$

$$= a\left\{\frac{\partial}{\partial x}(xr^{n-1}) + \frac{\partial}{\partial y}(yr^{n-1}) + \frac{\partial}{\partial z}(zr^{n-1})\right\}$$

$$= a\left[r^{n-1} + x(n-1)r^{n-2}\frac{\partial r}{\partial x} + r^{n-1} + y(n-1)r^{n-2}\frac{\partial r}{\partial y} + r^{n-1} + z(n-1)r^{n-2}\frac{\partial r}{\partial z}\right]$$

Comment: $\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}$

$$= \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\Rightarrow = a\left\{3r^{n-1} + (n-1)\,x\,r^{n-2}\frac{x}{r} + (n-1)\,y\,r^{n-2}\frac{y}{r} + (n-1)\,z\,r^{n-2}\frac{z}{r}\right\}$$

$$= a\left\{3r^{n-1} + (n-1)\,r^{n-3}\left(x^2+y^2+z^2\right)\right\}$$

$$= a\left\{3r^{n-1} + (n-1)\,r^{n-1}\right\}$$

$$= \boxed{a r^{n-1}(n+2) = \vec{\nabla}\cdot\left(a\hat{r}r^n\right)}$$

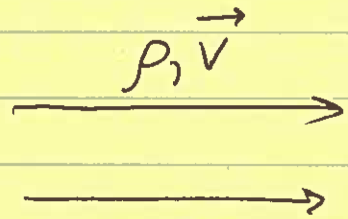
$$\boxed{\Rightarrow \vec{\nabla}\cdot\left(a r^n \hat{r}\right) = 0 \text{ if } n = -2} \rightarrow f(r) = a\frac{\hat{r}}{r^2}\,!$$

However, ~~if~~ we need to be a little careful; we must treat $r = 0$ with some care. We will find that $\vec{\nabla}\cdot\left(a\hat{r}r^n\right) = 0$ at $r \neq 0$, but that it goes to ∞ at $r = 0$

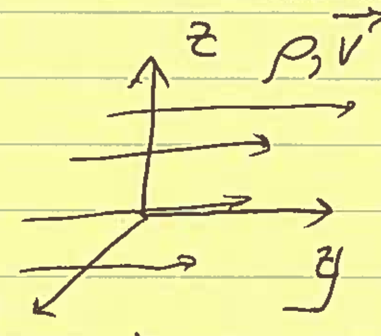
Important: $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$, $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q\hat{r}}{|\vec{r}|^2}$

Physically, how can we interpret the divergence?

Consider a fluid flow

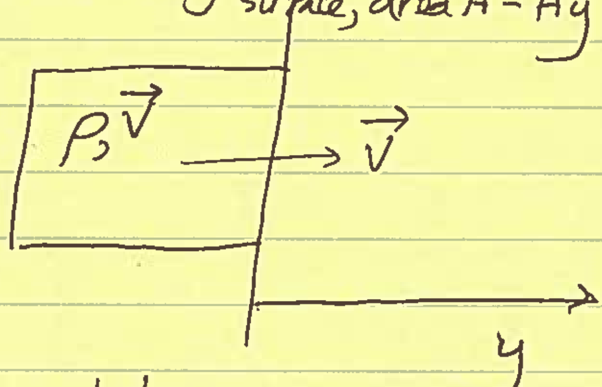


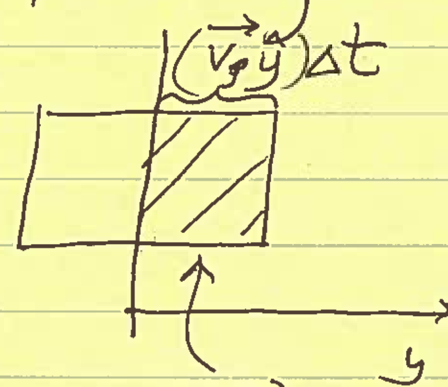
in Cartesian coordinates



ⓐ let's define flux (flow of fluid) from fluid

(i) Imagine slit in the xz plane at $y=0$
surface, area $\vec{A} = A\hat{y}$





at $t=0$, after Δt $\quad L = \vec{V}\cdot\hat{y}\,\Delta t$

(ii) How much mass flowed across A in Δt?

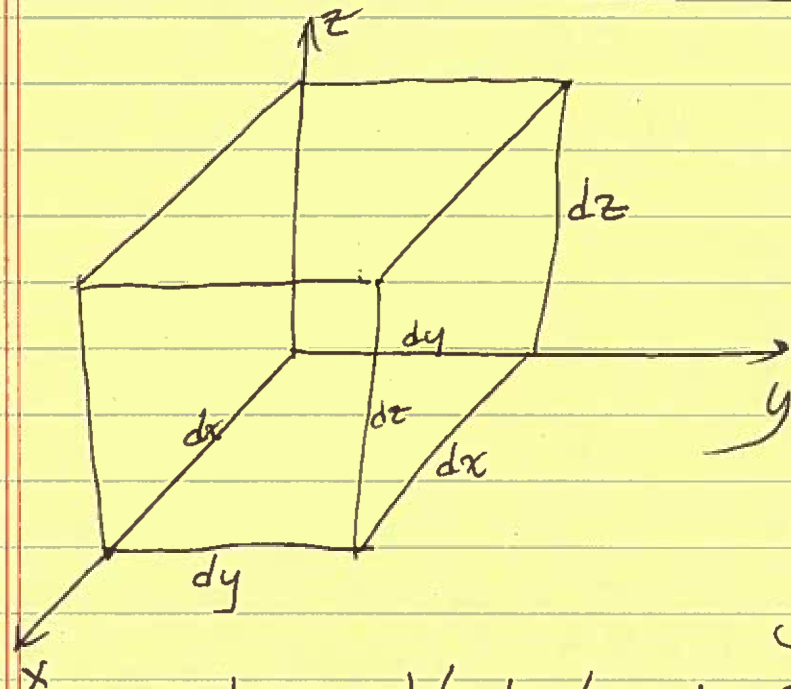
$$\Delta M = \text{Volume} \times \rho = \underbrace{A \times (V_y \Delta t)}_{\text{volume}}\,\rho$$

$$\rightarrow \frac{\Delta m}{\Delta t} = A\rho V_y$$ ≡ amount of mass that flowed across surface in Δt

we define flux as $\quad \rho\,\vec{V}\cdot\hat{A}$

$$\boxed{\frac{1}{A}\frac{\Delta m}{\Delta t} = \rho V_y \equiv \text{flux}}$$

Consider a box in Cartesian space



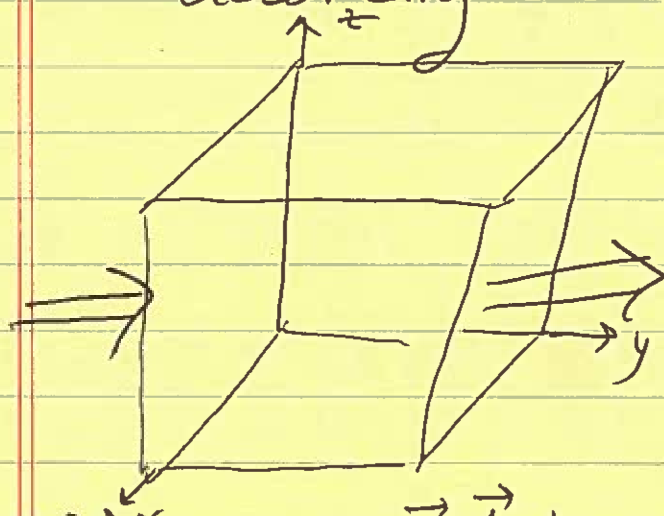
Each area is A so that $A_y\hat{e}_y$, $A_z\hat{e}_z$, $A_x\hat{e}_x$ as before

~~Each side~~

where

$$\boxed{\begin{aligned} A_x &= dy\,dz \\ A_y &= dx\,dz \\ A_z &= dx\,dy \end{aligned}}$$

(i) Okay, let's look at flow in $\hat{y}$ direction and ask how does the mass change according to this flow,



$$\underbrace{\rho_1 \vec{V}_1 \cdot \vec{A}_y \Delta t}_{\text{mass in } \Delta t,\ \hat{y}} \qquad \rho_2 \vec{V}_2 \cdot \vec{A}_y \Delta t$$

mass in: $\rho_1 \vec{V}_1 \cdot \vec{A}_y \Delta t$

mass out: $-\rho_2 \vec{V}_2 \cdot \vec{A}_y \Delta t$

$$\Delta m = \rho(x,y,z)\, v_y(x,y,z)\, A_y \Delta t$$
$$\quad - \rho(x,y+\Delta y,z)\, v_y(x,y+\Delta y,z)\, A_y \Delta t$$

$$= \rho(x,y,z)\, v_y(x,y,z)\, A_y \Delta t$$
$$\quad - \left[\rho(x,y,z) + \frac{\partial \rho}{\partial y}\Delta y\right]\left[v_y(x,y,z) + \frac{\partial v_y}{\partial y}\Delta y\right] A_y \Delta t$$

$$= -\left[\frac{\partial \rho}{\partial y} v_y(x,y,z) + \rho(x,y,z)\frac{\partial v_y}{\partial y}\right]\Delta y\, A_y \Delta t$$

← drop 2nd order terms

$$\boxed{\Delta m = -\frac{\partial}{\partial y}\left[\rho v_y\right]\Delta y\, A_y \Delta t}$$

(ii) Expand answer to all 3 directions

$$\Delta m = \cancel{A}\, V \Delta t\left[-\left(\frac{\partial}{\partial y}\rho v_y\right)\cancel{\Delta y} - \left(\frac{\partial}{\partial x}\rho v_x\right)\cancel{\Delta x} - \left(\frac{\partial}{\partial z}\rho v_z\right)\cancel{\Delta z}\right]$$

where $V = dx\,dy\,dz$

$$\Rightarrow \Delta \rho = \frac{\Delta m}{V} = \Delta t\left[-\vec{\nabla}\cdot(\rho\vec{v})\right]$$

$$\Longrightarrow \frac{\Delta \rho}{\Delta t} = -\vec{\nabla}\cdot(\rho\vec{v}) \longrightarrow \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{v} = 0}$$

Equation of Continuity

mass flux

$$\Rightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

(i) In general, we encounter continuity equations all of the time and we write them generally as

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J}_\rho = 0$$

some density of a quantity

flux of that quantity

(ii) In fluids, we define the momentum density,

$$\vec{\wp} = \rho\vec{v} = \frac{\text{momentum}}{\text{volume}}$$

and we write

$$\frac{\partial \rho\vec{v}}{\partial t} = \frac{\partial\vec{\wp}}{\partial t} = -\vec{\nabla}\cdot\overrightarrow{\overrightarrow{S}}$$

stress tensor

flux of momentum

if only fluid motions are considered, we have

$\boxed{\rho\vec{v}}$ ← box that contains momentum density

→ box flows in some direction $\vec{v}$

quantity fluxed

$$\Rightarrow \overrightarrow{\overrightarrow{S}} = (\rho\vec{v})\,\vec{v}$$

speed (velocity) of fluxing

note that $\overleftrightarrow{S}$ is a tensor

box has 3 components

$\boxed{\rho(v_x, v_y, v_z)} \longrightarrow$ flow (v_x, v_y, v_z) has 3 components

and $\overleftrightarrow{S} = \rho \vec{v}\vec{v}$ can be written

$$\boxed{S_{ij} = \rho v_i v_j}$$

(iii) for an EM field, the E & B fields carry momentum and the Maxwell stress tensor is defined

$$\overleftrightarrow{S} = \epsilon_0 \vec{E}\vec{E} + \frac{\vec{B}\vec{B}}{\mu_0} - \frac{1}{2}\overleftrightarrow{\mathbb{1}}\left(\epsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$$

unit tensor

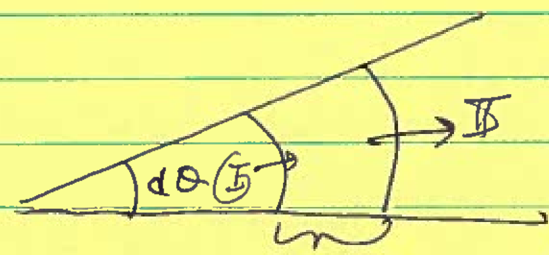
or

$$\boxed{S_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2}\delta_{ij}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)}$$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0, \text{ Continuity Eqn.}}$$

(e) <u>Comment</u>: In our derivation, we were apparently cavalier w/ our handling of the area elements, (dy dz), (dx, dz), (dy dz). In Cartesian coordinates, this okay, but in ~~spherical~~ cylindrical coordinates this is not okay, e.g.,

cylindrical coordinates



$$\text{at II}, \quad (\rho v_r)_{r+dr}\,\underbrace{(r\,d\theta\,dz)}_{II} \qquad (r+dr)$$

$$\text{I} \quad (\rho v_r)(r)\,\underbrace{r_I\,d\theta\,dz}$$

Area elements change w/ r ~~and~~ $\vec{\nabla}\cdot(\rho\vec{v})$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r)$$

because of the coordinate system