Physics 421
Test 1
Thursday, 2018 October 18

Answer 4 of the following 6 questions. You may use your text and class notes while working on the test.

VECTOR IDENTITIES

TRIPLE PRODUCTS

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

PRODUCT RULES

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

SECOND DERIVATIVES

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_a^b (\nabla f) \cdot d\mathbf{l} = f(b) - f(a)$

Divergence Theorem: $\int_{\text{volume}} (\nabla \cdot \mathbf{A})\, d\tau = \oint_{\text{surface}} \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int_{\text{surface}} (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint_{\text{line}} \mathbf{A} \cdot d\mathbf{l}$

VECTOR DERIVATIVES

CARTESIAN. $d\mathbf{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$; $d\tau = dx\,dy\,dz$

Gradient. $$\nabla t = \frac{\partial t}{\partial x}\hat{i} + \frac{\partial t}{\partial y}\hat{j} + \frac{\partial t}{\partial z}\hat{k}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl. $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{k}$$

Laplacian. $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

SPHERICAL. $d\mathbf{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient. $$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\phi}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl. $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\phi}$$

Laplacian. $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

CYLINDRICAL. $d\mathbf{l} = dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z}$; $d\tau = r\,dr\,d\phi\,dz$

Gradient. $$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl. $$\nabla \times \mathbf{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial \phi}\right]\hat{z}$$

Laplacian. $$\nabla^2 t = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

Question 1

A particle of mass m moves in a force field given by

$$\mathrm{F} = \hat{e}_r f(r) \tag{1}$$

where $\hat{e}_r$ is the unit vector in the radial direction, $f(r)$ is a scalar function, and r is the distance from the origin, $r = \sqrt{x^2 + y^2 + z^2}$. The force always passes through a fixed point and has magnitude that depends only on distance from the fixed point. A force with these properties is called a *Central Force.*

a. Show that the field F is conservative.

b. The angular momentum of a particle is given as

$$\mathbf{l} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times \frac{m\mathbf{v}}{\mathrm{dt}} = \mathbf{r} \times m\frac{\mathrm{d}\mathbf{r}}{\mathrm{dt}} \tag{2}$$

where $\mathbf{r}$ and $\mathbf{v}$ are the position and velocity of the mass, respectively. Show that for any force of the form given above

$$\frac{\mathrm{d}\mathbf{l}}{\mathrm{dt}} = 0. \tag{3}$$

The angular momentum is thus conserved for motion in a central field.

c. For the case where $f(r) = \frac{K}{r^3}$, find the scalar potential for F.

a) find $\vec{\nabla}\times\vec{F} = \vec{\nabla}\times \hat{e}_r f(r)$

looking at $\vec{\nabla}\times$ in spherical coordinates, see that only the following act on $\hat{e}_r f(r)$,

$$\vec{\nabla}\times\vec{F} = \frac{1}{r\sin\theta}\underbrace{\frac{\partial}{\partial\phi}\left[\cancel{\sin\theta}\, f(r)\right]}_{\to 0}\hat{\theta} + \frac{1}{r}\Big[\underbrace{-\frac{\partial}{\partial\theta}f(r)}_{\to 0}\Big]\hat{\phi}$$

$\Rightarrow \vec{F}$ is curlless $\Rightarrow \hat{e}_r f(r)$ is conservative

b) $\vec{l} = \vec{r}\times m\dfrac{d\vec{r}}{dt}$

$$\to \frac{d\vec{l}}{dt} = \frac{d}{dt}\left(\vec{r}\times m\frac{d\vec{r}}{dt}\right) = \cancelto{0}{\frac{d\vec{r}}{dt}\times m\frac{d\vec{r}}{dt}} + \vec{r}\times m\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$$

$$= m\vec{r}\times\frac{d^2\vec{r}}{dt}$$

$$= m\vec{r}\times\vec{a}, \quad \vec{a} = \frac{\vec{F}}{m} = \frac{f(r)}{m}\hat{e}_r$$

$$\longrightarrow \vec{r}\times\vec{a} = 0$$

set to 0

$\Phi(r) - \Phi(\infty)$

$$\text{c)} \int_\infty^r d\Phi = -\int_\infty^r \vec{F}\cdot d\vec{r} = -\int_\infty^r f(r)\,dr = -\int_\infty^r \frac{K}{r^3}dr = \frac{K}{2r^2}\Big|_\infty^r = \frac{K}{2r^2} - 0$$

$$\rightarrow \Phi(r) = \frac{K}{2r^2} + \Phi(\infty)$$

usually taken to be 0

Question 2

The parabolic coordinate system (u, v, ϕ) transformations are

$$x = uv\cos\phi \quad (4)$$
$$y = uv\sin\phi \quad (5)$$
$$z = \frac{1}{2}\left(u^2 - v^2\right) \quad (6)$$

a. Find the arc length ds in terms involving u, v, and ϕ, du, dv, and dϕ, and the unit vectors $\hat{i}, \hat{j}, \hat{k}$.

b. Find ds^2

c. Find an expression for the velocity.

d. Find the scale factors.

e. Find the Laplacian.

a) $d\vec{s} = \hat{i}dx + \hat{j}dy + \hat{k}dz$

$= \hat{i}\left[du\, v\cos\phi + u\,dv\cos\phi - uv\sin\phi\, d\phi\right]$
$+ \hat{j}\left[du\, v\sin\phi + u\,dv\sin\phi + uv\cos\phi\, d\phi\right]$
$+ \hat{k}\left[u\,du - v\,dv\right]$

$= du\left[\hat{i}v\cos\phi + \hat{j}v\sin\phi + \hat{k}u\right]$
$+ dv\left[\hat{i}u\cos\phi + \hat{j}u\sin\phi - \hat{k}v\right]$
$+ d\phi\left[-\hat{i}uv\sin\phi + \hat{j}uv\cos\phi\right]$

b) $ds^2 = d\vec{s}\cdot d\vec{s}$

$= du^2\left(v^2\cos^2\phi + v^2\sin^2\phi + u^2\right)$
$+ dv^2\left(u^2\cos^2\phi + u^2\sin^2\phi + v^2\right)$
$+ d\phi^2\left(u^2v^2\sin^2\phi + u^2v^2\cos^2\phi\right)$

$= du^2(u^2+v^2) + dv^2(u^2+v^2) + d\phi^2(u^2v^2)$

c) $\vec{v} = \frac{d\vec{s}}{dt}$

d) $h_u = \sqrt{u^2+v^2},\ h_v = \sqrt{u^2+v^2},\ h_\phi = uv$

e) $$\vec{\nabla}\cdot\vec{\nabla} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1h_3}{h_2}\frac{\partial}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial x_3}\right)\right]$$

$$= \frac{1}{uv(u^2+v^2)}\left[\frac{\partial}{\partial u}\left(\frac{uv\sqrt{u^2+v^2}}{\sqrt{u^2+v^2}}\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{uv\sqrt{u^2+v^2}}{\sqrt{u^2+v^2}}\frac{\partial}{\partial v}\right) + \frac{\partial}{\partial \phi}\left(\frac{u^2+v^2}{uv}\frac{\partial}{\partial \phi}\right)\right]$$

$$= \frac{\cancel{v}}{\cancel{u}\cancel{v}(u^2+v^2)}\left[\frac{\partial}{\partial u}u\frac{\partial}{\partial u}\right] + \frac{\cancel{u}}{\cancel{u}v(u^2+v^2)}\left[\frac{\partial}{\partial v}v\frac{\partial}{\partial v}\right] + \frac{1}{(uv)^2}\frac{\partial^2}{\partial \phi^2}$$

$$= \frac{1}{(u^2+v^2)}\left[\frac{1}{u}\frac{\partial}{\partial u}u\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial}{\partial v}v\frac{\partial}{\partial v}\right] + \frac{1}{(uv)^2}\frac{\partial^2}{\partial \phi^2}$$

Question 3

Given the scalar function,

$$\phi(x,y) = \left(x^2 - y^2\right) \tag{7}$$

a. Draw constant ϕ curves where ϕ = -4, -1, 0, 1, and 4.

b. Draw (and add) the vector $-\nabla\phi$ to your picture at points $(x,y) = (\pm1,\pm1)$, $(0,\pm2)$, and $(\pm2,0)$ and the draw some paths along which energy would flow on your plot if $\phi(x,y)$ represents a temperature distribution. Energy flows from high temperatures to low temperatures.

c. Find the magnitude and the direction in which $\phi(x,y)$ decreases most rapidly at point (-3,2).

d. Find the rate of change of $\phi(x,y)$ with distance at (1,2) in direction (3,-1).

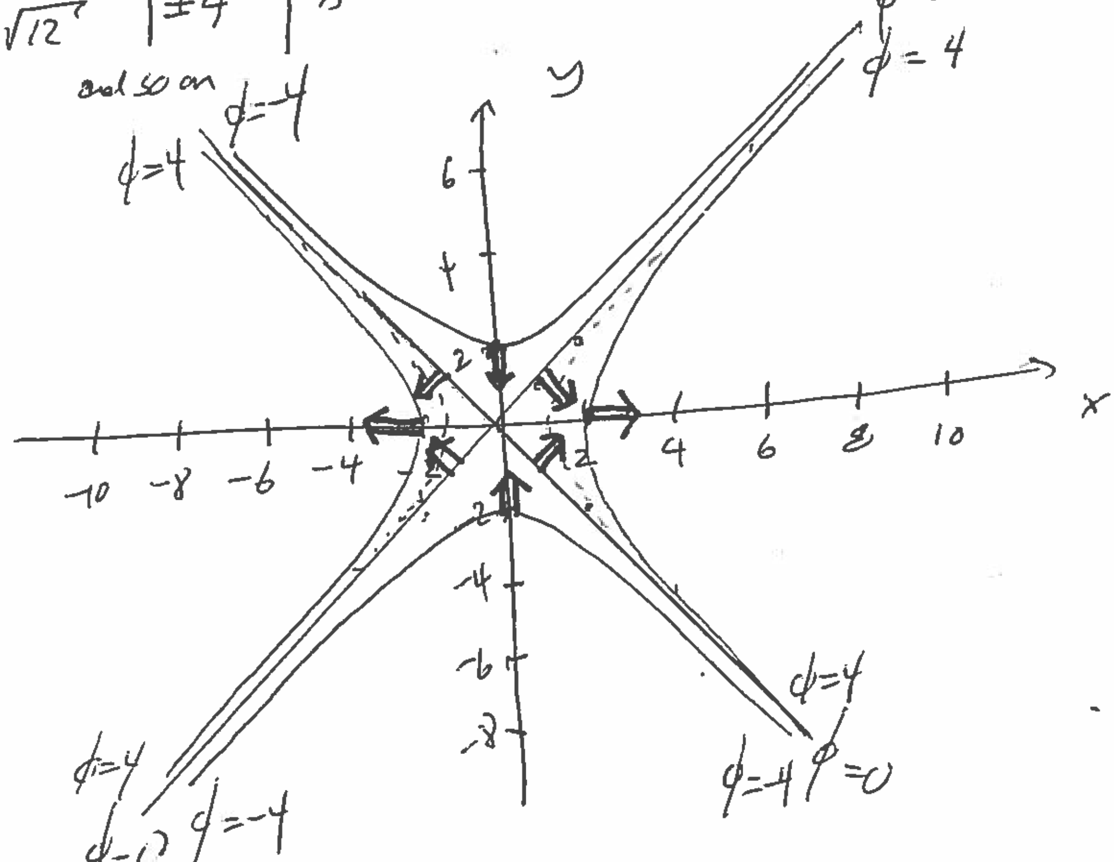
a)

y	x	ϕ
0	± 2	$+4$
$\pm 2i$	0	"
$\sqrt{12}$	± 4	"

build a table →

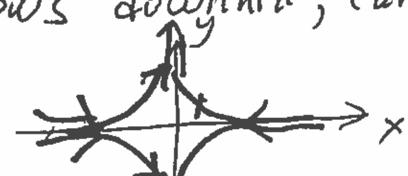
and so on

note: $\boxed{y^2 = x^2 - \phi}$



b) $\vec{\nabla}\phi = 2x\hat{i} - 2y\hat{j}$ ($\Rightarrow$ arrows)

c) energy flows "downhill", from hig T to low T



c) $\vec{\nabla}\phi = 2x\hat{i} - 2y\hat{j}$

(i) magnitude, $\sqrt{(2x)^2+(2y)^2} = \sqrt{36+16} = \sqrt{52}$

(ii) direction, $\vec{\nabla}\phi = -6\hat{i} - 4\hat{j}$

$\rightarrow \dfrac{-6\hat{i}-4\hat{j}}{\sqrt{52}}$

d) "directional derivative"

$$\vec{\nabla}\phi \cdot \frac{(3\hat{i}-\hat{j})}{\sqrt{10}}$$

evaluated at $(1,2)$

$$= (2\hat{i} - 4\hat{j}) \cdot \left(\frac{3\hat{i}-\hat{j}}{\sqrt{10}}\right)$$

$$= \frac{6-4}{\sqrt{10}}$$

$$= \frac{2}{\sqrt{10}}$$

$$= \sqrt{\frac{2}{5}}$$

Question 4

If A, B, and C are three vectors which are not parallel to the same plane, show that any vector V can be expressed as the linear combination of A, B, and C, that is, we can write

$$V = aA + bB + cC \tag{8}$$

where a, b, and c are constants. Show that

$$a = \frac{[VBC]}{[ABC]},\ b = \frac{[AVC]}{[ABC]},\ \text{and}\ c = \frac{[ABV]}{[ABC]} \tag{9}$$

In the above, brackets indicae triple scalar product, that is,

$$[ABC] = A \cdot (B \times C) \tag{10}$$

(i) $\vec{V} = a\vec{A} + b\vec{B} + c\vec{C}$, find a, b, c

(a) $\vec{V} \times \vec{B} = a\vec{A} \times \vec{B} + b\cancel{(\vec{B} \times \vec{B})}^{0} + c(\vec{C} \times \vec{B})$

$\vec{C} \cdot (\vec{V} \times \vec{B}) = a\vec{C} \cdot (\vec{A} \times \vec{B}) + c\cancel{\vec{C} \cdot (\vec{C} \times \vec{B})}^{0}$

$\rightarrow a = \dfrac{\vec{C} \cdot (\vec{V} \times \vec{B})}{\vec{C} \cdot (\vec{A} \times \vec{B})} = \dfrac{[CVB]}{[CAB]} = \dfrac{[VBC]}{[ABC]}$

cyclic

(b) $\vec{V} \times \vec{A} = a\cancel{\vec{A} \times \vec{A}}^{0} + b(\vec{B} \times \vec{A}) + c(\vec{C} \times \vec{A})$

$\vec{C} \cdot (\vec{V} \times \vec{A}) = b\vec{C} \cdot (\vec{B} \times \vec{A}) + c\cancel{\vec{C} \cdot (\vec{C} \times \vec{A})}^{0}$

$\rightarrow b = \dfrac{[CVA]}{[CBA]} = \dfrac{[CVA]}{-[CAB]} = \dfrac{-[CAV]}{-[CAB]} = \dfrac{[AVC]}{[CAB]}$

reverse

(c) $\vec{V} \times \vec{A} = a\cancel{(\vec{A} \times \vec{A})}^{0} + b(\vec{B} \times \vec{A}) + c(\vec{C} \times \vec{A})$

$\vec{B} \cdot (\vec{V} \times \vec{A}) = b\cancel{\vec{B} \cdot (\vec{B} \times \vec{A})}^{0} + c\vec{B} \cdot (\vec{C} \times \vec{A})$

$\rightarrow c = \dfrac{[BVA]}{[BCA]} = \dfrac{[ABV]}{[ABC]}$

Question 5

Evaluate directly and then by using the Divergence Theorem,

$$\oint_S F \cdot d\sigma, \tag{11}$$

where $d\sigma$ is the area element. Perform your integration over S, the unit sphere, for field $F = z\hat{e}_r$.

(a) $\oint z\hat{e}_r \cdot \hat{e}_r r^2 \sin\theta \, d\phi \, d\theta$

$= \oint \underbrace{r\cos\theta}_{z} \, r^2 \sin\theta \, d\theta \, d\phi$

$= r^3 2\pi \int_0^\pi \cos\theta \sin\theta \, d\theta$

$= 2\pi r^3 \int_0^\pi \cos\theta \, d(-\cos\theta)$

$= -2\pi r^3 \left.\frac{\cos^2\theta}{2}\right|_0^\pi$

$= 0$

(b) $\oint \vec{F} \cdot d\vec{\sigma} = \int_V \overbrace{\vec{\nabla} \cdot}^{\vec{\nabla}\cdot \text{ for radial direction}} \vec{F} \, d^3x = \int_V \frac{1}{r^2}\frac{\partial}{\partial r} r^2 z \, d^3x$

$= \int \frac{1}{r^2}\frac{\partial}{\partial r} r^3 \cos\theta \, d^3x$

$= \int 3\cos\theta \left[r^2 \sin\theta \, d\theta \, d\phi \, dr \right]$

$= 0$, from above solution of $\int_0^\pi \cos\theta \sin\theta \, d\theta$

Question 6

Find the line integral $\oint F \cdot dr$ for the vector field

$$F = y\hat{\imath} + xz\hat{\jmath} + z\hat{k} \tag{12}$$

using the path given by the unit circle in the xy plane centered on the origin in the following ways.

a. Evaluate the line integral directly.

b. Apply the 2-dimensional Stokes' Theorem (from *Green's Theorem in the plane*) using the surface area of a flat disk $x^2 + y^2 \leq 1$ for $z = 0$.

c. Apply Stokes' Theorem using the surface area of hemispherical shell $x^2 + y^2 + z^2 = 1$ for $z \geq 0$.

Helpful integral:

$$\int \sqrt{1-x^2}dx = \frac{1}{2}\left(x\sqrt{1-x^2} + \sin^{-1} x\right) \tag{13}$$

a) $\int \vec{F} \cdot d\vec{r} = \int (y\,dx + xz\,dy + z\,dz)$ over the unit circle in xy plane centred on the origin ($\rightarrow dz = 0$, $z=0$)

$= \int (y\,dx)$

unit circle, $x^2+y^2=1 \rightarrow y=\sqrt{1-x^2}$

$= \int_{-1}^{-1} \sqrt{1-x^2}\,dx$ ← integral only takes account of $\frac{1}{2}$ and loops so need $\cancel{*2}$, top half of loop

$= \left[x\sqrt{1-x^2} + \sin^{-1} x\right]_{+1}^{-1}$

$= \left[0 + \sin^{-1} 1 - 0 - \sin^{-1}(-1)\right]$

$= -\frac{\pi}{2} - \left(+\frac{\pi}{2}\right)$

$= \pi$

b) find $\vec{\nabla} \times \vec{F} = (-x\hat{\imath} + 0\hat{\jmath} + z\hat{k} - \hat{k}) = (-x\hat{\imath} - \hat{k})$ ← for flat disk in xy plane

$\Rightarrow \int (\vec{\nabla} \times \vec{F}) \cdot d\vec{\sigma} = \int -d\sigma = -\pi$ ← area of unit circle

c) (i) $\vec{\nabla} \times \vec{F} = -x\hat{i} + (z-1)\hat{k}$

(ii) $d\vec{\sigma} = \hat{e}_r \sin\theta \, d\theta \, d\phi$

$= (\hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta)\sin\theta \, d\theta \, d\phi$

$$\Rightarrow \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} d\theta \left[-x\sin\theta\cos\phi + (z-1)\cos\theta\right]\sin\theta$$

$$= \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} \left[-\sin^2\theta\cos^2\phi + (\cos\theta - 1)\cos\theta\right]\sin\theta$$

<u>integrate ϕ</u>

$$= \int_0^{\frac{\pi}{2}} d\theta \left[-\sin^2\theta\,\pi + (\cos\theta-1)\cos\theta\, 2\pi\right]\sin\theta$$

$$= \pi\int_1^0 \left[-\underbrace{\sin^2\theta}_{1-\cos^2\theta} + (\cos\theta - 1)\cos\theta\, 2\right] d(-\cos\theta)$$

$$= -\pi\left[\frac{2\cos^3\theta}{3} - \cos^2\theta\right]_1^0 - \pi\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_1^0$$

$$= -\pi\left[+\frac{2}{3} + 1\right] - \pi\left[+1 + \frac{1}{3}\right]$$

$$= -\pi\left[1 - 1 + \frac{2}{3} + \frac{1}{3}\right]$$

$$= -\pi$$