Physics 421
Test 2
Thursday, 2018 November 15

Answer 4 of the following questions. You may use your text and class notes while working on the test.

Question 1

The function

$$f(x) = x^2 \text{ for } x = [-\pi, \pi] \quad (1)$$

repeats with period 2π.

a. Find the Fourier series for $f(x)$.

b. Using your result from Part (a), find

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \quad (2)$$

(a) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

(i) $f(x) = f(-x) \rightarrow$ even, only need cos terms

(ii) $\int_{-\pi}^{\pi} \frac{a_0}{2} dx = \int_{-\pi}^{\pi} f(x) dx = \frac{x^3}{3}\Big|_{-\pi}^{\pi} = \frac{2\pi^3}{3} = \frac{a_0}{2} 2\pi \Rightarrow \boxed{a_0 = \frac{2}{3}\pi^2}$

(iii) $\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} a_n \cos nx \cos n'x \, dx = \int_{-\pi}^{\pi} f(x) \cos n'x \, dx$

$$\pi \delta_{nn'} a_n = \int_{-\pi}^{\pi} x^2 \cos nx \, dx$$

$$= \left[\cancel{\frac{x^2 \sin nx}{n}}\Big|_{-\pi}^{\pi} - \frac{2}{n} \int_{-\pi}^{\pi} x \sin nx \, dx \right]$$

$$= -\frac{2}{n}\left[-x\frac{\cos nx}{n}\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} \cos nx \, dx \right]$$

$$= -\frac{2}{n}\left[\frac{-\pi \cos n\pi}{n} - \left(\frac{\pi \cos n\pi}{n}\right) + \cancel{\frac{\sin nx}{n^2}\Big|_{-\pi}^{\pi}} \right]$$

$$= \frac{4\pi}{n^2}(\cos n\pi)$$

$$\pi \delta_{nn'} a_n = \frac{4\pi}{n^2}(-1)^n \Longrightarrow \boxed{a_n = \frac{4}{n^2}(-1)^n}$$

(iv) $\boxed{f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos nx}$

b) $\overline{|f(x)|^2} = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\right]^2$

$$= \left(\frac{\pi^2}{3}\right)^2 + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

ad $\overline{|f(x)|^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{1}{2\pi}\frac{x^5}{5}\Big|_{-\pi}^{\pi} = \frac{2\pi^5}{2\pi 5} = \frac{\pi^4}{5}$

$$\Rightarrow \frac{\pi^4}{5} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$8\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{9-5}{45}\pi^4 \Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}}$$

Question 2

The function

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \quad (3)$$

represents a pulse centered on $x = 0$.

a. Find the Fourier cosine transform of $f(x)$. Why is the cosine transform appropriate?

b. Take the inverse Fourier cosine transform to show that

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}d\omega \quad (4)$$

c. From Part (b) and $f(x)$, show that

$$\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}d\omega = \begin{cases} 0, & |x| > 1 \\ \pi/4, & |x| = 1 \\ \pi/2, & |x| < 1 \end{cases} \quad (5)$$

a) $C(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos kx\,dx$

$= \sqrt{\frac{2}{\pi}}\int_0^1 \cos kx\,dx$

$C(k) = \sqrt{\frac{2}{\pi}}\frac{\sin k}{k}$; $f(x)$ is even $\Rightarrow$ cos trasform

b) $f(x) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\sin k}{k}\cos kx\,dk$

$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin k\cos kx}{k}dk$

c) $\int_0^\infty \frac{\sin k\cos kx}{k}dk = \frac{\pi}{2}f(x) = \begin{cases} 0, & |x| > 1 \\ \frac{\pi}{4}, & |x| = 1 \\ \frac{\pi}{2}, & |x| < 1 \end{cases}$

$f(x) = 0, |x| > 1$

$\frac{1}{2}$ Jump

$f(x) = 1, |x| < 1$

Question 3

a. Two vectors are orthogonal if their dot product is zero, that is, A· B = 0. When we say that two functions, f(x) and g(x), are orthogonal over the domain $[a, b]$, what does orthogonal mean in this context?

b. Show that solutions to the one-dimensional Helmholtz equation,

$$\frac{\partial^2}{\partial x^2} F(x) + \lambda F(x) = 0, \tag{6}$$

are orthogonal over $[a, b]$ if $F(a) = F(b) = 0$ or if $dF(a)/dx = dF(b)/dx = 0$ when $\lambda > 0$.

a) $\int_a^b f^* g \, dx = 0$; orthogonal

b) write $f'' = \frac{\partial^2}{\partial x^2} f$ to save effort

c) f_1 & f_2 are solutions to Helmholtz equation

(i) $f_2^*(f_1'' + \lambda_1 f_1)$ & $f_1^*(f_2'' + \lambda_2 f_2)$

take complex conjugate and then subtract

(ii) $\int_a^b (f_2^* f_1'' - f_1 f_2^{*\prime\prime}) dx + \int_a^b (\lambda_1 f_2^* f_1 - \lambda_2^* f_1 f_2^*) dx = 0$

$\int_a^b f_1 f_2^{*\prime\prime} dx = f_1 f_2^{*\prime} \Big|_a^b - \int_a^b f_2^{*\prime} f_1' dx$

$= f_1 f_2^{*\prime} \Big|_a^b - f_2^* f_1' \Big|_a^b + \int_a^b f_2^* f_1'' dx$

(iii) $f_1 f_2^{*\prime} \Big|_a^b - f_2^* f_1' \Big|_a^b + (\lambda_1 - \lambda_2^*) \int_a^b f_1 f_2^* dx = 0$

$f_1 f_2^{*\prime} \Big|_a^b - f_2^* f_1' \Big|_a^b = (\lambda_2^* - \lambda_1) \int_a^b f_1 f_2^* dx$

if $\begin{cases} f_1(b) = f_1(a) = 0 \\ f_2^*(b) = f_2^*(a) = 0 \end{cases}$

or

$\begin{cases} f_2^{*\prime}(b) = f_2^{*\prime}(a) = 0 \\ f_1'(b) = f_1'(a) = 0 \end{cases}$

$\Rightarrow 0 = (\lambda_2^* - \lambda_1) \int_a^b f_1 f_2^* dx$

(iv) if $\lambda_2^* \neq \lambda_1 \Rightarrow \int_a^b f_1 f_2^* dx = 0$

orthogonal

(v) if $\lambda_2^* = \lambda_1 \Rightarrow \int_a^b f_1 f_2^* dx$ is anything except ∞

Question 4

a. Show that the Fourier transform of the n-th derivative of the function $f(x)$,

$$\frac{\partial^n f(x)}{\partial x^n}, \tag{7}$$

is $(ik)^n C(k)$ where $C(k)$ is the Fourier transform of $f(x)$.

b. Solve the one-dimensional diffusion equation,

$$\kappa^2 \frac{\partial^2}{\partial x^2} u(x,t) = \frac{\partial}{\partial t} u(x,t) \tag{8}$$

using Fourier transform techniques. Let the initial internal energy distribution be written $u(x,0)$. You may leave your answer for $u(x,t)$ as an integral.

c. If the initial energy distribution was described as a δ-function located at $x = a$, that is, $u(x,0) = u_o \delta(x-a)$, what is $C[u(x,0)]$, the Fourier transform of $u(x,0)$? What is $C[u(x,t)]$, the Fourier transform of $u(x,t)$?

a) $C(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\left[\frac{\partial f}{\partial x}\right] dx\, e^{-ikx}}$

let $u = e^{-ikx}$, $dV = \frac{\partial f}{\partial x} dx$

$2\pi C(k) = \cancel{f e^{-ikx}\Big|_{-\infty}^{\infty}}^{\,0} - \int_{-\infty}^{\infty} f(-ik) e^{-ikx} dx$

F.T. of $\frac{\partial f}{\partial x}$

$= ik \underbrace{\int_{-\infty}^{\infty} f e^{-ikx} dx}_{2\pi C(f)}$

$\Rightarrow C\left(\frac{\partial f}{\partial x}\right) = ikC(f)$

$\Rightarrow \boxed{C\left(\frac{\partial^n f}{\partial x^n}\right) = (ik)^n C(f)}$

b) $K^2 \frac{\partial^2}{\partial x^2} u = \frac{\partial}{\partial t} u$

$K^2 \int_{-\infty}^{\infty} \left(\frac{\partial^2 u}{\partial x^2}\right) e^{-ikx} dx = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial t} u\right) e^{-ikx} dx$

$(ik)^2 K^2 2\pi C(k,t) = \frac{\partial}{\partial t} 2\pi C(k,t)$

$-k^2 K^2 C(k,t) = \frac{\partial}{\partial t} C(k,t)$

$-k^2 K^2 t = \ln\left(\frac{C(k,t)}{C(k,0)}\right)$

$$C(k,t) = \underbrace{C(k,0)}_{\text{FT of } u(x,0)} e^{-k^2\kappa^2 t}$$

$$\Rightarrow u(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{-k^2\kappa^2 t} e^{ikx}\, dk$$

FT of $u(x,0)$

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{-k^2\kappa^2 t} e^{ikx}\, dk}$$

c) $u(x,0) = u_0\,\delta(x-a)$

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_0\,\delta(x-a)\, e^{-ikx}\, dx$$

$$\boxed{A(k) = \frac{u_0}{2\pi} e^{-ika}}$$

$$\rightarrow C(k,t) = \frac{u_0}{2\pi} e^{-ika} e^{-k^2\alpha^2 t}$$

$$\boxed{C(k,t) = \frac{u_0}{2\pi} e^{-(ika + k^2\alpha^2 t)}}$$

Question 5

The Klein-Gordon equation is

$$\nabla^2 u(x,t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u(x,t) + \lambda^2 u(x,t) \tag{9}$$

where t is time, v is a characteristic speed, and λ is a constant.

a. Use the method separation of variables to find the general solution to the one-dimensinoal Klein-Gordon equation.

b. If $u(x,t) = 0$ at $x = 0$ and l, show that $u(x,t)$ oscillates with frequencies given by

$$\omega_n = v\sqrt{(n\pi/l)^2 + \lambda^2} \tag{10}$$

where n is an integer.

c. Can the solutions of the Klein-Gordon equation be broken down into linear combinations of two oppositely directed traveling waves as in the d'Alembert solution for the wave equation? Support your answer.

a) $\frac{\partial^2}{\partial x^2}u = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u + \lambda^2 u$

let $u = f(x)T(t)$

$T(t)\frac{\partial^2 f}{\partial x^2} = \frac{f(x)}{v^2}\frac{\partial^2}{\partial t^2}T + \lambda^2 fT$

$\frac{1}{f}\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2 T}\frac{\partial^2 T}{\partial t^2} + \lambda^2$

for convenience

a) $\frac{1}{f}\frac{\partial^2 f}{\partial x^2} - \lambda^2 = \frac{1}{v^2 T}\frac{\partial^2 T}{\partial t^2} = -k^2 \Rightarrow T = a\cos kvt + b\sin kvt$

b) $\frac{1}{f}\frac{\partial^2 f}{\partial x^2} - \lambda^2 = -k^2 \rightarrow \frac{\partial^2 f}{\partial x^2} - (\lambda^2 - k^2)f = 0$

$\Rightarrow f = A\cos\sqrt{k^2-\lambda^2}\,x + B\sin\sqrt{k^2-\lambda^2}\,x$

(i) at $x = 0, l$, $u(x,t) = 0$

$\rightarrow A = 0, \sqrt{k^2-\lambda^2}\,l = n\pi$

$k^2 - \lambda^2 = \left(\frac{n\pi}{l}\right)^2$

$k = \sqrt{\lambda^2 + \left(\frac{n\pi}{l}\right)^2}$

$\rightarrow T = a\cos\left(\sqrt{\lambda^2 + \left(\frac{n\pi}{l}\right)^2}\,vt\right) + b\sin\left(\sqrt{\lambda^2 + \left(\frac{n\pi}{l}\right)^2}\,vt\right)$

and $\omega = \sqrt{\lambda^2 + \left(\frac{n\pi}{\ell}\right)^2}\, v$

c) general sol^{n}

$$\begin{pmatrix} \cos\sqrt{k^2-\lambda^2}\,x \\ \sin\sqrt{k^2-\lambda^2}\,x \end{pmatrix}\begin{pmatrix} \cos kvt \\ \sin kvt \end{pmatrix}$$

$$\Rightarrow \begin{cases} \cos\sqrt{k^2-\lambda^2}\,x \cos kvt, \\ \cos\sqrt{k^2-\lambda^2}\,x \sin kvt, \\ \sin\sqrt{k^2-\lambda^2}\,x \cos kvt, \\ \sin\sqrt{k^2-\lambda^2}\,x \sin kvt \end{cases}$$

ok try $\sin\left[\overbrace{\sqrt{k^2-\lambda^2}}^{\xi}x - kvt\right] - \sin\left[\overbrace{\sqrt{k^2-\lambda^2}}^{\xi}x + kvt\right]$

$= \cos\xi x \sin(-kvt) + \sin\xi x \cos(-kvt)$

$\pm\left[\cos\xi x \sin kvt + \sin\xi x \cos kvt\right]$

$$= \begin{cases} 2\cos\xi x \sin kvt & \text{"}-\text{"} \\ 2\sin\xi x \cos kvt & \text{"}+\text{"} \end{cases}$$

and can also find other terms by using $\cos(\xi x \pm kvt)$ waves

so if we define $\sqrt{k^2-\lambda^2}\left(x \pm \frac{kvt}{\sqrt{k^2-\lambda^2}}\right)$, we again get that

solutions are 2 propagating waves

Question 6

A string stretched between $x = 0$ and l is initially held in the shape

$$\psi(x,0) = \begin{cases} \sin 2\pi(x/l) &, \quad x < (l/2) \\ 0 &, \quad x > (l/2) \end{cases} \tag{11}$$

The string is released (from rest) with its subsequent motion governed by the one-dimensional wave equaation,

$$\nabla^2\psi(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(x,t) \tag{12}$$

a. Find the general solution for the one-dimensional wave equation using separation of variables.

b. Given the boundary conditions, $\psi(0,t) = \psi(l,t) = 0$, and that the string was released from rest, find $\psi(x,t)$.

a) $\frac{\partial^2}{\partial x^2}\psi(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(x,t)$

Let $\psi(x,t) = f(x)\,T(t)$

$\rightarrow T(t)\frac{\partial^2}{\partial x^2}f(x) = \frac{f(x)}{c^2}\frac{\partial^2 T(t)}{\partial t^2}$

factor out $\psi(x,t)$

$\underbrace{\frac{1}{f(x)}\frac{\partial^2}{\partial x^2}f(x)}_{\text{independent of } t} = \underbrace{\frac{1}{c^2 T(t)}\frac{\partial^2 T(t)}{\partial t^2}}_{\text{independent of } x} \Rightarrow = -k^2$ ← some constant

b) Solve each ODE

$\rightarrow \frac{1}{f(x)}\frac{\partial^2}{\partial x^2}f(x) = -k^2 \rightarrow f(x) = a\cos kx + b\sin kx$

and

$\rightarrow \frac{1}{c^2 T(t)}\frac{\partial^2}{\partial t^2}T(t) = -k^2 \rightarrow T(t) = c\cos kct + d\sin kct$

$\Rightarrow \boxed{\psi(x,t) = (a\cos kx + b\sin kx)(c\cos kct + d\sin kct)}$

c) Apply BCs

(i) $\psi(0,t) = \psi(l,t) = 0$

$\rightarrow a = 0$ $\quad \hookrightarrow kl = n\pi \rightarrow k = \frac{n\pi}{l}$

$\Rightarrow \psi(x,t) = b\sin\left(\frac{n\pi}{l}x\right)\left(c\cos\frac{n\pi}{l}ct + d\sin\frac{n\pi}{l}ct\right)$

(ii) string released from rest $\rightarrow \dot{\psi}(x,0)=0$

$$\rightarrow \dot{\psi}(x,t) = b_n \sin\left(\frac{n\pi}{\ell}x\right)\left[-c_n \sin\frac{n\pi}{\ell}ct + d_n \cos\frac{n\pi}{\ell}ct\right]\frac{n\pi c}{\ell}$$

at $t=0$, $\dot{\psi}(x,0)=0$

$$\rightarrow \dot{\psi}(x,0) = b_n \sin\left(\frac{n\pi}{\ell}x\right)\left[0 + d_n\right]\frac{n\pi c}{\ell}$$

and d must be 0

(iii) $$\psi(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) c_n \cos\left(\frac{n\pi c}{\ell}t\right)$$

(iv) at $t=0$, $\psi(x,0) = \begin{cases} \sin\frac{2\pi x}{\ell}, & x < \ell/2 \\ 0, & x > \ell/2 \end{cases}$

(v) find b_n at $t=0$ combine $b_n c_n$ into a new b_n

$$\int_0^\ell \psi(x,0)\sin\frac{n'\pi x}{\ell}dx = \sum_{n=1}^{\infty} b_n \int_0^\ell \sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{n'\pi x}{\ell}\right)dx$$

$$\int_0^{\ell/2} \sin\frac{2\pi x}{\ell}\sin\frac{n'\pi x}{\ell}dx = \frac{\ell}{2}b_n\delta_{nn'}$$

$$\left[\frac{\sin\left(\frac{2\pi}{\ell}-n'\frac{\pi}{\ell}\right)x}{2\left(\frac{2\pi}{\ell}-n'\frac{\pi}{\ell}\right)} - \frac{\sin\left(\frac{2\pi}{\ell}+n'\frac{\pi}{\ell}\right)x}{2\left(\frac{2\pi}{\ell}+n'\frac{\pi}{\ell}\right)}\right]_0^{\ell/2} = \frac{\ell}{2}b_n\delta_{nn'}$$

$$\left[\frac{\sin\left(\frac{\pi}{2}[2-n]\right)}{\frac{2\pi}{\ell}(2-n)} - \frac{\sin\left(\frac{\pi}{2}[2+n]\right)}{\frac{2\pi}{\ell}(2+n)}\right] = \frac{\ell}{2}b_n \rightarrow b_n = \frac{1}{\pi}\left\{\frac{\sin\frac{\pi}{2}(2-n)}{(2-n)} - \frac{\sin\frac{\pi}{2}(2+n)}{(2+n)}\right\}$$

$$\rightarrow \boxed{\psi(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{n\pi c t}{\ell}\right)}$$

Question 7

A metal bar has temperature distribution $T(x,t)$. At time $t=0$, the bar has temperature distribution,

$$T(x,0) = T_\circ e^{-(x/x_\circ)^2}. \tag{13}$$

For $t>0$, energy flows in the bar, and the temperature distribution evolves in a manner described by the one-dimensional diffusion equation,

$$\frac{\partial^2}{\partial x^2}T(x,t) = \frac{1}{\alpha^2}\frac{\partial}{\partial t}T(x,t) \tag{14}$$

where α is a constant.

a. Use separation of variables to find the base solution for the one-dimensional diffusion equation.

b. Find $A(k)$ the Fourier transform of the initial temperature distribution, $T(x,0)$. What is the Fourier transform for $T(x,t)$? A helpful result is that the integral of a Gaussian is

$$\int_{-\infty}^{\infty} e^{-x^2/x_\circ^2}\mathrm{d}x = x_\circ\sqrt{\pi} \tag{15}$$

c. Give an expression for $T(x,t)$. Your answer for $T(x,t)$ may be in integral form.

a) $\frac{\partial^2}{\partial x^2}T(x,t) = \frac{1}{\alpha^2}\frac{\partial}{\partial t}T(x,t)$

let $T(x,t) = f(x)g(t)$

$\rightarrow g(t)\frac{\partial^2 f(x)}{\partial x^2} = \frac{1}{\alpha^2} f(x)\frac{\partial g(t)}{\partial t}$

divide by $T(x,t)$

$\frac{1}{f(x)}\frac{\partial^2 f(x)}{\partial x^2} = \frac{1}{\alpha^2 g(t)}\frac{\partial g(t)}{\partial t} = -k^2 \leftarrow$ constant, $k^2>0$

$\Rightarrow$ (i) $\frac{1}{f}\frac{\partial^2 f}{\partial x^2} = -k^2 \rightarrow f(x) = a\cos kx + b\sin kx$

(ii) $\frac{1}{\alpha^2 g(t)}\frac{\partial g(t)}{\partial t} = -k^2 \Rightarrow \frac{1}{g(t)}\frac{\partial g(t)}{\partial t} = -k^2\alpha^2$

$\ln\frac{g(t)}{g(0)} = -k^2\alpha^2(t-0)$

$g(t) = g(0)e^{-k^2\alpha^2 t}$

and $\boxed{T(x,t) = (a\cos kx + b\sin kx)g(0)e^{-\alpha^2 k^2 t}}$

b) find $A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} T(x,0)\, e^{-ikx}\, dx$

$$= \frac{T_0}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{x}{x_0}\right)^2} e^{-ikx} = 0$$

$$= \frac{T_0}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{x}{x_0}\right)^2 - ikx}\, dx$$

complete the square, $-\frac{1}{x_0^2}\left[\underbrace{x^2 + ikx x_0^2}_{(x+c)^2 - c^2}\right]$

$$\Rightarrow c = i\frac{k}{2}x_0^2$$

$$= \frac{T_0}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{x_0^2}\left(x + i\frac{k}{2}x_0^2\right)^2} e^{-\frac{k^2}{4}x_0^2}\, dx$$

$$= \frac{T_0}{2\pi} e^{-\frac{1}{4}k^2x_0^2}\int_{-\infty}^{\infty} e^{-\xi^2/x_0^2}\, d\xi$$

where we set $\xi = \left(x + i\frac{k}{2}x_0^2\right)$

$$= \frac{T_0}{2\pi} e^{-\frac{1}{4}k^2x_0^2}\left[x_0\sqrt{\pi}\right]$$

$$\boxed{A(k) = \frac{T_0 x_0}{2\sqrt{\pi}} e^{-\left(\frac{k x_0}{2}\right)^2}}$$

c) $T(x,t) = \int_{-\infty}^{\infty} A(k)\, e^{-\alpha^2 k^2 t} \cos kx\, dk$

by symmetry ($f(x,0)$ is even), use cos transform