PHYSICAL REVIEW D VOLUME 47, NUMBER 4 15 FEBRUARY 1993

Inelastic billiard ball in a spacetime with a time machine

E. V. Mikheeva
Sternberg State Astronomical Institute, Moscow 119899, Russia

I. D. Novikov
Copenhagen University Observatory, Oster Voldgade 3, DK-1350 Copenhagen, Denmark;
NORDITA, Blegdamsvej 17, DK-2100 Copenhagen, Denmark;
and Astro Space Center of the P. N. Lebedev Physical Institute, Moscow, Russia
(Received 13 July 1992)

The nonrelativistic motion with self-collision of an inelastic billiard ball in spacetime with a time machine is discussed. We consider the wormhole-type time machine, assuming that $\epsilon \equiv$(radius of wormhole mouth)/(distance between mouths) $\ll 1$, and that (radius of ball)/(distance between wormhole mouths)$=O(\epsilon^2)$. The coefficient of friction of the balls is of order ϵ, and the balls can have an arbitrary amount of inelasticity. Solutions are sought with an accuracy up through order ϵ^4. We demonstrate that the generic class of initial data has self-consistent solutions of the equations of motion. Up to the order studied the friction does have an effect, but the inelasticity has no effect whatsoever.

PACS number(s): 04.20.Cv

I. INTRODUCTION

In the paper by Echeverria, Klinkhammer, and Thorne [1], the motion with self-collision of a nonrelativistic elastic billiard ball in spacetime with a time matching has been investigated. The time machine was treated as a wormhole that takes the ball backward in time. These investigations have been performed in the framework of the hypothesis of the principle of self-consistency (PSC). This PSC was declared and discussed in papers [2–5]. It states that the only solution to the laws of physics that can occur locally in the real Universe are those which are globally self-consistent. Echeverria, Klinkhammer, and Thorne have found that generic classes of initial data have multiple, and even infinite numbers of self-consistent solutions to the equation of motion, and they have found no evidence for the existence of generic initial data with no self-consistent solutions.

The purpose of this paper is to investigate a more realistic, more complex case, namely, the nonrelativistic motion with self-collision of an inelastic billiard ball in spacetime with a time machine. The billiard ball collisions are treated in the approximation where the parameter $\epsilon \equiv$(radius of wormhole mouth)/(distance between mouths) $\ll 1$. We assume also that (radius of ball)/(distance between wormhole mouths)$=O(\epsilon^2)$. The balls are assumed to have an arbitrary amount of inelasticity, and their coefficient of friction is of order ϵ. The solutions are sought with an accuracy up through order ϵ^4.

We will demonstrate that in this case the generic class of initial data has self-consistent solutions to the equations of motion. At least one solution is found to this order ϵ^4 for all initial data, and two solutions are found for all "dangerous" (see the definition in the paper [1]) initial data. Up to the order studied, the inelasticity has no effect whatsoever, but friction does have an effect. The explanation of this fact is given in the concluding Sec. V. To probe the effects of inelasticity itself one needs to carry the calculations to higher order.

II. THE FORMULATION OF THE PROBLEM

In this paper we discuss the wormhole-type time machine. The simplest toy model of this time machine is the following.

In flat, Minkowskii spacetime one cuts out the world tubes of two equal balls that are at rest in some Lorentz coordinate system, and identifies the surfaces of the balls (it is the wormhole with vanishingly short length), with a time delay ΔT between them. We shall call these balls the mouths of the time machine. Throughout this paper we measure spatial distances in units of the separation between the centers of the mouths in the external space and time in units of ΔT. We denote by A the radii of the two mouths, and by R the radius of the billiard ball. In the problem under discussion the ball enters mouth **A** (see Fig. 1), exits from the mouth **B**, thereby traveling backward in time, and collides with itself in the past. We restrict attention to the initial trajectories of a billiard ball being coplanar with the line of centers of the mouths, and, for simplicity, to solutions in which the ball traverses the time machine only once. It corresponds to the class-I and class-II solutions in the analysis of Echeverria, Klinkhammer, and Thorne [1].

These classes of solutions are small perturbations of the self-inconsistent solution, in the sense that the ball's path is displaced by only enough to permit the ball to undergo a glancing collision rather than a head-on collision.

We suppose that the time machine–ball system does not interact with the external matter. We shall presume also that the ball is small enough ($R \ll 1$) that we can ignore the tidal force exerted on the ball by the mouths of the time machine. The ball is treated as a "test object" that moves through the fixed wormhole geometry. The "time machine traversal rules" are the same as in [1] [see (2.1a), (2.1b) in the paper [1]]. They follow from energy conservation and simple geometrical consideration, and can be summarized by the following: (a) the absolute value of velocity of the ball is not changed by the traversal, (b) the relations between angles of the enter and exit are clear from Fig. 1.

Our purpose is to discuss inelastic collision between younger and older versions of the billiard ball and to obtain self-consistent solutions to the equations of motion.

An inelastic impact is characterized by two parameters. The first parameter e characterizes the recovery of a ball after the collision. It is defined as the ratio of the components of the relative velocities which are normal to the contacting plate after and before collision.

The equality $e=0$ corresponds to sticking together; $e=1$ corresponds to elastic impact. Throughout this paper we shall use another parameter, $a\equiv(1+e)/2$.

The second parameter is the coefficient of friction f. It is defined as

$$f\equiv\frac{\text{magnitude of frictional force}}{\text{magnitude of normal thrust}}\ ,$$

for the case of ideal smooth surfaces $f=0$, and there is no change in the tangent components of velocities. The case $f\to\infty$ corresponds to absolutely rough surfaces, and the final tangent component of the relative velocity is equal to zero.

For standard billiard balls $e=0.8$–0.95; $f\approx0.04$ (see [6]).

In the next section we present a set of equations that govern self-consistent solutions for an ideal elastic ball. We shall give the solutions to this set for the case $R\ll A\ll1$ in the form which is different from [1], and with an accuracy that is enough for the discussion of the inelastic case. This last case will be discussed in Sec. IV.

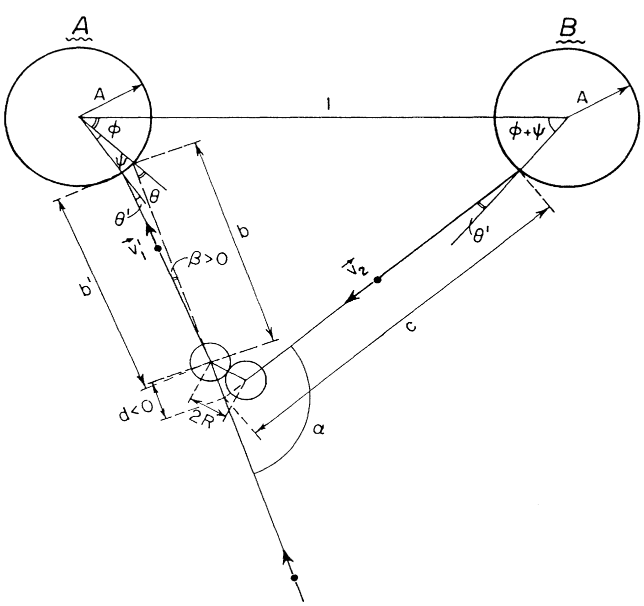

FIG. 1. Geometry of a self-consistent solution. Note that $|\mathbf{v}_1|=|\mathbf{v}_2|$.

III. THE CASE OF ELASTIC BILLIARD BALL

Following to the idea of Echeverria, Klinkhammer, and Thorne [1] one can write the set of equations that govern self-consistent, coplanar solutions for an ideal elastic ball:

$$\begin{aligned} &v_2=v_1\sin[(\alpha+\beta)/2]/\sin[(\alpha-\beta)/2]\ ,\\ &\frac{|d|}{v_1}=\frac{2R}{(v_1^2+v_2^2+2v_1v_2\cos\alpha)^{1/2}}\ ,\\ &\frac{d}{v_1}=1-\frac{b'+c}{v_2}\ , \end{aligned}\tag{1}$$

$$\begin{aligned} &\alpha=2(\phi+\Theta-\Theta')-\beta\ ,\\ &c=\frac{\sin(\phi+\Theta)}{\sin\alpha}\left[1-2A\cos\phi+2A\frac{\sin(\psi/2)\cos\gamma_1}{\sin\gamma_2}\right]\\ &\quad-2A\frac{\sin(\psi/2)\cos\gamma_3}{\sin\gamma_2}\ ,\\ &d=b-\frac{\sin\gamma_2}{\sin\alpha}\left[1-2A\cos\phi+2A\frac{\sin(\psi/2)\cos\gamma_1}{\sin\gamma_2}\right]\ ,\\ &\gamma_1=\tfrac{1}{2}(\Theta-3\Theta'-\beta)\ ,\\ &\gamma_2=\Theta+\phi-2\Theta'-\beta\ ,\\ &\gamma_3=\tfrac{1}{2}(\Theta-\Theta'-\beta)+\phi\ ,\\ &\Theta'=\arcsin\left[\sin(\Theta-\beta)-\frac{b}{A}\sin\beta\right]\ ,\\ &b'=b\left[\cos\frac{\Theta+\beta+\Theta'}{2}\Big/\cos\frac{\Theta-\beta+\Theta'}{2}\right]\ ,\\ &\psi=\Theta-\beta-\Theta'\ . \end{aligned}\tag{2}$$

We denote by $\mathbf{v}_1$ the younger ball's velocity before the collision, by $\mathbf{v}_1'$ its velocity as it leaves the collision, by $\mathbf{v}_2$ the older ball's velocity before the collision, and by $\mathbf{v}_2'$ its velocity as it leaves the collision, $|\mathbf{v}_1'|=|\mathbf{v}_2|$. The definitions of all other variables are clear from Fig. 1 (see also Figs. 2 and 3). The set (1) of equations follows from energy and momentum conservation, the geometry of the balls relative to each other and relative to their trajectories at the moment of the collision, and the chronology of the motion, namely, from the demand that the older ball return to the event of the collision at the same moment as the younger one left it.

The set (2) describes various geometrical parameters and the wormhole traversal rules. We shall consider v_2, b, and β as unknowns. They are functions of the parameters v_1, Θ, and ϕ of the ball's initial trajectory (and other parameters of the model which are fixed).

Following Echeverria and Klinkhammer [7], we introduce new variables λ_b, λ_1, and λ_2 (instead of b, v_1, and v_2 correspondingly):

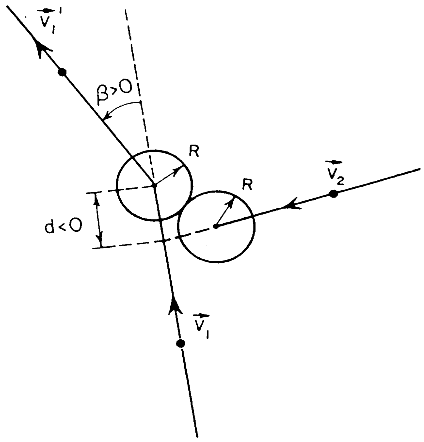

FIG. 2. Geometry of the impact of *class I*.

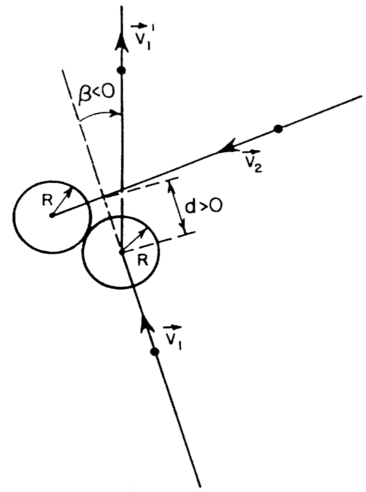

FIG. 3. Geometry of the impact of *class II*.

$$b=\frac{\sin(\phi-\Theta)}{\sin 2\phi}(1-2A\cos\phi)+\frac{R(2\lambda_b-1)}{\cos\phi},$$
$$v_1=\frac{\cos\Theta}{\cos\phi}(1-2A\cos\phi)+\frac{2R(2\lambda_1-1)}{\cos\phi}, \tag{3}$$
$$v_2=\frac{\cos\Theta}{\cos\phi}(1-2A\cos\phi)+\frac{2R(2\lambda_2-1)}{\cos\phi}.$$

Now we consider λ_b, λ_2, and β as unknowns. They are functions of λ_1, ϕ, and Θ.

We shall derive the solutions to Eqs. (1)–(3) for the physically reasonable case $R \ll A \ll 1$. To be definite we assume that $R=O(A^2)$.

There are two classes of self-collisions for the ball (see [1]). The solutions to (1)–(3) up through third order are, for *class I*,

$$\lambda_2=\lambda_1\left[1+\frac{2A\cos\phi}{\tan^2\phi-\tan^2\Theta}\right],$$
$$\lambda_b=\frac{\lambda_1}{\sin^2\phi}\left[1+\frac{2A\cos\phi}{\tan^2\phi-\tan^2\Theta}\right], \tag{4}$$
$$\beta=\frac{8AR\lambda_1\sin\phi}{\cos\Theta(\tan^2\phi-\tan^2\Theta)},$$

and, for *class II*,

$$1-\lambda_2=(1-\lambda_1)\left[1+\frac{2A\cos\phi}{\tan^2\phi-\tan^2\Theta}\right],$$
$$1-\lambda_b=\frac{1-\lambda_1}{\sin^2\phi}\left[1+\frac{2A\cos\phi}{\tan^2\phi-\tan^2\Theta}\right], \tag{5}$$
$$\beta=-\frac{8AR(1-\lambda_1)\sin\phi}{\cos\Theta(\tan^2\phi-\tan^2\Theta)}.$$

We shall need to know the solutions to (1)–(3) up through fourth order. We search for these solutions in the form of corrections to (4) and (5):

$$\lambda_2=\lambda_1\left[1+\frac{2A\cos\phi}{\tan^2\phi-\tan^2\Theta}\right](1+x),$$
$$\lambda_b=\frac{\lambda_1}{\sin^2\phi}\left[1+\frac{2A\cos\phi}{\tan^2\phi-\tan^2\Theta}\right](1+z), \tag{6}$$
$$\beta=\frac{8AR\lambda_1\sin\phi}{\cos\Theta(\tan^2\phi-\tan^2\Theta)}(1+y).$$

The solutions for set (4) are

$$x=\frac{4A^2\cos^2\phi\cos 2\Theta}{\cos^2\Theta(\tan^2\phi-\tan^2\Theta)^2},$$
$$y=\frac{2A\cos\phi}{\tan^2\phi-\tan^2\Theta}(1+\tan^2\phi-2\tan^2\Theta), \tag{7a}$$
$$z=\frac{4A^2\cos^2\phi\cos 2\Theta}{\cos^2\Theta(\tan^2\phi-\tan^2\Theta)^2}+R\cos\phi\left[\frac{\sin 2\phi}{\sin(\phi+\Theta)}+\frac{16\lambda_1\sin^2\phi(S\cos^2\phi-J)}{\cos^2\Theta(\tan^2\phi-\tan^2\Theta)^2}\right],$$

and, for set (5)

$$x=\frac{4A^2\cos^2\phi\cos 2\Theta}{\cos^2\Theta(\tan^2\phi-\tan^2\Theta)^2},$$
$$y=\frac{2A\cos\phi}{\tan^2\phi-\tan^2\Theta}(1+\tan^2\phi-2\tan^2\Theta), \tag{7b}$$
$$z=\frac{4A^2\cos^2\phi\cos 2\Theta}{\cos^2\Theta(\tan^2\phi-\tan^2\Theta)^2}-R\cos\phi\left[\frac{\sin 2\phi}{\sin(\phi+\Theta)}+\frac{16(1-\lambda_1)\sin^2\phi(S\cos 2\phi-J)}{\cos^2\Theta(\tan^2\phi-\tan^2\Theta)^2}\right],$$

where

$$S=\frac{\sin^2(\phi-\Theta)}{\sin^3 2\phi}\left[\frac{\cos(\phi+\Theta)}{\cos^2\Theta\sin 2\phi}+\frac{\sin\phi}{\cos^3\Theta}+\frac{3\cos 2\phi\sin(\phi+\Theta)}{\cos^2\Theta\sin^2 2\phi}-\frac{\sin(\phi-\Theta)}{\cos^2\Theta}-\frac{\sin(\phi-\Theta)\cos 2\phi\sin\Theta}{\sin 2\phi\cos^3\Theta}-\frac{\sin(\phi-\Theta)}{\cos^2\Theta\sin^2 2\phi}\right],$$

$$J=\frac{\sin(\phi+\Theta)\sin^2(\phi-\Theta)}{\sin^5 2\phi\cos^3\Theta}[\sin(2\phi+\Theta)\sin 2\phi+\cos\Theta+3\cos\Theta\cos^2 2\phi]\ . \tag{8}$$

IV. INELASTIC BILLIARD BALL

Using the standard approach to description of the collision of inelastic billiard balls (see [6]), and the same ideas as in Sec. III for the derivation of a complete set of equations that govern self-consistent solutions, one can obtain the system

$$\bar{v}=(1+f^2)^{1/2}av_{20}\cos\Theta_{20}\ ,$$
$$\tan\Theta_1=f,\quad f<2\tan\Theta_{20}/(9a)\ , \tag{9}$$

$$\bar{v}^{\,2}=v_1^2+v_2^2-2v_1v_2\cos\beta\ ,$$
$$v_{20}^2=v_1^2+v_2^2-2v_1v_2\cos\alpha\ , \tag{10}$$
$$\bar{v}v_{20}\cos(\Theta_{20}-\Theta_1)=v_1^2+v_2^2\cos(\alpha-\beta)-v_1v_2(\cos\alpha+\cos\beta)\ ,$$

$$\frac{|d|}{\sin s}=\frac{2R}{\sin\alpha}=\frac{|\delta|}{\sin(\gamma+|\beta|)}\ ,$$
$$s=\Theta_{20}-i\ , \tag{11}$$
$$\sin i=\frac{v_1\sin\alpha}{v_{20}}\ ,$$
$$\delta=v_2-(c+b')\ .$$

The definitions of the additional variables are clear from Fig. 4.

Sets (9)–(11) are the analogy of set (1). We have to solve these together with (2) and (3). We assume $f\ll 1$ and, to be definite, $f=O(A)$. We do not make any special assumptions about the parameter a, but it should be not very small, $a\gg A$. For *class I* collisions we search for the solutions to sets (9)–(11), (2), and (3) in the form

$$\lambda_2=\lambda_1\left[1+\frac{2A\cos\phi}{\tan^2\phi-\tan^2\Theta}\right](1+x)(1+X)\ ,$$
$$\lambda_b=\frac{\lambda_1}{\sin^2\phi}\left[1+\frac{2A\cos\phi}{\tan^2\phi-\tan^2\Theta}\right](1+z)(1+Z)\ , \tag{12}$$
$$\beta=\frac{8RA\lambda_1\sin\phi}{\cos\Theta(\tan^2\phi-\tan^2\Theta)}(1+y)(1+Y)\ ,$$

where x, y, and z are known from (7a), and X, Y, and Z are unknowns.

The solutions are $Y=0$ up through order $O(A)$, and

$$X=Z=-\frac{2Af}{\sin\phi(\tan^2\phi-\tan^2\Theta)}\ . \tag{13}$$

An analogous procedure for *class II* collisions gives the solutions

$$X=Z=\frac{2Af}{\sin\phi(\tan^2\phi-\tan^2\Theta)}\ . \tag{14}$$

We would like to emphasize that in this approximation the self-consistent solution does not depend on the parameter a.

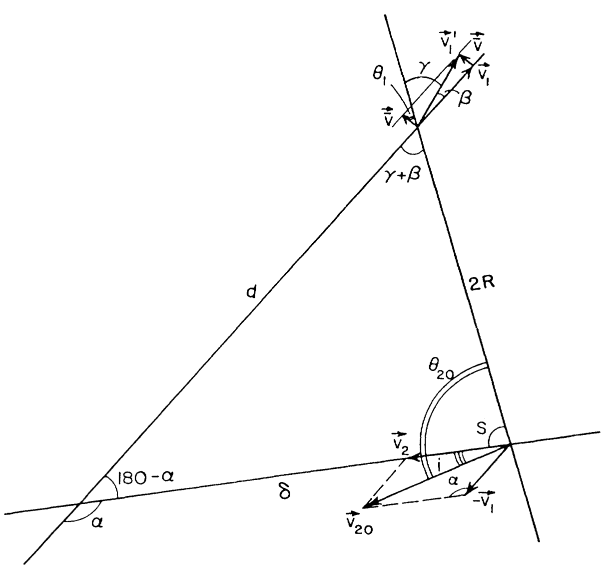

FIG. 4. Some details of the geometry of the self-consistent solution for inelastic impact.

V. CONCLUSIONS

The fact that the solution does not depend (to the order studied) on the parameter a, characterizing the recovery of a ball after the collision, has a rather simple explanation. Indeed, inelasticity dominates friction in the case of a head-on collision and vice versa in the case of a glancing collision. In our consideration, the ball is undergoing a glancing collision rather than a head-on collision. Under this condition friction is essential but inelasticity gives the effect of the next order because the deformation of the balls is negligible.

We have demonstrated that self-consistent solutions of *class I* and *class II* for the inelastic billiard ball problem exist as well as for the elastic one.

In a proper time the ball was subjected to two collisions: the first one when it was "younger" and the second one after the passage through the time machine. In the *class I* and *class II* solutions the trajectory between these two collisions is slightly displaced (typically of the order of the ball's radius R) with respect to the self-inconsistent solutions (when we naively continue the initial trajectory beyond the first collision).

There is another type of self-consistent solution—when the trajectory of the motion of a ball between two collisions is quite different from the self-inconsistent one. In this type of self-consistent solution the ball can traverse the time machine many times between two collisions.[1] We have not discussed this type of solution in the paper. We suspect, but have not proved, that there are analogous self-consistent solutions for the inelastic billiard ball. Also, we have not discussed here the Jinne-type self-consistent solutions. This type of solution, taking into account inelasticity, was proposed and discussed by Lossev and Novikov [8].

In the model of the wormhole with vanishingly short length of the throat which we consider in this paper, the motion along larger circles at the edges of the mouths is also the geodesical motion. It gives the possibility to add some points to the "wormhole traversal rules" taking into account the trajectories for which some parts before and after the traverse of the time machine are large circles on the edges of the wormhole. This new type of the self-consistent solution leads to important conclusions and is discussed in a separate paper (see [9]).

[1]In the time of the external observer it is one collision between younger and older versions of the ball.

ACKNOWLEDGMENTS

The authors are grateful to Kip S. Thorne for useful remarks. A part of this work was done when one of the authors (I.N.) took part in the 1992 physics-astrophysics program of the Aspen Center for Physics. This paper was supported in part by the Danish Natural Sciences Research Council Grant N 11-9640-1.

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