Math 211: Outcomes, Design Considerations and Content

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1 Preamble

We view Math 211 as a basis from which pre-service teachers can engage in the mathematical work of teaching, including providing age-appropriate explanations, formatively assessing student understanding, accurately assessing the correctness of non-standard solutions, providing differentiated instruction, asking accessible questions which provoke thought and discussion, and modifying pace and emphasis accounting for student understanding and the demands of mathematics they will take later.

Serving as a basis for such work does not mean that we suggest structuring the course around exactly this work. Indeed, in the transition to the Common Core it can be safely assumed that pre-service teachers will have little if any experience in learning mathematics in the deeper ways we will be demanding of students, so it is essential to provide such experience to them as adult learners of mathematics itself. Moreover, within the model of a liberal education, students engage in core disciplines first as foundational before moving on to how that discipline is addressed within their own area. In this context, since teachers must draw on their own adult-level understanding as they navigate with child-level language and arguments, our choice is to focus more on the adult-level understanding at first before further preparation which addresses the interplay of both levels.

We elaborate on the basis we see Math 211 providing below, both in terms of outcomes and content, providing explanations for our choices. We also elaborate with tasks we find representative of the level of accomplishment we are aiming for. Some tasks are more classroom and homework oriented, and some would be good for use on exams; we do not indicate which.

2 Outcomes and further design considerations

1. **Understand the Common Core development of the Operations and Algebraic Thinking (OA) and Number and Operations in Base Ten Progressions (NBT) in Grades K-5.**

   For OA, this means understanding how to establishing properties of operations and what algebraic thinking is. For NBT this means using properties to justify algorithms and strategies, and understanding the role of algorithms and strategies in mathematical development.

The inclusion of this material is self-evident, since development of skill with and sense of numbers is by far the most important part of elementary mathematics. We should emphasize that we are choosing to from the beginning adapt the preferred order of development, arguments and language of the Common Core, since it is mathematically rigorous, pedagogically based, and what future teachers will be working with for the foreseeable future.

Sample tasks:

- Add 34_{five} and 22_{five} using both the standard algorithm and a compensation strategy. For each way to calculate, write a series of equations which captures the steps of the algorithm or strategy.
- Use rectangular arrays to justify the distributive property.
- Divide 932 by 4 by sorting base-ten blocks representing 932 into four piles, starting with hundreds and performing exchanges as needed. Then perform the standard algorithm and explain how these are reflecting the exact same underlying mathematics.

2. **Understand and engage in a range of mathematical reasoning.**

Mathematics educators need to be well-versed in a range of reasoning which can support understanding. In order roughly from least to most rigorous, we see some such reasoning as follows.

- Examples and discussion. It takes some mathematical work to recognize an example as germane or not to a general kind of statement (e.g. is $532 + 309 = 509 + 332$ an example of commutativity?). While providing examples does not serve as an argument (except when providing counterexamples) this kind of work, sometimes called contextualizing and decontextualizing, is an important step in reasoning.
- Finding models, making sketches, and solution planning. These are all skills important to launch full immersion into reasoning.
- Analyzing examples. This is an intermediate step between providing relevant examples and providing a fully generalized example. A key step is understanding whether an example works is fully generalizable.
- Identify relevant definitions. Relevant definitions (including model-based definitions) are an essential part of mathematical argument.
- Arguments with pictures. These can range in level of rigor and completeness depending on how they are structured.
- General argument based on examples. This type of argument is a bit of a lost art (historically, it was used by mathematicians such as Pascal). If one starts with an example and then explains fully how relevant features of that example hold in general, one can provide a fully rigorous argument.
- Arguments with variables and more formalism. While these are the lingua franca for modern mathematics, and in particular essential for undergraduate math majors to master, they should be employed strategically in this context.

We should delineate between the first four and the last three, which can support fully rigorous argument.

Understanding this range helps motivate the deeper adult-level understanding we desire (see motivation as a design consideration below), and of course provides a basis for the multi-leveled mathematics which happens in a classroom. Having students work through this range can also help students progress in their own argument-making, so it would be justifiable as part of the curriculum even if the only outcome desired were the last kind of arguments listed, namely complete formal arguments.

Sample tasks:

- Prove that an odd number added to an odd number gives an even number in each of the three following ways: proof by picture (and explanation), generalized example, and using variables.
- Provide a complete argument that a number is divisible by 9 if and only if the sum of its (base-ten) digits are.

3. **Understand through experience the Mathematical Practices.**

Reasoning is addressed above as particularly important, but the full range of mathematical practices such as problem-solving, modeling, being precise, and seeing structure are essential for students to experience. (We leave it to the interested reader to look up the Mathematical Practices as described in the Common Core document.)

These practices serve both understanding what is ultimately challenging material (contrary to what some may be led to believe by the word "elementary") and serve future teachers in understanding the outcomes desired in mathematics education reform.

Sample tasks:

- Take the number 23. Double it, double that, etc. and make a list of the first five numbers you get. Then add the first (that is 23), second and fifth numbers and compare that with 23×19. Explain what you see, explicitly citing properties of operations as needed.
- Explain how Muppets (who have four fingers on each hand) can determine multiples of seven by "putting down a finger." Justify your answer.

4. **Begin to see the coherence and global structure of mathematics, as embodied in the Standards.**

A full understanding of this coherence and global structure is likely to be out of reach, but appropriate opportunities for reflection on how the constructions of elementary mathematics fit together should be given. Two views of this structure are given by Cody Patterson and by the last author, Dev Sinha. Patterson has developed the "M^3 framework", identifying how topics in the Common Core are often introduced by their Meaning, further developed with relevant Machinery, and then built on through work such as applications and arguments which require Mastery. This development can be seen both within strands at a grade-level and coherently across grades. Sinha identifies three kinds of mathematical constructions: foundational (models, definitions), elaborative (examples, propositions), and refined (algorithms, theorems). One can then view the mathematical work of tasks (for kids) or proofs (for mathematicians) as connecting these through application and argument.

While these types of analyses are fairly advanced, and yet to be written up at an academic level much less a textbook level, it can help students to start to see the considerations which go into developing mathematics which is both rigorous and pedagogically sound. Here it is more important to provide examples, such as addressing the role of the number line and how it is introduced in second grade in order to serve for fraction reasoning in third and fourth.

Sample tasks:

- Ms. Taylor has been teaching second grade for twenty years and says "I don't see why adding on the number line is in second grade now. My students get more confused by the number line than they are helped by it." Explain to Ms. Taylor why while what she says about her students is valid, the writers of the CCSSM chose to include addition on the number line as part of second grade.
- Write a sequence of problems which call for the partitive missing factor model of division, which use the same (or very similar) modeling context but are appropriate to grades 3, 4, 5 and 6.

These outcomes, along with other background about the culture of the course, lead to design considerations which must be attended to. Students come in with preconceptions about the course (e.g. "elementary = easy"), which should be addressed carefully. If students see the greater difficulty than expected as arbitrary, rather than as arising from the demands of the subject, they are unlikely to engage as productively as needed. This is especially true given the lack of confidence, all too often based on a lack of accomplishment, which this cohort generally exhibits. This would speak to choosing activities which are clearly relevant to K-5 (K-8) education, and highlighting connections. But one must balance this with the primary outcome for pre-service teachers to engage in the mathematical practices themselves. It might feel good and safe to discuss K-5 student reasoning, but this cannot be the emphasized too much at the expense of material which is really "juicy" for adult learners.

Indeed, if the a goal is to promote the mathematical practices in this cohort, we should remember that mathematical practices require engagement in rich content, in this case a college-level view of elementary mathematics, with appropriate mindset, in the sense of Dweck. The choices made in developing a 211 syllabus, including content and course structure, should be made to maximize the amount of fuel (content) and spark (mindset) needed to light the desired fire (mathematical practices).

To be concrete about this interplay, consider choices made around grading of student work engaging in reasoning. An advisable choice would be to have grading which awarded significant partial credit for reasoning work along the range of reasoning listed in the second outcome. Such a grading scheme could for example award 2 points for a relevant example, 1 point for explanation of it, 2 points for all needed definitions, etc. out of 10 for a complete solution, which could just be the argument as a mathematician would give it. Such a framework would continually reinforce student understanding of this range of reasoning and would also reward effort through partial credit, as work on mindset would advise us to do.

3 Content

The choice of content for the Math 211 sequence is among the most difficult in the undergraduate curriculum. The topics must be engaged at a college level without obscuring their elementary nature. Material must be challenging, in particular in order to invite engagement with the mathematical practices, but needs to be inviting and accessible. While the cohort of students has few who self-identify as strong in math, the demands on reasoning and problem solving need to be significant in order to be true to the mathematical demands of teaching. The extent to which content can meet these constraints and support our outcomes above will determine to great degree how successful we are in achieving those outcomes.

As we weigh the relative merit of content, both content which has been part of these courses in the past and some new possibilities, we address questions such as the following: How does this topic fit with the outcomes? Why should we cover this topic? Or, why shouldn't we cover this topic? What kind of activities would we want to emphasize, for understanding and engagement? How much time should be spent covering this topic?

- **Justifying algorithms and strategies, including identifying when they work**

  The process of justifying arithmetic algorithms lies at the heart of shifting emphasis from procedural to conceptual. It forms the basis of being able to answer student questions of "why?" as well as other work such as understanding errors students make in learning those algorithms. This topic should be a consistent theme throughout Math 211, both for its importance as a topic within the Common Core and because of how readily one can invite 211 students to engage in the mathematical practices through tasks based in this work. Possible tasks include: being able to explain clearly the reasoning behind each step in an algorithm; identifying common error patterns; determining whether modified or alternate algorithms are correct or are essential different or not; responding to potential questions (as in the *Questions from the Classroom* problems); trying to apply an algorithm in an unfamiliar context especially with other number bases).

  We recommend that this work, along with establishing properties (see below) constitute the backbone of Math 211. A number of weeks (say two to four) should be spent on this material, probably best not done in "one shot".

- **Establishing properties**

  As with justifying algorithms, establishing properties is essential to conceptual emphasis. Both teachers and many college instructors tend to tell students what properties hold and demand that they be able to recall those properties at relevant times. In the Common

Core, students are to be given opportunities and tools to reason about properties, and it prescribes those tools. For example, in second grade students are to "Work with groups of equal objects to gain foundations for multiplication" and in particular rectangular arrays (2.OA.5). In third grade, students should understand the definition of multiplication as total numbers of quantities which come in equal groups (3.OA.1), which is equivalent to repeated addition. They should have plenty of opportunity to see the same products coming in different order, for example possibly filling out multiplication tables or playing games to achieve fluency (3.OA.7). Then when appropriate, teachers can call attention to the rectangular arrays and either count by-column and then by-row, or "rotate 90 degrees" to see the commutative property. (Note that we don't recommend the formalism of Cartesian products - see "Set theory".) This all serves to help students "Understand properties of multiplication" (cluster heading above 3.OA.5).

Since Common Core aligned instruction needs to engage such reasoning, and our students have not seen such reasoning, establishing properties is of fundamental importance in this class. We recommend at least three to five class periods of exclusive focus on this topic, as well as regular revisiting of it as properties are used throughout the course.

- **Arithmetic in other bases**

Arithmetic in other bases are is a staple of courses for pre-service elementary teachers (and was made famous as part of the "New Math" in a song of Tom Leher). The rationale for treating arithmetic in other bases are fairly clear, especially given the desire to deeply understand elementary mathematics in a college context. Working in other bases strengthens the understanding of the base ten system, in particular by working through the familiar algorithms. Explanations of how and why the standard algorithms work can be more informative when seen in a new context. When combined with other topics, such as divisibility, other bases afford some good opportunities for conjecture followed by reasoning.

Possible tasks include: translating numbers between base ten and other bases (possibly justifying the division-wth-remainder algorithm after treatment of division!); for a given number write the preceding and succeeding numbers in different bases; write the first ten to twenty counting numbers in different bases; illustrate numbers in other bases with blocks; determine which of a list of numbers is least/greatest; find missing digits (for example, find x so that $x2_{\text{seven}} = 44_{\text{ten}}$); perform arithmetic in other bases, with particular emphasis on explaining what's going on with the standard algorithms; draw block models for arithmetic in other bases.

Even though non-ten number bases aren't part of the Common Core, we recommend working in other bases throughout a number of topics (number systems, algorithms and strategies, and divisibility). Evidence of understanding in other bases translates to strong evidence of deeper understanding of our base-ten system.

- **Modeling and its role throughout the material**

  More engaging, realistic and involved modeling is one of the main shifts in the Common Core. While "full" modeling is a strong point of emphasis in high school, increased realistic context and engaging in some parts of the modeling process will be present throughout K-8. At the elementary level, modeling can illuminate the understanding of concepts in OA and NBT as well as provide opportunity to observe the coherence and global structure of mathematics. Moreover, students should see - perhaps through examples they are working themselves - that sometimes a modeling context can provide an opportunity for mastery level application of material and sometimes a modeling context can help a beginner's understanding of material.

  We recommend that modeling be included throughout the course as many opportunities arise. We do not see a need to address it exclusively as a separate concept to be the main focus of any full class periods.

- **Algorithms vs. Strategies: what and why?**

  Recognizing the roles of algorithms and strategies is a central part of understanding the NBT and OA strands; see Outcome 1 above. In particular, students must see the rationale for teaching strategies other than the standard algorithm. Recognizing the usefulness of different methods will allow the students to creatively engage in Mathematical Practices as well as serve as a basis for discussions about coherence (e.g. a partial sums multiplication strategy is based on the distributive property).

  General discussion of this topic should take up about a class period near the beginning of the term, followed up by ongoing exposure to strategies and algorithms as well as references to the advantages and disadvantages of each type of method throughout the term.

- **Looking at Standards and Progressions documents**

  Since the CCSSM documents provide the structure for the courses that these students want to teach, it seems appropriate to encourage familiarity with these documents. Further, as suggested in Outcome 4, the standards documents can help make the coherence between and within strands more apparent while encouraging thoughtful and rigorous approaches to the individual concepts. This is particularly true for the Progressions documents. While studying the Standards Documents themselves may be a key to independent ongoing professional development (see Ma, *Knowing and Teaching Elementary Mathematics, Anniversary edition* page 131) reading them alone will not be adequate.

  We recommend that the Standards and corresponding Progressions documents (published by the Institute for Mathematics and Education hat the University of Arizona) be regularly referenced in the class, and possibly be used for required reading for example when discussing the OA and NBT strands. But we do not recommend exclusive focus on these at any point in the class.

- **Algebraic thinking**

  The choice of the term "algebraic" in the Standards in early grades is entirely intentional. It signals a focus on algebraic structure which comes well before formal algebraic manipulations with variables. It provides strong rationale for development of both strategies and algorithms, and their justification. In the context of a college course, one can expect full use of algebra. But in order to communicate the nature of algebraic thinking in early grades, it is helpful to provide students with opportunities to engage in it without algebra. The "double 23" task described at the end of Outcome 3 provides one example. Using tape diagrams to solve problems which would typically (in this country) be approached through algebra is another possibility. Finally, establishing divisibility rules requires plenty of algebraic thinking (and algebra, in proofs which use variables).

  While we don't see algebraic thinking as a topic of exclusive focus, we recommend that throughout the course the aim of engaging algebraic thinking be explained (e.g. when strategies and properties are discussed). And we also recommend that throughout the course opportunities be given to engage in algebraic thinking, both with variables and in settings where such thinking can happen "with just numbers."

- **Divisibility rules**

  While not part of the Common Core, the establishing of divisibility rules (such as the fact that a number is divisible by three if and only if the sum of its digits is divisible by three) provides some great opportunities in this setting. Justifying divisibility rules engages college-level reasoning, and reinforces the meaning of the base-ten number system. It requires careful use of definitions, and allows students to reason in a context where they can readily generate examples. Success in establishing these rules is thus a clear indication of student mastery of some of the most important threads in the class. Finding such rules can also be a good opportunity to generate and establish conjectures, especially if such rules are explored in other bases.

  We suggest that divisibility rules be used as a culmination topic, towards the end of the class, spending one and a half or two weeks on them. We also recommend that proof through general argument around an example is more accessible than proof using variables. If one prefers the latter, then students should engage in work where variables refer to digits of multi digit numbers throughout the term to be prepared for argument with them.

- **Historical (and other) number systems**

  Historical number systems provide a good opportunity to address understanding and motivation. The students in 211 will be thinking about the base ten number system probably more deeply than they ever have (and maybe even for the first time). The comparison of the base ten system using Hindu-Arabic numerals with other historical systems provides an accessible point of entry to this discussion.

Possible tasks include: translating between the various number systems (probably with basic facts about systems given on tests so as to not require knowing them from memory); writing the preceding and succeeding numbers to a given number (to illustrate the different grouping processes of the different systems); finding the smallest or largest number that can be written with a given quantity of numerals (again, the student would be thinking about grouping).

While this topic is not directly addressed in the CCSS, and it is not logically necessary for the remainder of the course material, this topic can well serve some pedagogical functions, in particular framing and motivating the exploration of the base ten system. The same can be said for systems such as poker chip number systems, which also have the advantage of being hands-on. We suggest these be included for a week or less.

- **Formal logic, logic puzzles**

  Certainly we expect prospective teachers to be able to use logical arguments, but the outcomes and design considerations suggest a limited use (if any) of these topics. The Common Core prescribes that reasoning be situated within "the canon", and that prescription should be heeded for this class as well. We have found formal logic to be of limited pedagogical value, especially when compared with mathematical reasoning within intrinsically interesting and valuable content. Take for example the development of truth tables and contrast that with justifying the distributive law using rectangles. While logic puzzles certainly allow for the engagement in mathematical practices they do little to promote the understanding of the CCSS development of OA and NBT and the overall coherence of mathematical topics they will teach. We prefer to highlight the richness of the topics that these students will teach, rather than implicitly reinforcing the notion that the foundational math they will teach is too easy, and we must resort to other topics for them, as adults, to be fully engaged.

  We recommend not including formal logic or puzzles in the course. Instead reasoning through a range of modes, most of which are less formal, is to be integral to the fabric of the course.

- **Problem solving**

  Similar to formal logic, problem solving is something we would rather see woven into the fabric of the course than treated as a separate topic. In particular, no topics should be introduced simply because they are fertile ground for good problems (e.g. combinatorics). There can be plenty of problem solving located within the most important material for the course.

Problem solving frameworks (such as Polya's) could be useful, in particular giving students who are not confident problem solvers some help in getting started and staying productive, which is essential in maintaining a growth mindsets. But as with logic, this kind of structure provided by the instructor need not be of a formal nature.

- **Sets**

  The terminology of sets is sometimes used in textbooks for this course. It can be useful for students to have familiarity with the language of sets, even if it will only be used in reading the textbook, though some documents such as the NCTM's *Focal Points* seem to use this language. Study of sets can also give practice wielding technical language and writing precise, mathematical descriptions.

  But the language of sets is not used in the Common Core, and where the language of sets is used in textbooks (for example in models for the binary operations, in distinguishing the different types of numbers, and in probability) it can easily be replaced by intuitive notions of sets and set properties. When we reflect on student needs and difficulties in this class, such as struggle in articulating a coherent picture of the number types (integer, rational, irrational), it doesn't seem that the source of this missing knowledge is an inadequate understanding of the technical language of sets, but rather a lack of experience with the mathematics attendant to arithmetic (base ten representation, estimation, etc.).

  Thus we recommend that formal language around sets not be treated, and class time be used to address other topics in more depth.