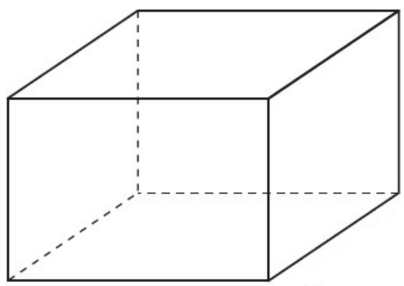
Math 251
Final Exam Answers - Spring 2013

(1) (10 pts) Wire with a total length of 300 inches will be used to construct the edges of a rectangular box, and thus provide the frame for the box. The bottom of the box must have a width that is twice the length. Find the maximum volume that such a box can have.



Solution: Let w be the width of the bottom. Then $2w$ is the length. Let h be the height. The total length of the edges is

$$4h + 4w + 4(2w) = 4(h + 3w) = 300.$$

So $h + 3w = 75$, thus $h = 75 - 3w$.

The volume is

$$V(w) = w \cdot (2w) \cdot h = 2w^2(75 - 3w) = 150w^2 - 6w^3.$$

The quantity w must be between 0 and 25. So the maximum of the volume will be either at the endpoints (in this case the volume at the endpoints is zero, so that is the minimum) or at a critical point.

$$V'(w) = 300w - 18w^2$$

This is 0 when $w = 0$ and when $300 = 18w$ which is when $w = 50/3$.

$$V(50/3) = 2(2500/9)(25) = 125,000/9.$$

(2) (5 pts) Evaluate each limit, justifying your answers.

(a) $\lim_{x \to 0^+} e^{1/x}$

Solution: Since $1/x$ is approaching ∞, the limit is the same as $\lim_{u \to \infty} e^u = \infty$.

(b) $\lim_{x \to 0^-} e^{1/x}$

Solution: Since $1/x$ is approaching $-\infty$, the limit is the same as

$$\lim_{u \to -\infty} e^u = \lim_{v \to \infty} \frac{1}{e^v} = 0.$$

(c) $\lim_{x \to 0} e^{1/x}$

Solution: This limit does not exist since the left hand and right hand limit are not the same.

(3) (5 pts) Find the absolute maximum and absolute minimum values of the function

$$f(x) = x^3 - 12x + |x|$$

on the interval $[0, 4]$.

Solution: Since this function is continuous on a closed interval, the extrema occur at an end point (0 or 4), a point where the derivative doesn't exist (0) or at a critical point. On $(0, 4]$ the derivative is $3x^2 - 11$, so the only critical point in our interval is $x = \sqrt{11/3}$.

$f(0) = 0$, $f(4) = 20$. The derivative is negative for $x < \sqrt{11/3}$, so f is decreasing on $(0, \sqrt{11/3})$. The derivative is positive for $x > \sqrt{11/3}$ so the function is increasing on $(\sqrt{11/3}, 4)$.

Thus the maximum value is $f(4) = 20$. The minimum value is

$$f(\sqrt{11/3}) = \frac{11}{3}\sqrt{11/3} - 11\sqrt{11/3} = -\frac{22}{3}\sqrt{11/3}.$$

(4) (4 pts) Let $a(t)$ be a function satisfying $a'(t) = 3a(t) - 3$. Let $f(t) = \ln(a(t))$. Give $f'(t)$. You may need to write this in terms of $a(t)$.

Solution:

$$\frac{d}{dt}f(t) = \frac{d}{dt}\ln(a(t)) = \frac{1}{a(t)}a'(t) = \frac{3a(t)-3}{a(t)} = 3 - \frac{3}{a(t)}.$$

(5) (8 pts) The cost to produce x units of a certain product is given by $C(x) = 10,000 + 8x + \frac{1}{16}x^2$. Find the value of x that gives the minimum average cost. Average cost is given by $\frac{C(x)}{x}$. Be sure to justify that the value of x does give a minimum.

Solution: We write $A(x)$ for the average cost.

$$A(x) = \frac{10000}{x} + 8 + \frac{1}{16}x$$

We want to minimize $A(x)$ on $(0, \infty)$. We notice that the limit of $A(x)$ as x approaches either ∞ or 0 from above is ∞. We analyze the derivative to locate the minimum.

$$A'(x) = -\frac{10000}{x^2} + \frac{1}{16}.$$

This is zero when $x^2 = 160,000$. Since $x > 0$, this corresponds to $x = 400$. The derivative is negative on $(0, 400)$ and positive on $(400, \infty)$ so $A(x)$ has its minimum at $x = 400$.

(6) (8 pts) Compute the following limits exactly (no rounding). You must show your work. In some cases the correct answer may be $\infty, -\infty$ or the limit may not exist.

(a) $\displaystyle\lim_{\theta\to 0}\frac{\theta}{\sin(\pi\theta)}$

Solution: We can rewrite this in terms of the familiar limit $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$:

$$\lim_{\theta\to 0}\frac{\theta}{\sin(\pi\theta)} = \lim_{\theta\to 0}\frac{\pi\theta}{\sin(\pi\theta)}\frac{1}{\pi} = \lim_{u\to 0}\frac{u}{\sin(u)}\frac{1}{\pi} = \lim_{u\to 0}\frac{1}{\frac{\sin(u)}{u}}\frac{1}{\pi} = \frac{1}{1}\frac{1}{\pi} = \frac{1}{\pi}.$$

(b) $\displaystyle\lim_{x\to\infty} xe^{-3x}$

Solution:

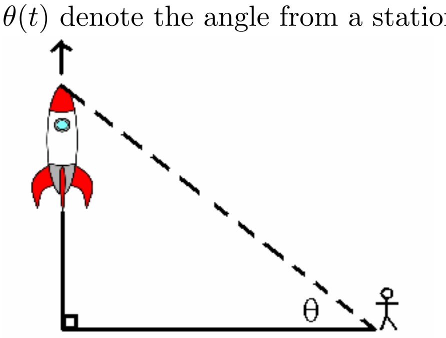
$$\lim_{x\to\infty} xe^{-3x} = \lim_{x\to\infty}\frac{x}{e^{3x}} = \lim_{x\to\infty}\frac{1}{3e^{3x}} = 0$$

where the next to last equality is by L'Hospital's rule.g

(c) $\displaystyle\lim_{t\to 0}\frac{\cos(t)}{t^2}$

Solution: The numerator approaches 1 and the denominator approaches 0 from above. So the limit is ∞.

(7) (10 pts) A rocket is launched at time $t = 0$ and after launch it is moving upward at a constant speed. Let $\theta(t)$ denote the angle from a stationary observer up to the base of the rocket after t minutes.



Do the following:

(a) Find $\theta(0)$ and $\displaystyle\lim_{t\to\infty}\theta(t)$.

Solution: $\theta(0) = 0$. $\lim_{t\to\infty}\theta(t) = \pi/2$.

(b) Determine whether $\theta(t)$ is increasing or decreasing and whether $\theta(t)$ is concave up or concave down.

Solution: θ is increasing since the rocket is moving up and the angle increases as the rocket moves up. The graph of $\theta(t)$ is concave down because the rate of increase decreases as t increases.

One can analyze the derivative and second derivative as follows: If v is the velocity at which the rocket is moving up, and d is the distance from the observer to the launch point of the rocket, then

$$\tan(\theta(t)) = \frac{vt}{d}$$

Then taking derivatives

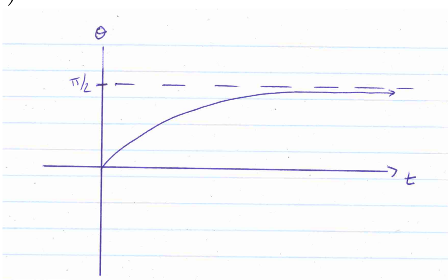
$$\sec^2(\theta(t))\theta'(t) = \frac{v}{d}.$$

This allows one to see that $\theta'(t)$ is positive on $(0, \infty)$. Taking derivatives again,

$$2\sec(\theta(t))\sec(\theta(t))\tan(\theta(t))[\theta'(t)]^2 + \sec^2(\theta(t))\theta''(t) = 0.$$

Since every term in the above equation is known to be positive on $(0, \infty)$ (where θ is in $(0, \pi/2)$) except $\theta''(t)$, the only way for the equation to be satisfied is if $\theta''(t) < 0$. This tells us that the concavity is down.

(c) Use the above information to sketch a graph of $\theta(t)$ for $t \geq 0$.



(8) (10 pts) For a certain pain medication, the size of the dose D depends on the weight of the patient W. We can write $D = f(W)$, where D is measured in milligrams and W is measured in pounds.

(a) Interpret $f(150) = 125$ and $f'(150) = 3$ in terms of this pain medication.
Solution: $f(150) = 125$ means that for a 150 pound patient, the dose is 125 milligrams. $f'(150) = 3$ means that for each increase in weight of one pound above 150, the dose should increase by about 3 milligrams (and for each decrease of one pound below 150, the dose should decrease by 3 milligrams).

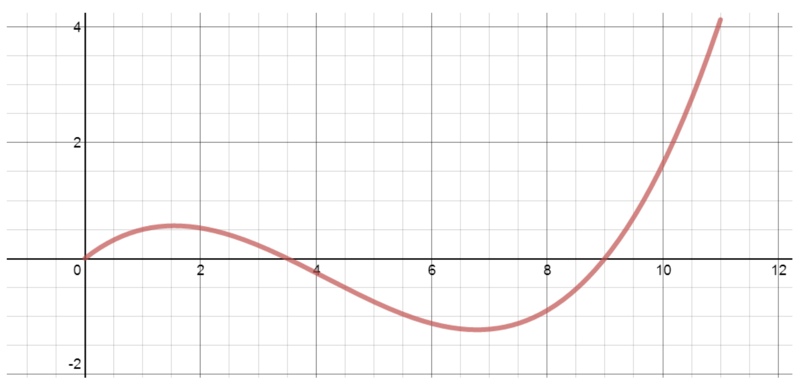
(b) Use the tangent line to $y = f(x)$ at $x = 150$ to estimate $f(155)$.
Solution: The tangent line is

$$y = 3(x - 150) + 125.$$

So for $x = 155$, we get $y = 3 * 5 + 125 = 140$ milligrams.

(c) Given that $f''(W) < 0$, is your estimation in (b) too big or too small? Justify your answer. (Hint: draw a picture.)
Solution: The estimate is too small since the graph is convex down, and thus below the linear approximation.

(9) (8 pts) Below is the graph of the derivative $g'(x)$ on the interval $[0, 11]$

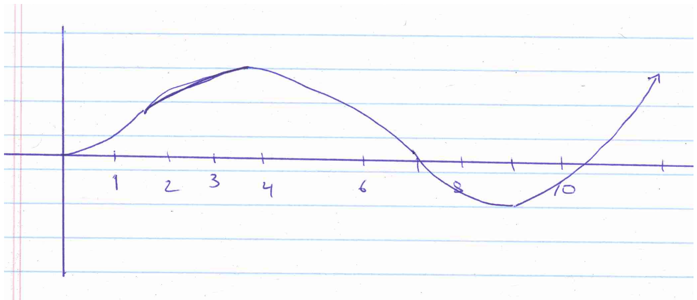


Be careful, the questions below are about $g(x)$, but the graph is for $g'(x)$.

(a) For what intervals is $g(x)$ increasing?
Solution: g is increasing on $(0, 3.5)$ since g' is positive there. g is decreasing on $(3.5, 9)$ since g' is negative there. g is increasing on $(9, \infty)$ since g' is positive there.

(b) For what x is the graph of $g(x)$ concave down? Give your answer in interval notation.
Solution: g'' is positive on $(0, 1.5)$ and $(7, \infty)$ since g' is increasing on these intervals. So g is concave up on these intervals. g'' is negative on $(1.5, 7)$ since g' is decreasing there. So g is concave down on that interval.

(c) Give a reasonable sketch for $g(x)$ satisfying $g(0) = 0$. Make sure to label the x-axis.



(10) (12 pts) Find the indicated derivatives. You do not need to simplify.

(a) For $g(t) = 1 + \tan\left(\frac{t-3}{2}\right)$, find $g'(t)$.

Solution:

$$g'(t) = \sec^2(\frac{t-3}{2})\frac{1}{2}$$

(b) For $f(r) = (r^3 - 3r)e^{2r} + e^2$, find $f'(r)$.
Solution:

$$f'(r) = (3r^2 - 3)e^{2r} + (r^3 - 3r)2e^{2r}.$$

(c) Find $\frac{dy}{dx}$ for $y = \sqrt{\frac{ax+1}{bx+1}}$ and constants a, b.
Solution:

$$\frac{dy}{dx} = (\frac{1}{2})(\frac{ax+1}{bx+1})^{-1/2}\frac{a(bx+1) - b(ax+1)}{(bx+1)^2}.$$

(d) Find $\frac{dy}{du}$ for $y = \arcsin(u^2)$.
Solution:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^4}}2u$$

(11) (10 pts) A water tank has the shape of an inverted cone (point down) with base radius 250 cm and height of 900 cm. Initially, the tank is full of water. If the water level is falling at the rate of 7 cm/min, how fast is the tank losing water when the water is 520 cm deep? Include units in your answer.

Solution: By similar triangles, the radius of the cone of water (inside the conical tank is related to the height by

$$\frac{250}{900} = \frac{r}{h} \text{ so } r = \frac{250}{900}h.$$

The volume of a cone is $\frac{1}{3}\pi r^2 h$. So when the water is at height h, the volume of water is

$$V(h) = \frac{1}{3}\pi(\frac{250}{900}h)^2 h = \frac{\pi * 250^2}{3 * 900^2}h^3 \text{ (cubic centimeters).}$$

Taking derivatives with respect to t, we get

$$\frac{d}{dt}V = \frac{\pi * 250^2}{3 * 900^2}3h^2\frac{dh}{dt} \text{ (cubic centimeters)/(minute)}$$

We solve by putting in $h = 520$, and $\frac{dh}{dt} = -7$. This gives

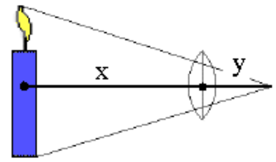
$$\frac{\pi * 250^2}{900^2}3 * 520^2 * (-7) \text{ (cubic centimeters)/(minute)}$$

Factoring out some common terms, we get

$$-\frac{\pi 175}{27}260^2 \text{ (cubic centimeters)/(minute)}$$

(12) (10 pts) A candle is placed x cm from a convex lens. If the distance of the focused image is y cm from the lens (see diagram), then x and y are related by the equation

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{L}$$



where the constant L denotes the focal length of the lens.

(a) Find $\frac{dy}{dx}$.
Solution: Solving for y,

$$\frac{y}{x}+1=\frac{y}{L} \text{ so } 1=\frac{y}{L}-\frac{y}{x}=y\left(\frac{1}{L}-\frac{1}{x}\right)$$

so

$$y=\frac{1}{(\frac{1}{L}-\frac{1}{x})}=\frac{Lx}{x-L}$$

Thus

$$\frac{dy}{dx}=\frac{L(x-L)-Lx}{(x-L)^2}=\frac{-L^2}{(x-L)^2}$$

(b) Using your answer from (a), justify that $\frac{dy}{dx}<0$.
Solution: Since L is not zero, the numerator is -1 times a perfect square, so is negative and the denominator is a perfect square, so is positive (as long as $x<L$ so that the function is defined).

(c) If the candle is moved farther away from the lens, will the focused image move towards or away from the lens? Justify your answer.
Solution: If the candle is moved away from the lens, then x increases. This means that y decreases since $\frac{dy}{dx}<0$, so the focussed image moves towards the lens. Note though that y does not approach zero: as x goes to infinity, y approaches L.