# MATH 307: Introduction to Proof
# Master Syllabus
# Fall 2014

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## 1 Preamble

This course is meant to adequately ease the transition from our lower division (200-level) courses, that are of computational nature to upper division (300/400-level) which are heavily proof-oriented.
Like any specialized area of human activity, Mathematics has its own language and commonly used terms that make it less accessible to the outsider. For example, two Mathematicians in discussion may be heard uttering phrases similar to the following.
"*I have a counter-example to your recent conjecture published in the Journal of Esoteric and Obscure Pure Mathematics.*"
"*Ah, but your proof relies on Zorn's lemma. Can you find a proof that is independent of the Axiom of Choice?*"
"*If you factor out the nilpotent elements, your group is torsion-free, unless it has finite order*"
Whereas these statements make perfect sense to a professional Mathematician, to an outsider (especially students entering MATH 307), they are almost certainly completely opaque.

## 2 Course Objectives

One main objective of this course is to introduce students to the symbols and terminologies needed to understand the language of Mathematics. A second (at most likely more important than the previous) objective of the course is to initiate students to Mathematical proofs, with constant emphasis on the structure of the most common proof formats
At the end of the course, successful students should be comfortable with the following tasks:

1.1. **Students should know how to interpret and employ "if and only if" and "if-then" statements.** They should be able to understand that an "if and only if" statement is equivalent to two "if-then" statements. They should then be able to employ "if-then" statements both as statements and in the course of proofs. Students should be able to distinguish between an "if-then" statement, its converse, its negation and its contrapositive.

1.2. **Students should know what counterexamples are.** They should be able to generate examples illustrating a general statement and say whether an example falsifies the statement. They should know that counterexamples are often what is required to disprove a statement. They should be able to be systematic about finding counterexamples by employing the contrapositive. They should also be clear about the fact that while examples can provide good insights in the course of finding a proof, no amount of examples would actually constitute a proof itself. A good illustrative example is the Euler polynomial $n^2 + n + 41$, which produces prime numbers for the first forty nonnegative integers $n$, but not prime in general.

1.3. **Students should know what definitions, theorems, and proofs are.** They should know that one reads these in different "modes." For example, the logical statements in proofs should be read critically, while definitions are to be accepted by fiat. Students should understand how each of these kinds of statements pertains to examples.

1.4. **Students should be fluent with simple induction proofs.** They should be able to identify base cases and induction steps, to understand that induction proofs involve labeled statements (not equations), to be familiar with strong induction and with cases in which base cases are more complicated, and to employ induction arguments without falling into circular reasoning. Good practice examples of proofs by induction may be drawn from MATH 346 (the fundamental theorem of arithmetic is one such result). An important result that one can prove as well is the binomial formula.

1.5. **Students should be able to interpret quantifiers.** They should be able to identify the different roles that variables play. They should be able to negate statements with quantifiers. They should be beginning to employ quantifiers appropriately in their own arguments. While the most popular quantifiers are $\forall$ and $\exists$, students should also be familiar with $\exists!$ and some context where it shows up (bijection, inverses,...). Students should be comfortable proving "there exists a unique" type statements. In planning lectures on this aspect, instructors may draw natural examples from definitions of concepts from MATH 315. Some of these concepts include upper/lower bounds, supremum/infimum, convergent sequence, Cauchy sequence, bounded functions, continuous functions,...etc.

# 3 Course content and structure

While there are a number of possible ways in which the course can be run, and pretty much any content could be justified, experience suggests some constraints on these choices.

2.1. **Practice and feedback are essential.** Precise mathematical language is generally foreign, in some cases completely foreign, to these students. They will try to mimic you, the text, and other students, but invariably they will make many mistakes. It will take some time before they naturally connect abstract statements to relevant examples, so their own ability to catch mistakes is severely limited. The more frequently these mistakes can be corrected, the more the students can progress.

2.2. **The mathematics needs to be focussed somehow.** The current default text, Mathematical Thinking: Problem-Solving and Proofs," was chosen because it has a large number of generally good exercises. In particular, it has exercise about logic which are fairly compelling. It also has a wide range of topics so it can accommodate different choices of emphasis. On the other hand, students have reported to me that this textbook is very hard to read on their own. My personal opinion about the textbook is that there are a number of good textbooks that cover well the topics we want to teach. In addition to the above, some of the textbooks I have used (or prepared notes from) include:
1. Introduction to Mathematical Structures and Proofs, by Larry Gerstein. This is probably the most complete in terms of covering abstract notions that are needed in most upper division courses. But, it does not

seem to devote much energy in explaining the structures of Mathematical proofs.

2. Understanding Mathematical Proof, by J. Taylor and R. Garnier.
As the title suggests, the book focusses on different structures of various Mathematical proofs. It presents formal proofs, and offers detailed comments of such proofs. But, it does not cover some of the abstract notions that I feel are important for subsequent courses.

3. Mathematical Reasoning: Writing and Proof, by Ted Sundstrom.
This covers essentially the same topics as 1, but also tries to address more directly the writing of Mathematical proofs.

My policy on textbook is to recommend that students get any of these textbooks for their own studies. With that in mind, I put together the homework assignments (usually from these texts) and pass them out to students.

2.3. **Some ideas of possible topics.** While in principle anything could be used, demands of subsequent study and experience in running the class has led to favor the following topics. In covering these topics, one should deliberate select results whose proofs would cover the main techniques of proofs (direct proofs, proof by contrapositive, proofs by contradictions). One should also draw examples and problems in the contexts that would be favorable to facilitate transition into analysis, algebra, number theory, linear algebra, geometry, topology,...

-Elementary logic, basic propositional and predicates calculi. Examples of logical equivalences should be chosen to cover the logic of the different techniques of proofs. For example, verifying that a conditional statement is equivalent to its contrapositive.

-Sets: Cover set operations (union, intersections, relative complement, power sets, symmetric difference, cartesian product,...), the equality of sets as double-inclusion. Prove the set counterpart of the rules of the propositional calculus (distributivity, DeMorgan's laws, absorption laws,...)

-Functions. Cover definitions and properties (injective, surjective, bijective functions).The image of a subset of the domain under a function, and the pre-image of a subset of the codomain under a function. Students find these concepts extremely complicated. A good exercise consists in proving that a function $f$ is injective if and only if for every subset $X$ of the domain, $f^{-1}(f(X)) = X$. Another one that I have

found extremely challenging to students is to consider the case when $f$ is bijective, and prove that $f^{-1}(Y) = f^{-1}(Y)$, where the LHS is the pre-image under $f$ while the RHS is the direct image under $f^{-1}$. The challenging part is convincing students that there is something to prove here.
-Sums of arithmetic and geometric sequences.
-Absolute values and inequalities.
-Counting problems (for example, counting the subsets of a finite set).
- The binomial formula and applications.
- Equivalences relations and partitions.

2.4. **Most common statements** An attempt should be made to address proving statements of the form:
- $\forall x, P(x)$
- $\exists x, P(x)$
- $\forall x, \exists y, P(x, y)$
- $\exists x, \forall y, P(x, y)$
- $\forall x, \exists! y, P(x, y)$

Pay special attention to the cases where the propositional functions $P(x)$ and $P(x, y)$ are conditional, since most definitions of Mathematical concepts arise in this form.

2.5. **Other topics.** Some important topics that one might consider covering include:
- Countable and uncountable sets.
- Direct products (arbitrary) of sets.