Math 211: Content and Practice Outcomes

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We view Math 211 as a basis from which pre-service teachers can engage in the mathematical work of teaching, including

- providing age-appropriate explanations,
- formatively assessing student understanding,
- accurately assessing the correctness of non-standard solutions,
- providing differentiated instruction,
- asking accessible questions which provoke thought and discussion,
- modifying pace and emphasis to account for student understanding and the demands of mathematics they will take later.

Serving as a basis for such work does not mean that we suggest structuring the course around exactly this work. Indeed, in the transition to the Common Core it can be safely assumed that pre-service teachers will have little if any experience in learning mathematics in the deeper ways we will be demanding of students, so it is essential to provide such experience to them as adult learners of mathematics itself. Moreover, within the model of a liberal education, students engage in core disciplines first as foundational before moving on to how that discipline is addressed within their own area. Finally teachers must draw on their own adult-level understanding – which is more concise, abstract and connected than child-level understanding – as they navigate with child-level language and arguments. Our choice is to focus more on the adult-level understanding in this course.

We will be using these outcomes for a proficiency-based grading system, but as primary goals they inform all of the content, instruction and other design aspects of the course.

1 Content-focused Outcomes

1. **Numeral Systems**

   (a) Recognize and use place value. For example, be able to identify whether or not a numeral system employs place value.

   (b) Use a variety of numeral systems for elementary mathematics. For example, perform multiplication in a chip system, or skip count in a different base.

   (c) Systematically use exchanges in a variety of numeral systems. One example is to find all representations of a number in a lazy chip system, justifying one's list is complete. Another example is to explain where and how exchanges arise in arithmetic in a different number system.

   (d) Know from memory and use the definition of base-b numeral system. For example, justify the repeated division method to translate between bases and the rules of divisibility in different bases.

2. **Meaning, Method and Mastery.**

   (a) Use the terms meaning, method and mastery to analyze K-12 learning trajectories. For example, understand how arguments which establish properties of arithmetic are connecting meaning to method.

   (b) Use the terms meaning, method and mastery to analyze one's own Math 211 learning. For example, explain how understanding the meaning of what a variable is comes into play when one establishes divisibility rules.

3. **Addition and Subtraction**

   (a) Identify and apply various meanings for addition and subtraction such as put-together/ take-apart and comparison and explain the connection between addition and subtraction using these meanings. For example, tell if a word problem is best represented by a "take away" or "compare" representation of subtraction or show which unknown quantity in a "compare" context suggests addition.

   (b) Use appropriate visual models and manipulatives for each meaning of addition and subtraction. In particular, use the number line in different ways depending on the meaning. For example show "hops backward" on the number line for the "take from" meaning of subtraction but distance between two numbers for the "compare" meaning. Another example is to identify contextual problems which present a challenge because "key-word" translation would lead one astray, and use a visual model such as a "number bond" to address the issues.

   (c) Perform and justify computation algorithms and strategies in a variety of numeral systems using models as well as the definition of base numeral systems and able to distinguish between strategies and algorithms in each case. One example would be to use three distinct strategies/algorithms for subtraction of two-digit numbers in base three. Another example would be to expand what is happening in the standard addition algorithm step-by-step using properties of arithmetic.

4. **Multiplication and Division**

   (a) Identify and apply various meanings of multiplication, in particular through equal groups, repeated addition and skip counting, rectangular arrays, and scaling on the number line. Be able to show these meanings are equivalent.

   (b) Use meanings to justify properties. For example, use an area model labeled by variables to justify the distributive property.

   (c) Recognize structure in the multiplication table. For example, identify multiplication facts which would be more difficult in different bases and explain why they are so.

   (d) Identify and apply various meanings of division, including the partitive, measurement and repeated subtraction models and explain the connection of division to multiplication with these meanings. One example is to produce contextualized problems which give rise to division through each of these meanings, another is to explain why division by zero is not defined for any number.

   (e) Perform and justify multiplication and division computation algorithms and strategies in a variety of numeral systems using models as well as the definition of base numeral systems. For example, explain how long division works using area models.

5. **Divisibility**

   (a) Justify basic divisibility methods. For example, give two arguments for the fact that if b and c are multiples of a then $b + c$ is, one using equal groups and one using the definition of divisibility and algebra.

   (b) Use and justify divisibility rules.

2 Practice-Focused outcomes

As you engage in this content, we expect you to engage in mathematical practices. A good analogy is to be made with music, where there is a distinction between listening to music and producing it. Accounting for these practices will happen more in class rather than in exams or writing assignments. Nonetheless, they are essential in your achieving proficiency in the content above.

1. **Understand and engage in a range of mathematical reasoning.**

   Math teachers need to be well-versed in a range of reasoning which can support understanding. In order roughly from least to most rigorous, we see some such reasoning as follows.

   (a) Background examples and discussion. It takes some mathematical work to recognize an example as germane or not to a general kind of statement (e.g. is $532 + 309 = 509+332$ an example of commutativity?). While providing examples does not serve as an argument (except when providing counterexamples) this kind of work, sometimes called contextualizing and decontextualizing, is an important step in reasoning.

   (b) Finding models, making sketches, and solution planning. These are all skills important to launch full immersion into reasoning.

   (c) Analyzing examples. This is an intermediate step between providing relevant examples and providing a fully generalized example. A key step is understanding whether an example that works is fully generalizable.

   (d) Identify relevant definitions. Relevant definitions (including model-based definitions) are an essential part of mathematical argument.

   (e) Arguments with pictures. These can range in level of rigor and completeness depending on how they are structured.

   (f) Generalized examples. This type of argument is a bit of a lost art (historically, it was used by mathematicians such as Pascal). If one starts with an example and then explains fully how relevant features of that example hold in general, one can provide a fully rigorous argument.

   (g) Arguments with variables and more formalism. While these are the lingua franca for modern mathematics, and in particular essential for undergraduate math majors to master, they should be employed strategically in this context.

   We should delineate between the first four and the last three. The first four are key ingredients of reasoning, which we would expect any students with some effort to be able to summon when relevant. But it is only the last three which can support fully rigorous argument, and thus be considered complete as justification on homework.

   For example, a student proficient in the practice of argument would be able to go from understanding the proof that an even number added to an even number is even to a proof that a multiple of three added to a multiple of three gives another multiple of three. They would be able to engage multiple such proofs, for example proof by picture (and explanation), generalized example, and using variables. Similarly, having seen that a number is a multiple of four if its last two base-ten digits are, they would be able to show that the same holds for multiples of eight in base twelve.

2. **Gain experience in the Mathematical Practices.**

   Reasoning is addressed above as particularly important, but the full range of mathematical practices such as problem-solving, modeling, being precise, and seeing structure are essential for students to experience. We strongly recommend looking at the Mathematical Practices in the Common Core document, but we include them here for quick reference.

   (a) Make sense of problems and persevere in solving them.
   (b) Reason abstractly and quantitatively.
   (c) Construct viable arguments and critique the reasoning of others.
   (d) Model with mathematics.
   (e) Use appropriate tools strategically.
   (f) Attend to precision.
   (g) Look for and make use of structure.
   (h) Look for and express regularity in repeated reasoning.

   These practices serve both understanding what is ultimately challenging material (contrary to what some may be led to believe by the word "elementary") and serve future teachers in understanding the outcomes desired in mathematics education reform.

   For example, students engaging in the practice of seeing structure (g) will be able to explain why one can multiply numbers by adding together numbers on a list obtained by repeated doubling. They will persevere (a) when engaging in long-division in other bases, and for example see there must be an arithmetic error by checking their answer through multiplication and then finding their error. They will attend to precision (f) in how they describe numbers in other bases, and checking that answers given are valid in that base.

3. **Link concrete to abstract**

   Proficient Math 211 students will be able to take a general statement and provide an example. Conversely, they will look at some collection of examples or other phenomena and be able to formulate a statement which captures essential features of the collection. Such students understand the interplay between conjectures and theorems.

   These students will see how meanings are developed, first in more concrete ways connecting to previous experience and then more abstract ways. They will understand how connecting different approaches reinforces all, and by doing so creates a robust meaning. They will begin to identify what meanings can serve student understanding in some situations, as a foundation for a lifelong refinement of this teaching practice.

   For example, a student proficient in linking concrete to abstract would be able to give at least four different models for numbers and be able to connect any two of them.

4. **Begin to see the coherence and global structure of mathematics**

   A full understanding of this coherence and global structure is an ambitious aspiration. There are two concrete ways in which students in Math 211 can make significant steps towards such vision. One is a view which is also mentioned in our content standards. It is the Meaning-Method-Mastery framework, first developed by Cody Patterson. This development can be seen both within isolated strands of mathematics and as the strands interact.

   Another way in which students will see coherence is through awareness of Algebraic Thinking in elementary mathematics. The Common Core prescribes using of arithmetic as a venue to reflect on some of the main ideas of algebra. For example, by understanding the role of "canceling" (or "compensation") in arithmetic strategies, students are better able to see a purpose for similar moves in algebra. Math 211 students will start to see these connections, for example, through giving adult-level descriptions including variables for phenomena in arithmetic.

   A student engaging in the practice of seeing coherence might, for example, explain why the number line is important in early grades, when it doesn't necessarily serve understanding of the mathematics at hand. They may write a sequence of problems which call for the partitive missing factor model of division, which use the same (or very similar) modeling context but are appropriate to grades 3, 4, 5 and 6. They could explain the connection between "FOIL" and multiplication of two-digit numbers.