Lecture 2: Introduction to Crossed Products and More Examples of Actions

N. Christopher Phillips

University of Oregon

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- Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
- Lecture 2 (13 July 2016): Introduction to Crossed Products and More Examples of Actions.
- Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 4 (18 July 2016): Crossed Products by Actions with the Rokhlin Property.
- Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
- Lecture 6 (20 July 2016): Applications and Problems.

A rough outline of all six lectures

- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
- Crossed products by actions with the tracial Rokhlin property.
- Applications of the tracial Rokhlin property.

Recall: Group actions on C*-algebras

Definition

Let G be a group and let A be a C*-algebra. An *action of G on A* is a homomorphism $g \mapsto \alpha_g$ from G to $\mathrm{Aut}(A)$.

That is, for each $g \in G$, we have an automorphism $\alpha_g \colon A \to A$, and $\alpha_1 = \mathrm{id}_A$ and $\alpha_g \circ \alpha_h = \alpha_{gh}$ for $g, h \in G$.

We saw some examples coming from actions on compact Hausdorff spaces. We also saw the inner action: if $g \mapsto z_g$ is a (continuous) homomorphism from G to the unitary group $U(A)$ of a unital C*-algebra A, then $\alpha_g(a) = z_g a z_g^*$ defines an action of G on A. (We write $\alpha_g = \mathrm{Ad}(z_g)$.)

Exercise

Prove that $g \mapsto \mathrm{Ad}(z_g)$ really is a continuous action of G on A.

Finally, we looked at one example of an infinite tensor product action (next slide).

An infinite tensor product action

Let $A_n = (M_2)^{\otimes n}$, the tensor product of n copies of the algebra M_2 of 2×2 matrices. Thus $A_n \cong M_{2^n}$. Define

$$\varphi_n \colon A_n \to A_{n+1} = A_n \otimes M_2$$

by $\varphi_n(a) = a \otimes 1$. Let A be the (completed) direct limit $\varinjlim_n A_n$. (This is just the 2^∞ UHF algebra.) Define a unitary $v \in M_2$ by

$$v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define $z_n \in A_n$ by $z_n = v^{\otimes n}$. Define $\alpha_n \in \mathrm{Aut}(A_n)$ by $\alpha_n = \mathrm{Ad}(z_n)$, that is, $\alpha_n(a) = z_n a z_n^*$ for $a \in A$. Then α_n is an inner automorphism of order 2. Using $z_{n+1} = z_n \otimes v$, one can easily check that $\varphi_n \circ \alpha_n = \alpha_{n+1} \circ \varphi_n$ for all n (diagram on next slide). and it follows that the α_n determine an order 2 automorphism α of A. Thus, we have an action of $\mathbb{Z}_2$ on A.

Exercise: Prove that $\varphi_n \circ \alpha_n = \alpha_{n+1} \circ \varphi_n$.

Infinite tensor product action example (continued)

Recall: $A_n = (M_2)^{\otimes n} \cong M_{2^n}$.

$\varphi_n \colon A_n \to A_{n+1} = A_n \otimes M_2$ is $\varphi_n(a) = a \otimes 1$, and $A = \varinjlim_n A_n$.

$v = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in U(M_2)$, and $z_n = v^{\otimes n} \in U(A_n)$.

$\alpha_n \in \mathrm{Aut}(A_n)$ is $\alpha_n = \mathrm{Ad}(z_n)$.

Commutative diagram to define the order 2 automorphism $\alpha \in \mathrm{Aut}(A)$:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\varphi_0"] \arrow[d, "\alpha_0"] & M_2 \arrow[r, "\varphi_1"] \arrow[d, "\alpha_1"] & M_4 \arrow[r, "\varphi_2"] \arrow[d, "\alpha_2"] & M_8 \arrow[r, "\varphi_3"] \arrow[d, "\alpha_3"] & \cdots \arrow[r] & A \arrow[d, "\alpha"] \\
\mathbb{C} \arrow[r, "\varphi_0"] & M_2 \arrow[r, "\varphi_1"] & M_4 \arrow[r, "\varphi_2"] & M_8 \arrow[r, "\varphi_3"] & \cdots \arrow[r] & A
\end{tikzcd}$$

The action of $\mathbb{Z}_2$ is not inner (see later), although it is "approximately inner" (that is, a pointwise limit of inner actions).

General infinite tensor product actions

We had $A = \varinjlim_n (M_2)^{\otimes n}$ with the action of $\mathbb{Z}_2$ generated by the direct limit automorphism

$$\varinjlim_n \mathrm{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes n}$$

We write this automorphism as

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

In general, one can use an arbitrary group, one need not choose the same unitary representation in each tensor factor (indeed, the actions on the factors need not even be inner), and the tensor factors need not all be the same size (indeed, they can be arbitrary unital C*-algebras, provided one is careful with tensor products).

If all the factors are finite dimensional matrix algebras (not necessarily of the same size) and the action in each factor is inner, the action is frequently called a "product type action".

More examples of product type actions

We will later use the following two additional examples:

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_3,$$

and

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad}\big(\mathrm{diag}(-1, 1, 1, \ldots, 1)\big) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_{2^n+1}.$$

In the second one, there are supposed to be 2^n ones on the diagonal, giving a $(2^n + 1) \times (2^n + 1)$ matrix.

The tensor product of copies of conjugation by the regular representation

Let G be a finite group. Set $m = \mathrm{card}(G)$. Let G act on the Hilbert space $l^2(G) \cong \mathbb{C}^m$ via the left regular representation. That is, if $g \in G$, then g acts on $l^2(G)$ by the unitary operator $(z_g \xi)(h) = \xi(g^{-1}h)$ for $\xi \in l^2(G)$ and $h \in G$. Now let G act on $M_m \cong L(l^2(G))$ by conjugation by the left regular representation: $g \mapsto \mathrm{Ad}(z_g)$. Then take $A = \varinjlim_n (M_m)^{\otimes n}$ (which is the m^∞ UHF algebra), with the action of G given by

$$g \mapsto \bigotimes_{n=1}^{\infty} \mathrm{Ad}(z_g).$$

The first example we gave of a product type action is the case $G = \mathbb{Z}_2$. The left regular representation of $\mathbb{Z}_2$ is generated by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{rather than} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

but these two matrices are unitarily equivalent. Using this, one can show (see later) that the two product type actions are “essentially the same”.

Crossed products

Let G be a locally compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on a C*-algebra A. There is a *crossed product C*-algebra* $C^*(G, A, \alpha)$, which is a kind of generalization of the group C*-algebra $C^*(G)$. Crossed products are quite important in the theory of C*-algebras.

One motivation (mentioned already): Suppose G is a semidirect product $N \rtimes H$. The action of H on N gives an action $\alpha\colon H \to \mathrm{Aut}(C^*(N))$, and one has $C^*(G) \cong C^*(H, C^*(N), \alpha)$. (Exercise: Prove this when H and N are finite.) Thus, crossed products appear even if one is only interested in group C*-algebras and unitary representations of groups.

Another motivation (not applicable to finite groups acting on spaces): The noncommutative version of X/G is the fixed point algebra A^G. In particular, for compact G, one can check that $C(X/G) \cong C(X)^G$. For noncompact groups, often X/G is very far from Hausdorff and A^G is far too small. The crossed product provides a much more generally useful algebra, which is the “right” substitute for the fixed point algebra when the action is free.

The universal property of the crossed product

Recall the group case. If G is a discrete group, then multiplication in $\mathbb{C}[G]$ is $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$, and adjoint is $(a \cdot u_g)^* = \overline{a} u_{g^{-1}}$. If w is a unitary representation of G on H, the unital *-homomorphism $\pi_w\colon \mathbb{C}[G] \to L(H)$ is $\pi_w\left(\sum_{g\in G} a_g \cdot u_g\right) = \sum_{g\in G} a_g \cdot w_g$.

Theorem

The assignment $w \mapsto \pi_w$ is a bijection from unitary representations of G on H to unital *-homomorphisms $\mathbb{C}[G] \to L(H)$.

Unitary representations are replaced as follows:

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a topological group G on a C*-algebra A. A *covariant representation* of (G, A, α) on a Hilbert space H is a pair (v, σ) consisting of a unitary representation $v\colon G \to U(H)$ and a representation $\sigma\colon A \to L(H)$ satisfying the *covariance condition*

$$v_g \sigma(a) v_g^* = \sigma(\alpha_g(a))$$

for all $g \in G$ and $a \in A$.

The universal property of the crossed product (continued)

$\alpha\colon G \to \mathrm{Aut}(A)$ is an action of the group G on the C*-algebra A, $v\colon G \to U(H)$ is a unitary representation, and $\sigma\colon A \to L(H)$ is a *-homomorphism.

(v, σ) is *covariant* if $v_g \sigma(a) v_g^* = \sigma(\alpha_g(a))$ for all $g \in G$ and $a \in A$.

Recall: $\pi_w\left(\sum_{g\in G} a_g \cdot u_g\right) = \sum_{g\in G} a_g \cdot w_g$, and $w \mapsto \pi_w$ is a bijection from unitary representations to unital *-homomorphisms. To keep things simple, we state the crossed product version only in the unital case.

We will define a crossed product $C^*(G, A, \alpha)$ such that, for (v, σ) as above and with σ unital (in particular, A is unital), there is a unital *-homomorphism $\pi_{v,\sigma}\colon C^*(G, A, \alpha) \to L(H)$, and $(v, \sigma) \mapsto \pi_{v,\sigma}$ is a bijection from covariant representations to unital *-homomorphisms.

Defining the crossed product

Assume G is finite. Recall: $\mathbb{C}[G]$ is all formal linear combinations $\sum_{g \in G} a_g \cdot u_g$ with $a_g \in \mathbb{C}$ for $g \in G$. Multiplication in $\mathbb{C}[G]$ is $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$, and adjoint is $(a \cdot u_g)^* = \overline{a} u_{g^{-1}}$.

Now let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C*-algebra A. As a vector space, $C^*(G, A, \alpha)$ is the group ring $A[G]$, consisting of all formal linear combinations of elements in G with coefficients in A:

$$A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g \colon c_g \in A \text{ for } g \in G \right\}.$$

The multiplication and adjoint are given by:

$$(a \cdot u_g)(b \cdot u_h) = (a[u_g b u_g^{-1}]) \cdot u_g u_h = (a\alpha_g(b)) \cdot u_{gh}$$

$$(a \cdot u_g)^* = \alpha_g^{-1}(a^*) \cdot u_{g^{-1}}$$

for $a, b \in A$ and $g, h \in G$, extended linearly. There is a unique norm which makes this a C*-algebra. (See below.)

Defining the crossed product (continued)

Recall: Multiplication in $\mathbb{C}[G]$ is $(a \cdot u_g)(b \cdot u_h) = (ab) \cdot u_{gh}$, and adjoint is $(a \cdot u_g)^* = \overline{a} u_{g^{-1}}$.

$\alpha \colon G \to \mathrm{Aut}(A)$ is an action of a finite group G on a unital C*-algebra A. Multiplication in $A[G]$ is

$$(a \cdot u_g)(b \cdot u_h) = (a[u_g b u_g^{-1}]) \cdot u_g u_h = (a\alpha_g(b)) \cdot u_{gh}$$

and adjoint is

$$(a \cdot u_g)^* = \alpha_g^{-1}(a^*) \cdot u_{g^{-1}}.$$

The definition of multiplication is based on the idea that conjugating $b \in A$ by u_g should implement the automorphism α_g, just like in the definition of a covariant representation. The definition of the adjoint is what is forced by this idea and the requirement that the group elements be unitary: $u_g^* = u_{g^{-1}}$.

Exercise: Prove that these definitions make $A[G]$ a *-algebra over $\mathbb{C}$. (You don't need A to be unital, and, provided you use only finite linear combinations in the definition of $A[G]$, you don't need G to be finite.)

Defining the crossed product: the general discrete case

$\alpha \colon G \to \mathrm{Aut}(A)$ is an action of a discrete group G on a C*-algebra A. (Don't assume G is finite, and don't assume A is unital.) The *skew group ring* is

$$A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g \colon c_g \in A, \ c_g = 0 \text{ for all but finitely many } g \in G \right\}.$$

Multiplication in $A[G]$ is

$$(a \cdot u_g)(b \cdot u_h) = (a[u_g b u_g^{-1}]) \cdot u_g u_h = (a\alpha_g(b)) \cdot u_{gh}$$

and adjoint is

$$(a \cdot u_g)^* = \alpha_g^{-1}(a^*) \cdot u_{g^{-1}}.$$

Exercise: Prove that $A[G]$ is a *-algebra over $\mathbb{C}$.

If G is discrete but not finite, $C^*(G, A, \alpha)$ is the completion of $A[G]$ in a suitable norm. (In general, there are several choices, but only one gives the right universal property.)

General locally compact case: See the appendix.

The universal property of the crossed product

Recall: If G is finite and $w \colon G \to U(H)$ is a unitary representation, then $\pi_w \colon \mathbb{C}[G] \to L(H)$ is $\pi_w \left(\sum_{g \in G} a_g \cdot u_g \right) = \sum_{g \in G} a_g \cdot w_g$. Moreover, $w \mapsto \pi_w$ is a bijection from unitary representations to unital *-homomorphisms. Also recall:

$$A[G] = \left\{ \sum_{g \in G} c_g \cdot u_g \colon c_g \in A, \ c_g = 0 \text{ for all but finitely many } g \in G \right\}.$$

Suppose $v \colon G \to U(H)$ and $\sigma \colon A \to L(H)$ are a covariant representation of (G, A, α) on H. Then define $\pi_{v,\sigma} \colon A[G] \to L(H)$ by

$$\pi_{v,\sigma} \left(\sum_{g \in G} c_g \cdot u_g \right) = \sum_{g \in G} \sigma(a_g) \cdot v_g.$$

This is just the extension of $a \mapsto \sigma(a)$ and $u_g \mapsto v_g$.

Theorem

For G finite and A unital, $(v, \sigma) \mapsto \sigma_{v,\sigma}$ is a bijection from unital covariant representations of (G, A, α) on H to unital *-homomorphisms $A[G] \to L(H)$.

Theorem

For G finite and A unital, $(v, \sigma) \mapsto \pi_{v,\sigma}$ is a bijection from unital covariant representations of (G, A, α) on H to unital *-homomorphisms $A[G] \to L(H)$.

Exercise: Prove this theorem.

All the calculations are algebra; no analysis is needed. The key to the algebra is to compare the definition of the product in $A[G]$ (recall that $u_g a u_g^* = \alpha_g(a)$) with the condition $v_g \sigma(a) v_g^* = \sigma(\alpha_g(a))$ in the definition of a covariant representation. To recover v from $\pi_{v,\sigma}$, look at $\pi_{v,\sigma}(u_g)$. To recover $\sigma(a)$, look at $\pi_{v,\sigma}(a \cdot u_1) = \pi_{v,\sigma}(a)$.

You don't need G to be finite.

Exercise: Keep G finite, but no longer assume that A is unital. Assume that you know $A[G]$ is a C*-algebra. Prove the theorem with "unital *-homomorphisms" replaced by "nondegenerate representations". ($\rho \colon B \to L(H)$ is nondegenerate if $\overline{\rho(B)H} = H$.)

Now you need some analysis: since $u_g \notin A[G]$, you will need to use an approximate identity for A.

The norm on $A[G]$

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. We construct a C* norm on the skew group ring $A[G]$.
Recall:

$$(au_g)(bu_h) = a\alpha_g(b)u_{gh} \qquad \text{and} \qquad (au_g)^* = \alpha_g^{-1}(a^*)u_{g^{-1}}.$$

Fix a faithful representation $\sigma_0 \colon A \to L(H_0)$ of A on a Hilbert space H_0. Set $H = l^2(G, H_0)$, the set of all $\xi = (\xi_g)_{g \in G}$ in $\bigoplus_{g \in G} H_0$, with the scalar product

$$\left\langle (\xi_g)_{g \in G}, (\eta_g)_{g \in G} \right\rangle = \sum_{g \in G} \langle \xi_g, \eta_g \rangle.$$

(Exercise: Prove that H is a Hilbert space.) Then define $\sigma \colon A[G] \to L(H)$ as follows. For $c = \sum_{g \in G} c_g u_g$ and $h \in G$,

$$(\sigma(c)\xi)_h = \sum_{g \in G} \sigma_0(\alpha_h^{-1}(c_g))(\xi_{g^{-1}h})$$

Exercise: If A and σ_0 are unital, find a representation $v \colon G \to U(H)$ and a unital representation $\sigma \colon A \to L(H)$ such that (v, σ) is covariant and $\pi = \pi_{v,\sigma}$. (This is one way one is really supposed to construct a C* norm.)

The norm on $A[G]$ (continued)

Recall:

$$(au_g)(bu_h) = a\alpha_g(b)u_{gh} \qquad \text{and} \qquad (au_g)^* = \alpha_g^{-1}(a^*)u_{g^{-1}}.$$

Also, for $c = \sum_{g \in G} c_g u_g$,

$$(\pi(c)\xi)_h = \sum_{g \in G} \sigma_0(\alpha_h^{-1}(c_g))(\xi_{g^{-1}h}).$$

If A is unital, then for $a \in A$ and $g \in G$, identify a with au_1 and get

$$(\pi(a)\xi)_h = \sigma_0(\alpha_{h^{-1}}(a))(\xi_h) \qquad \text{and} \qquad (\pi(u_g)\xi)_h = \xi_{g^{-1}h}.$$

(Use these formulas in the exercise on the previous slide.)

One can check that π is a *-homomorphism. We will just check the most important part, which is that $\pi(u_g)\pi(b) = \pi(\alpha_g(b))\pi(u_g)$. We have

$$\left[\pi(\alpha_g(b))\pi(u_g)\xi\right]_h = \sigma_0\big(\alpha_{h^{-1}}(\alpha_g(b))\big)(\pi(u_g)\xi)_h = \sigma_0\big(\alpha_{h^{-1}g}(b)\big)(\xi_{g^{-1}h})$$

and

$$\big(\pi(u_g)\pi(b)\xi\big)_h = \big(\pi(b)\xi\big)_{g^{-1}h} = \sigma_0\big(\alpha_{h^{-1}g}(b)\big)\xi\big)_{g^{-1}h}.$$

Exercise: Give a complete proof that π is a *-homomorphism.

The norm on $A[G]$ (continued)

Recall: for $c = \sum_{g \in G} c_g u_g$,

$$(\pi(c)\xi)_h = \sum_{g \in G} \sigma_0(\alpha_h^{-1}(c_g))(\xi_{g^{-1}h}).$$

If A is unital, then for $a \in A$ and $g \in G$,

$$(\pi(a)\xi)_h = \sigma_0(\alpha_{h^{-1}}(a))(\xi_h) \qquad \text{and} \qquad (\pi(u_g)\xi)_h = \xi_{g^{-1}h}.$$

For $c = \sum_{g \in G} c_g u_g$, it is easy to check that

$$\|\pi(c)\| \leq \sum_{g \in G} \|c_g\|$$

and not much harder to check that

$$\|\pi(c)\| \geq \max_{g \in G} \|c_g\|.$$

Exercise: Prove these two inequalities. (The second requires looking at what $\pi(c)$ does to suitable elements in H. It is related to $\|a\| \geq \max_{j,k} |a_{j,k}|$ for a matrix $a = (a_{j,k})_{j,k=1,2,\ldots,n}$.)

The norms on the right hand sides are equivalent, so $A[G]$ is complete in the norm $\|c\| = \|\pi(c)\|$.

More examples of group actions on spaces

Recall: Every action of a group G on a compact space X gives an action of G on $C(X)$.

- Let Z be a compact manifold, or a connected finite complex. (Much weaker conditions on Z suffice, but Z must be path connected.) Let $X = \widetilde{Z}$ be the universal cover of Z, and let $G = \pi_1(Z)$ be the fundamental group of Z. Then there is a standard action of G on X. Spaces with finite fundamental groups include real projective spaces (in which case this example was already on the first slide of examples) and lens spaces.
- The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ via the usual matrix multiplication. This action preserves $\mathbb{Z}^2$, and so is well defined on $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$. $\mathrm{SL}_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

  Restriction gives actions of these on $S^1 \times S^1$.

The rotation algebras

Let $\theta \in \mathbb{R}$. Recall the irrational rotation algebra A_θ, the universal C*-algebra generated by two unitaries u and v satisfying the commutation relation $vu = e^{2\pi i\theta} uv$. If $\theta_1 - \theta_2 \in \mathbb{Z}$, then $A_{\theta_1} = A_{\theta_2}$. (The commutation relation is the same.) Some standard facts, presented without proof:

- If $\theta \notin \mathbb{Q}$, then A_θ is simple. In particular, *any* two unitaries u and v in any C*-algebra satisfying $vu = e^{2\pi i\theta} uv$ generate a copy of A_θ.
- If $\theta \in \mathbb{Q}$, then A_θ is Type I. In fact, if $\theta = \frac{m}{n}$ in lowest terms, with $n > 0$, then A_θ is isomorphic to the section algebra of a locally trivial continuous field over $S^1 \times S^1$ with fiber M_n.
- In particular, if $\theta = 0$, or if $\theta \in \mathbb{Z}$, then $A_\theta \cong C(S^1 \times S^1)$.
- A_θ is the crossed product of the action of $\mathbb{Z}$ on S^1 generated by rotation by $e^{2\pi i\theta}$.
- There is a "natural" continuous field over S^1 whose fiber over $e^{2\pi i\theta}$ is A_θ. (Obviously it isn't locally trivial.)

The algebra A_θ is often considered to be a noncommutative analog of the torus $S^1 \times S^1$ (more accurately, a noncommutative analog of $C(S^1 \times S^1)$).

The gauge action on the rotation algebra

Recall: A_θ is the universal C*-algebra generated by two unitaries u and v satisfying the commutation relation $vu = e^{2\pi i\theta} uv$.

There is a unique action $\gamma \colon S^1 \times S^1 \to \mathrm{Aut}(A_\theta)$ such that

$$\gamma_{(\lambda,\zeta)}(u) = \lambda u \qquad \text{and} \qquad \gamma_{(\lambda,\zeta)}(v) = \zeta v$$

for $\lambda, \zeta \in S^1$. This essentially follows from the fact that λu and ζv satisfy the same commutation relation that u and v do. One must also check that $(\lambda, \zeta) \mapsto \gamma_{(\lambda,\zeta)}$ is a group homomorphism. (A bit of work is required to show that $(\lambda, \zeta) \mapsto \gamma_{(\lambda,\zeta)}(a)$ is continuous for all $a \in A_\theta$. Exercise: Do it. Hint: Show that it is true for a in the linear span of all $u^m v^n$, and then use an $\frac{\varepsilon}{3}$ argument.)

In particular, there are actions of $\mathbb{Z}_n$ on A_θ generated by the automorphism

$$u \mapsto e^{2\pi i/n} u \qquad \text{and} \qquad v \mapsto v$$

and by the automorphism

$$u \mapsto u \qquad \text{and} \qquad v \mapsto e^{2\pi i/n} v.$$

Appendix: The C*-algebra of a locally compact group, etc.

Let G be a locally compact group. We recall that nondegenerate representations of the group C*-algebra $C^*(G)$ on a Hilbert space H are in one to one correspondence with the unitary representations of G on H.

To construct $C^*(G)$, one starts with $L^1(G)$ (using left Haar measure μ) with convolution multiplication:

$$(a * b)(g) = \int_G a(h) b(h^{-1} g) \, d\mu(h).$$

(We omit the formula for the adjoint.) If G is discrete and $\delta_g \in l^1(G)$ is the standard basis vector corresponding to $g \in G$, this amounts to declaring that $\delta_g * \delta_h = \delta_{gh}$ and $\delta_g^* = \delta_{g^{-1}}$. A unitary representation $g \mapsto v_g$ of G on a Hilbert space H gives a nondegenerate *-representation π of $L^1(G)$ on H via the formula

$$\pi(a)\xi = \int_G a(g) v_g \xi \, d\mu(g).$$

(One must check many things about this formula.) If G is discrete, this is just $\pi(a) = \sum_{g \in G} a(g) v_g$, and in particular $\pi(\delta_g) = v_g$.

The group C*-algebra

For a locally compact group G and a unitary representation v of G on H, we set

$$\pi(a)\xi = \int_G a(g) v_g \xi \, d\mu(g)$$

for $a \in L^1(G)$ and $\xi \in H$. If G is discrete, this is just $\pi(a) = \sum_{g \in G} a(g) v_g$, and in particular $\pi(\delta_g) = v_g$.

Getting v from π is easy if G is discrete, since $v_g = \pi(\delta_g)$. In general, one must do some work with multiplier algebras; we omit the details.

We must still get a C*-algebra. To do this, define a C* norm on $L^1(G)$ by taking $\|a\|$ to be the supremum of $\|\pi(a)\|$ over all nondegerarate *-representations π of $L^1(G)$ on Hilbert spaces. Then complete in this norm.

If G is finite, this simplifies greatly. The sums $\pi(a) = \sum_{g \in G} a(g) v_g$ are finite sums and no completion is necessary (because $L^1(G)$ is finite dimensional). One only needs to find the C* norm. (It is equivalent to the L^1 norm, but not equal to it.)

The universal property of the crossed product

The crossed product $C^*(G, A, \alpha)$ (for G locally compact) is defined in such a way as to have a universal property which generalizes the universal property of the group C*-algebra $C^*(G)$. We give the statements for the general case.

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. A *covariant representation* of (G, A, α) on a Hilbert space H is a pair (v, σ) consisting of a unitary representation $v \colon G \to U(H)$ (the unitary group of H) and a representation $\sigma \colon A \to L(H)$ (the algebra of all bounded operators on H), satisfying the *covariance condition*

$$v_g \sigma(a) v_g^* = \sigma(\alpha_g(a))$$

for all $g \in G$ and $a \in A$. It is called *nondegenerate* if σ is nondegenerate.

A dense subalgebra of the crossed product

The skew group ring $A[G]$, used when G is discrete, is replaced by $C_c(G, A, \alpha)$, with product (using a left Haar measure μ on G and Banach space valued integration)

$$(ab)(g) = \int_G a(h) b(h^{-1} g) \, d\mu(h)$$

and (using the modular function Δ of G) adjoint

$$a^*(g) = \Delta(g)^{-1} \alpha_g(a(g^{-1})^*).$$

One can define a (non C*) norm by

$$\|a\|_1 = \int_G \|a(g)\| \, d\mu(g).$$

The completion is called $L^1(G, A, \alpha)$. (One can also define $L^1(G, A, \alpha)$ directly, using a more general version of Banach space valued integration.)

Exercise: Prove that $C_c(G, A, \alpha)$ is a normed *-algebra.

$C^*(G, A, \alpha)$ is the completion of $C_c(G, A, \alpha)$ (or $L^1(G, A, \alpha)$) in a suitable C* norm. (It is the universal enveloping C*-algebra of $L^1(G, A, \alpha)$.)

The universal property of the crossed product (continued)

Definition

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A, and let (v, σ) be a covariant representation of (G, A, α) on a Hilbert space H. Then the *integrated form* of (v, σ) is the representation $\pi \colon C_c(G, A, \alpha) \to L(H)$ given by

$$\pi(a)\xi = \int_G \sigma(a(g)) v_g \xi \, d\mu(g).$$

$C^*(G, A, \alpha)$ is then a completion of $C_c(G, A, \alpha)$, chosen to give:

Theorem

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Then the integrated form construction defines a bijection from the set of nondegenerate covariant representations of (G, A, α) on a Hilbert space H to the set of nondegenerate representations of $C^*(G, A, \alpha)$ on the same Hilbert space.