AN INTRODUCTION TO CROSSED PRODUCT C*-ALGEBRAS AND MINIMAL DYNAMICS

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Contents

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1. Introduction and Motivation

These notes are an introduction to group actions on C*-algebras and their crossed products, primarily by discrete groups and with emphasis on situations in which the crossed products are simple and at least close to the class of C*-algebras expected to be classifiable in the sense of the Elliott program. They are aimed at graduate students who have had a one semester or one year course on the general theory of C*-algebras. (We give more details on the prerequisites later in this section.) These notes are not intended as a reference work. Our emphasis is on explaining ideas and methods, rather than on giving complete proofs. For some results, different proofs are given at different locations in these notes, or special cases are proved of results which are proved later in greater generality by quite different methods. For others, some of the main ideas are explained and simpler versions of some of the relevant lemmas are proved, but we refer to the research papers for the full proofs. Other results and calculations are left as exercises; the reader is strongly encouraged to do many of these, to develop facility with the material. Yet other results, needed for the proofs of the theorems described here but not directly related to dynamics, are quoted with only some general description, or with no background at all.

Before giving a general outline, we describe some of the highlights of our treatment. We give a very large collection of examples of actions of groups on C*-algebras (Part 1), and we give a number of explicit computations of crossed products (Section 10). We give most or all of the proofs of the following results, including background:

- The reduced C*-algebra of a finitely generated nonabelian free group is simple (Theorem 6.6) and has a unique tracial state (Theorem 6.7).
- If G is an amenable locally compact group, then the map $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is an isomorphism (Theorem 9.7; proved using the Følner condition).
- If G is a discrete group, then the standard conditinal expectation from $C^*_{\mathrm{r}}(G, A, \alpha)$ to A is faithful (Proposition 9.16(4); this is hard to find in the literature).
- The crossed product of an AF algebra by a Rokhlin action of a finite group is AF (Theorem 13.15).
- The crossed product of a simple tracially AF C*-algebra by a tracial Rokhlin action of a finite group is tracially AF (Theorem 14.17).
- The reduced crossed product of a locally compact Hausdorff space by a minimal and essentially free action of a discrete group is simple (Theorem 15.10).

We give substantial parts of the proofs of the following results, including the relevant dynamics background:

- Let X be a finite dimensional infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism. Suppose that the image of $K_0(C^*(\mathbb{Z}, X, h))$ is dense in $\mathrm{Aff}\big(\mathrm{T}(C^*(\mathbb{Z}, X, h))\big)$. Then $C^*(\mathbb{Z}, X, h)$ has tracial rank zero. (See Theorem 16.1.) This includes the proof of Theorem 17.19, giving the recursive subhomogeneous structure of the orbit breaking subalgebra $C^*(\mathbb{Z}, X, h)_Y$ of Definition 16.18 when $Y \subset X$ is closed and $\mathrm{int}(Y) \neq \varnothing$, for which as far as we know a detailed proof has not been published.
- Let $h\colon X \to X$ be a minimal homeomorphism of a compact metric space. Assume that there is a continuous surjective map from X to the Cantor set. Then the radius of comparison of $C^*(\mathbb{Z}, X, h)$ is at most half the mean dimension of h. (See Theorem 23.14.)

We give a brief description of the contents. Parts 1 and 2 (Sections 2–10) are quite elementary in nature. Part 1 gives many examples of group actions on C*-algebras. Part 2 develops the theory of full and reduced group C*-algebras and full and reduced crossed products, with full details given for discrete groups and some indications of the theory for general locally compact groups. This part ends with a number of explicit computations of crossed products by discrete groups.

Part 3 (Sections 11–14) is about structure theory for crossed products of simple C*-algebras by finite groups. Section 11 discusses (giving some proofs, but not a complete presentation) some of the relevant structural properties of C*-algebras. In the rest of this part, we consider crossed products, primarily under the assumption that the action has the Rokhlin property or the tracial Rokhlin property. The presentation of the crossed product related machinery is fairly detailed but not complete, and a few results from other parts of the theory of C*-algebras are used with little indication of proof.

Part 4 (Sections 15–17) is a first look at minimal homeomorphisms of compact metric spaces and their crossed products. We give a complete proof of simplicity of reduced crossed products by essentially free minimal actions of discrete groups. When we turn to stronger structure theorems, for the case of actions of $\mathbb{Z}$, much more outside material is needed, and our presentation accordingly becomes much more sketchy.

In Part 5 (Sections 18–24), we discuss the machinery of large subalgebras, which is used to prove further results about the structure of crossed products by minimal homeomorphisms (and by free minimal actions of some other groups, as well as automorphisms of some noncommutative C*-algebras). Large subalgebras are motivated by the proofs in Section 16 and those sketched in Section 17. The theory here is considerably more technical, and uses considerably more material from outside the theory of crossed products. In particular, the Cuntz semigroup plays a key role in the statements of some results, and in the proofs of some results whose statement does not mention the Cuntz semigroup. Our presentation here is accordingly much less complete. In a number of cases, we give direct proofs of results which in the original papers are derived from stronger results with more complicated proofs, or we prove only special cases or simplified statements. These proofs are simpler, but are still not simple. The hope is that the presentation here can serve as an introduction to the machinery of large subalgebras, and enable beginners in the area to better understand the research papers using this method.

These notes are a greatly expanded version of lectures on crossed product C*-algebras given at the Ottawa Summer School in Operator Algebras, 20–24 August 2007. It contains additional material from lectures given at the Fields Institute in Fall 2007, from graduate courses given at the University of Oregon in Spring 2008 and Spring 2013 and at the University of Toronto in Winter 2014, and from lecture series given in Lisbon, Seoul, Shanghai, Barcelona, Kyoto, and Laramie.

These notes are still rough. There are surely many remaining misprints and some more serious errors. Some references are incomplete or missing entirely. There is no index. Even given the omissions discussed below, there should have been, as just one example, enough discussion of groupoids and their C*-algebras to identify the orbit breaking subalgebras (Definition 16.18) of crossed products, used in Parts 4 and 5, as C*-algebras of open subgroupoids of the transformation group groupoid. The author plans to keep a list of misprints, and a corrected and possibly expanded version of these notes, on his website.

Developments in the theory of Part 5, and even to some extent in the theory of Part 3, are quite rapid, and are faster than it is possible to keep up with in writing these notes. In particular, four extremely important developments are barely mentioned here. One is the use of versions of the tracial Rokhlin property (for both finite and countable amenable groups) which do not require the presence of projections. Several more sections could be written in Part 3 based on these developments. The second is the importance of stability under tensoring with the Jiang-Su algebra Z as a regularity condition. This condition is barely mentioned in Part 3, and deserves a much more substantial treatment there. Third, essentially nothing is said about higher dimensional Rokhlin properties, despite their importance even for finite groups and also as a competing method for obtaining results of some of the same kinds as in Part 5. Finally, essentially nothing is said about classifiability and related weaker conditions for crossed products of simple C*-algebras by infinite discrete groups, not even by $\mathbb{Z}$. Our discussion of crossed products of simple C*-algebras stops after considering finite groups, and the actions of infinite groups we consider almost all come from actions on compact metric spaces.

These notes assume the basic theory of C*-algebras, including:

- The basics of representation theory (including states and the Gelfand-Naimark-Segal construction).
- Type I C*-algebras.
- Some familiarity with nuclear C*-algebras.
- Direct limits and the usual examples constructed with them, such as UHF algebras, AF algebras, AT algebras, and AH algebras.
- Tensor products of Hilbert spaces.
- Some familiarity with minimal and maximal tensor products of C*-algebras.
- The basics of C*-algebras given by generators and relations and the usual elementary examples (such as M_n, $C(S^1)$, $C(S^1, M_n)$, the Toeplitz algebra, and the Cuntz algebras).
- Multiplier algebras.
- The Double Commutant Theorem.

We will give some exposition of the following topics, but not enough to substitute for a thorough presentation:

- Stable rank one.
- Real rank zero.

- Tracial rank zero.
- The Cuntz semigroup.
- Recursive subhomogeneous C*-algebras.
- Dimension theory for compact metric spaces.
- The mean dimension of a homeomorphism.
- Graph C*-algebras.

There will be occasional comments assuming other material, but which are not essential to the development:

- Larger values of topological stable rank, real rank, and tracial rank.
- K-theory. (C*-algebras satisfying the Universal Coefficient Theorem will be mentioned moderately often.)
- Morita equivalence.
- Groupoids and their C*-algebras.
- Partial actions and their crossed products.
- Free products and reduced free products.
- Quasitraces.

In a number of places, we make comments which refer to later material. We encourage the reader to jump back and forth. Some statements are given without proof: the proofs are either left as exercises or are beyond the scope of these notes. In Part 2, although the definitions related to group C*-algebras and crossed products are presented for actions of general locally compact groups, most of the proofs and examples are restricted to the discrete case, which is often considerably easier.

Items labelled "Exercise" are intended to be done by the reader. Items labelled "Problem" or "Question" are open questions.

By convention, all topological groups will be assumed to be Hausdorff. Homomorphisms of C*-algebras will be *-homomorphisms. We also use the following terminology.

Definition 1.1. A *Kirchberg algebra* is a separable nuclear purely infinite simple C*-algebra.

We don't assume that a Kirchberg algebra satisfies the Universal Coefficient Theorem.

We now give enough of the basic definitions related to group actions on C*-algebras and locally compact spaces that the discussion in the rest of this section will make sense.

Definition 1.2. Let G be a topological group, and let A be a C*-algebra. An *action of G on A* is a group homomorphism $\alpha\colon G \to \mathrm{Aut}(A)$, usually written $g \mapsto \alpha_g$, such that, for every $a \in A$, the function $g \mapsto \alpha_g(a)$, from G to A, is norm continuous.

The continuity condition is the analog of requiring that a unitary representation of G on a Hilbert space be continuous in the strong operator topology. It is usually much too strong a condition to require that $g \mapsto \alpha_g$ be a norm continuous map from G to the bounded operators on A. For example, let G be a locally compact group, and let $\alpha\colon G \to \mathrm{Aut}(C_0(G))$ be the action given by $\alpha_g(f)(k) = f(g^{-1}k)$ for $f \in C_0(G)$ and $g, k \in G$. We certainly want this action to be continuous. Suppose $g, h \in G$ with $g \neq h$. Then $\|\alpha_g - \alpha_h\| \geq 2$, as can be seen by choosing $f \in C_0(G)$ such that $f(g^{-1}) = 1$, $f(h^{-1}) = -1$, and $\|f\| = 1$. Indeed, one gets

$$\|\alpha_g - \alpha_h\| \geq \|\alpha_g(f) - \alpha_h(f)\| \geq |\alpha_g(f)(1) - \alpha_h(f)(1)| = |f(g^{-1}) - f(h^{-1})| = 2.$$

(The inequality $\|\alpha_g - \alpha_h\| \leq 2$ is easy.) Thus, if G is not discrete, then $g \mapsto \alpha_g$ is never norm continuous.

Of course, if G is discrete, there is no difference between the continuity conditions.

Isomorphism of actions is called conjugacy.

Definition 1.3. Let G be a group, let A and B be C*-algebras, and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions of G on A and B. A homomorphism $\varphi\colon A \to B$ is called *equivariant* if $\varphi \circ \alpha_g = \beta_g \circ \varphi$ for all $g \in G$. The actions α and β are called *conjugate* if there is an equivariant isomorphism $\varphi\colon A \to B$.

Equivariance means that the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} A & \xrightarrow{\alpha_g} & A \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ B & \xrightarrow{\beta_g} & B. \end{array}$$

Given $\alpha\colon G \to \mathrm{Aut}(A)$, we will construct in Section 8 below a crossed product C*-algebra $C^*(G, A, \alpha)$ and a reduced crossed product C*-algebra $C^*_{\mathrm{r}}(G, A, \alpha)$. (There are many other commonly used notations. See Remark 8.19. We may omit α if it is understood.) If A is unital and G is discrete, the crossed products are a suitable completion of the algebraic skew group ring $A[G]$, with multiplication determined by $gag^{-1} = \alpha_g(a)$ for $g \in G$ and $a \in A$. The main subject of these notes is some aspects of the structure of crossed products. Earlier sections give a large collection of examples of group actions on C*-algebras, and discuss the full and reduced group C*-algebras, which are the crossed products gotten from the trivial action of the group on $\mathbb{C}$.

Just as locally compact spaces give commutative C*-algebras, group actions on locally compact spaces give group actions on commutative C*-algebras.

Definition 1.4. Let G be a topological group, and let X be a topological space. An *action* of G on X is a continuous function $G \times X \to X$, usually written $(g, x) \mapsto g \cdot x$ or $(g, x) \mapsto gx$, such that $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$ and $1 \cdot x = x$ for all $x \in X$.

Discontinuous actions on spaces are of course also possible, but we will encounter very few of them.

Definition 1.5. Let G be a topological group, let X be a locally compact Hausdorff space, and let $(g, x) \mapsto gx$ be an action of G on X. We define the induced action of G on $C_0(X)$, say α, by $\alpha_g(f)(x) = f(g^{-1}x)$ for $g \in G$, $f \in C_0(X)$, and $x \in X$. (Exercise 1.6 asks for a proof that we really get an action.)

The inverse appears for the same reason it does in the formula for the left regular representation of a group. If G is not abelian, the inverse is necessary to get $\alpha_g \circ \alpha_h$ to be α_{gh} rather than α_{hg}. If $K \subset X$ is a compact open set, so that its characteristic function χ_K is in $C_0(X)$, then $\alpha_g(\chi_K) = \chi_{gK}$, not $\chi_{g^{-1}K}$.

We write $C^*(G, X)$ for the crossed product C*-algebra and $C^*_{\mathrm{r}}(G, X)$ for the reduced crossed product C*-algebra. We call them the transformation group C*-algebra and the reduced transformation group C*-algebra.

Exercise 1.6. Let G be a topological group, and let X be a locally compact Hausdorff space. Prove that the formulas given above determine a one to one correspondence between continuous actions of G on X and continuous actions of G on $C_0(X)$. (The main point is to show that an action on X is continuous if and only if the corresponding action on $C_0(X)$ is continuous.)

For the special case $G = \mathbb{Z}$, the same notation is often used for the action and for the automorphism which generates it. Thus, if A is a C*-algebra and $\alpha \in \mathrm{Aut}(A)$, one often writes $C^*(\mathbb{Z}, A, \alpha)$. For a homeomorphism h of a locally compact Hausdorff space X, one gets an automorphism $\alpha \in \mathrm{Aut}(C_0(X))$, and thus an action of $\mathbb{Z}$ on $C_0(X)$. We abbreviate this crossed product to $C^*(\mathbb{Z}, X, h)$.

We give some motivation for studying group actions on C*-algebras and their crossed products.

(1) Let G be a locally compact group obtained as a semidirect product $G = N \rtimes H$. The action of H on N gives actions of H on the full and reduced group C*-algebras $C^*(N)$ and $C^*_{\mathrm{r}}(N)$, and one has $C^*(G) \cong C^*(H, C^*(N))$ and $C^*_{\mathrm{r}}(G) \cong C^*_{\mathrm{r}}(H, C^*(N))$.
(2) Probably the most important group action is time evolution: if a C*-algebra A is supposed to represent the possible states of a physical system in some manner, then there should be an action $\alpha \colon \mathbb{R} \to \mathrm{Aut}(A)$ which describes the time evolution of the system. Actions of $\mathbb{Z}$, which are easier to study, can be though of as "discrete time evolution".
(3) Crossed products are a common way of constructing simple C*-algebras. Here are some of the more famous examples.
   - The irrational rotation algebras. See Example 10.25 below. They were not originally defined as crossed products.
   - The Bunce-Deddens algebras. See [39] or Section V.3 of [52]; one crossed product realization is Theorem VII.4.1 of [52], and another, for a specific choice of Bunce-Deddens algebra, and using an action of the dyadic rationals on the circle, can be found at the beginning of Section VIII.9 of [52].
   - The reduced C*-algebra of the free group on two generators. See Section VII.7 of [52]; simplicity is proved in Theorem 6.6.

   We will see other examples later.
(4) If one has a homeomorphism h of a locally compact Hausdorff space X, the crossed product $C^*(\mathbb{Z}, X, h)$ sometimes carries considerable information about the dynamics of h. The best known example is the result of [94] on minimal homeomorphisms of the Cantor set: isomorphism of the transformation group C*-algebras is equivalent to strong orbit equivalence of the homeomorphisms.
(5) For compact groups, equivariant indices take values in the equivariant K-theory of a suitable C*-algebra with an action of the group. When the group is not compact, one usually needs instead the K-theory of the crossed product C*-algebra, or of the reduced crossed product C*-algebra. (When the group is compact, this is the same thing.)

   In other situations as well, the K-theory of the full or reduced crossed product is the appropriate substitute for equivariant K-theory.

There are many directions in the theory of crossed products. These notes are biased towards the general problem of understanding the structure of crossed products by finite groups, by $\mathbb{Z}$, and by more complicated groups, in cases in which these crossed products are expected to be simple, and, in good cases, classifiable in the sense of the Elliott program. I should at least mention some of the other directions. Some of these are large and very active areas of research, some are small but active areas of research, in some it seems that most of the theory has been worked out, and some are just beginning. The list is not complete, and there is also interesting work which doesn't fit under any of these directions. Directions of work on group actions which don't involve crossed products (such as work on classification of actions) are mostly not mentioned. The references provided are not necessarily recent or representative of work in the subject; they are often just ones I have managed to find, sometimes with the help of people in the area. Moreover, some very active areas have very few references listed, perhaps only one or two books or survey articles.

- The relation between the structure of a nonminimal homeomorphism and the structure of its crossed product. See [280], [281], and [282]. The article [272] is one example of more recent work in this direction.
- The structure of crossed products of continuous trace C*-algebras by actions for which the induced action on the primitive ideal space is proper. See the textbook [236].
- Extensions of the notion of crossed product to coactions and actions of C* Hopf algebras ("quantum groups"), and the associated duality theory. The textbook [279] on quantum groups has a chapter on this subject. One of the classic papers is [12], which uses the formalism of multiplicative unitaries and, among other things, give a version of Takai duality for crossed products by quantum groups. For a recent survey of this area, see [56]. For one application (imprimitivity theorems, in connection with induction and restriction of representations of quantum groups), see [287], and the earlier paper [64].
- Crossed products twisted by cocycles. Cocycles can be untwisted by stabilization, so such crossed products are stably isomorphic to ordinary crossed products. See Corollary 3.7 of [192], with further applications in [193]. But for some purposes, one doesn't want to stabilize.
- Von Neumann algebra crossed products. There are several chapters on group actions and crossed products in Volume 2 [277] and Volume 3 [278] of Takesaki's three volume work on operator algebras. One direction with major recent activity is the classification of von Neumann algebra crossed products by ergodic measure preserving actions of countable nonamenable groups on probability spaces, including cases in which the group and the action can be recovered from the von Neumann algebra. See [119] for a recent survey. The papers [225], [288], and [289] are older surveys. Two of the important early papers in this direction are [223] and [224]. Two of many more recent important papers are [226] and [118].
- Smooth crossed products. See [252] and [253] for some of the foundations. See [175] and [71] for cyclic cohomology of crossed products by $\mathbb{Z}$ and $\mathbb{R}$, and see [216] for their K-theory.
- C*-algebras of groupoids, and crossed products by actions of groupoids on C*-algebras. The original book is [238]; a more recent book is [196]. There is much more work in this direction.
- Computation of the K-theory of crossed products, from the Pimsner-Voiculescu exact sequences [221], [221] their generalization [219] and the Connes

isomorphism [42] through the Baum-Connes conjecture. See [165] for a survey of the Baum-Connes conjecture and related conjectures.

- The Connes spectrum and its generalizations. See [179], [180], [181], and [141] for some of the early work for abelian groups. The Connes spectrum for compact nonabelian groups was introduced in [97], and for actions of compact quantum groups in [62]. These ideas have even been extended into ring theory, in which there is no topology [188].
- The ideal structure of crossed products, without assuming analogs of freeness or properness. Much of Williams' book [292] is related to this subject. A generalization to groupoids can be found in [239]. See [256], [63], and [66] for examples of more recent work. The Connes spectrum is also relevant here.
- Structural properties of crossed products which are inspired by those related to the Elliott program, but in cases in which neither the original algebra nor the crossed product is expected to be simple. (See [194] and [195] for some recent work, and [247], [95], [139], and [140] for a related direction.)
- Crossed products by endomorphisms, semigroups, and partial actions. The book [82] will appear soon, and is already available on the arXiv. A recent paper with some relation to problems considered here is [95].
- Semicrossed products: nonselfadjoint crossed products gotten from semigroup actions on C*-algebras. This area has a long history, starting with Arveson in the weak operator closed case [9] and with Arveson and Josephson in the norm closed case [10]. See [55] and [54] for two much more recent survey articles in the area, and [53] for a recent substantial paper.
- Crossed products by actions of locally compact groups on nonselfadjoint Hilbert space operator algebras. This is a very new field, in effect started in [133]. It already has applications to crossed products of C*-algebras; see [132].
- L^1 crossed products, so far mostly of $C(X)$ by $\mathbb{Z}$. See [58], [57], [145], and references in these papers.
- Algebraic crossed products of C*-algebras by discrete groups, so far mostly of $C(X)$ by $\mathbb{Z}$. See [269], [270], and [271].
- General Banach algebra crossed products. The beginnings of a general theory appear in [59].
- Crossed products of algebras of operators on L^p spaces. This is very recent. See [212].

We will not touch at all on many of these directions. However, work on the structure and classification of simple crossed products does not occur in isolation, and we will need some information from some of the other directions, including K-theory, groupoids, and partial actions.

The textbook references on crossed products that I know are Chapters 7 and 8 of [198] (very condensed; the primary emphasis is on properties of group actions rather than of crossed products), [292] (quite detailed; the primary emphasis is on ideal structure of general crossed products), and Chapter 8 of [52] (the primary emphasis is on crossed products, especially by $\mathbb{Z}$, as a means of constructing interesting examples of C*-algebras). There are no textbooks with primary emphasis on classification of crossed products or on crossed products by minimal homeomorphisms.

The "further reading" section in the introduction of [292] gives a number of references for various directions in the theory of crossed products which are treated neither in [292] nor here.

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Part 1. Group Actions

2. Examples of Group Actions on Locally Compact Spaces

This is the first of three sections devoted to examples of group actions.

In this section, we give examples of actions on commutative C*-algebras. In Section 3 we give a variety of examples of actions on noncommutative C*-algebras, and in Section 4 we give an additional collection of examples of actions that are similar to gauge actions.

Some general comments are in order. The main focus of the later part of these notes is group actions $\alpha \colon G \to \mathrm{Aut}(A)$ for a locally compact group G on a C*-algebra A such that the crossed product $C^*(G, A, \alpha)$ or reduced crossed product $C^*_{\mathrm{r}}(G, A, \alpha)$ (as defined in Sections 8 and 9) is at least as complicated as A itself. In particular, we usually want the (reduced) crossed product to be simple, and to be purely infinite if A is. There are many interesting and sometimes very important actions whose nature is quite different, and in our examples we do not discriminate: we give a very broad collection.

We make some comments (without proof) about the kinds of crossed products one gets. These don't make sense without knowing at least a little about crossed products (Sections 8 and 9), so it is useful to come back to the examples after reading much farther into these notes. Some of the comments made will be proved in the later part of these notes, but for many no proof will be given at all. For actions of compact groups, the crossed product is often closely related to the fixed point algebra A^G (or A^α when necessary to avoid confusion), given by

$$A^G = \big\{ a \in A \colon \alpha_g(a) = a \text{ for all } g \in G \big\}.$$

Instead of commenting on the crossed product, we therefore sometimes comment on the fixed point algebra.

There is one way in which we do discriminate. Crossed products only exist for actions of locally compact groups, because the group must have a Haar measure. With very few exceptions, we therefore only give examples of actions of locally compact groups.

We will also sometimes mention the Rokhlin property or related conditions on actions. Some of these are defined later. (The Rokhlin property for actions of finite groups is in Definition 13.1, and the tracial Rokhlin property for actions of finite groups is in Definition 14.1.) For some, however, no definition will be given in these notes.

Turning specifically to the commutative case, recall from Definition 1.5 and Exercise 1.6 that giving an action of a topological group G on a commutative C*-algebra $C_0(X)$ is the same as giving an action of G on the underlying space X. When G is locally compact, the crossed product C*-algebra $C^*(G, C_0(X))$ is usually abbreviated to $C^*(G, X)$. (See Definition 8.20.) As noted above, we are mostly interested in the case in which $C^*(G, X)$ is simple. So as to be able to make meaningful comments, we discuss several easy to state conditions on an action on a locally compact space which are related to simplicity of the crossed product. We will say more about these conditions in Section 15.

Definition 2.1. Let a topological group G act continuously on a topological space X. The action is called *minimal* if whenever $T \subset X$ is a closed subset such that $gT \subset T$ for all $g \in G$, then T is trivial, that is, $T = \varnothing$ or $T = X$.

Lemma 2.2. Let a topological group G act continuously on a topological space X. The action is minimal if and only if for every $x \in X$, the orbit $Gx = \{gx \colon g \in G\}$ is dense in X.

Proof. If there is $x \in X$ such that Gx is not dense, then $\overline{Gx}$ is a nontrivial G-invariant closed subset of X. For the converse, let $T \subset X$ be a nontrivial G-invariant closed subset of X. Choose any $x \in T$. Then $Gx \subset T$ and is therefore not dense. $\square$

Lemma 15.3 gives a number of weaker equivalent conditions for minimality for the special case $G = \mathbb{Z}$ and X is compact.

As shown by the action of $\mathbb{Z}$ on its one point compactification (in Example 2.15 below), it is not enough to require that one orbit be dense. There are special circumstances under which density of one orbit is sufficient, such as for an action of a subgroup by translation on the whole group. See Proposition 2.18 below.

It follows from Theorem 9.24(4) that minimality is a necessary condition for simplicity of $C^*_{\mathrm{r}}(G, X)$, and from Theorem 8.32 that minimality is a necessary condition for simplicity of $C^*(G, X)$. (As we will see in Section 9, $C^*_{\mathrm{r}}(G, X)$ is a quotient of $C^*(G, X)$, so we really only need to cite Theorem 9.24(4).)

Definition 2.3. Let a locally compact group G act continuously on a locally compact space X. The action is called *free* if whenever $g \in G \setminus \{1\}$ and $x \in X$, then $gx \neq x$. The action is called *essentially free* if whenever $g \in G \setminus \{1\}$, the set $\{x \in X \colon gx = x\}$ has empty interior.

Essential freeness makes sense in general, but for nonminimal actions it is not the most useful condition. One should at least insist that the restriction of the action to any closed invariant subset be essentially free in the sense of Definition 2.3. The action of $\mathbb{Z}$ on its one point compactification by translation is essentially free in the sense of Definition 2.3, but but does not satisfy the stronger condition, and its transformation group C*-algebra does not behave the way that a good version of essential freeness for nonminimal actions should imply.

Proposition 2.4. Let G be an abelian group. Then every minimal and essentially free action of G on a topological space X is free.

Proof. Let $(g, x) \mapsto gx$ be a minimal action of G which is not free. Then there is $h \in G \setminus \{1\}$ such that the closed set $T = \{x \in X \colon hx = x\}$ is not empty. We claim that T is invariant. To see this, let $g \in G$ and let $x \in T$. Then $h(gx) = g(hx) = gx$, so $gx \in T$. This proves the claim. By minimality, $T = X$. Therefore the action is not essentially free. $\square$

The actions in Example 2.35 and Example 2.38 below are minimal and essentially free but not free.

The following theorem (to be proved in Section 15) provides a very useful sufficient condition for simplicity of $C^*_{\mathrm{r}}(G, X)$.

Theorem 2.5 (Theorem 15.10)**.** Let a discrete group G act minimally and essentially freely on a locally compact space X. Then $C^*_{\mathrm{r}}(G, X)$ is simple.

This condition is not necessary; it follows from Theorem 6.6 that the trivial action of the free group on two generators on a one point space has a simple reduced crossed product.

The analog of minimality for actions on measure spaces is ergodicity.

Definition 2.6. Let $(X, \mathcal{B}, \mu)$ be a measure space, let G be a group, and let $(g, x) \mapsto gx$ be an action of G on X. For each $g \in G$, assume that the map $h^g \colon X \to X$, given by $h^g(x) = gx$, is measurable and preserves the measure μ. We say that the action is *ergodic* if whenever a measurable set $E \subset X$ satisfies $gE = E$ for all $g \in G$, then $\mu(E) = 0$ or $\mu(X \setminus E) = 0$.

The conditions on the action are just that the σ-algebra $\mathcal{B}$ and the measure μ are both G-invariant. That is, for all $g \in G$ and all $E \in \mathcal{B}$, we have $gE \in \mathcal{B}$ and $\mu(gE) = \mu(E)$. (Actually, all that one needs is that the measure class of μ is G-invariant, that is, that $\mu(gE) = 0$ if and only if $\mu(E) = 0$ for $E \in \mathcal{B}$ and $g \in G$.)

Definition 2.7. Let X be a compact metric space, let G be a topological group, and let $(g, x) \mapsto gx$ be an action of G on X. We say that the action is *uniquely ergodic* if there is a unique G-invariant Borel probability measure on X.

In Definition 2.7, it turns out that the measure μ is necessarily ergodic. More generally, the G-invariant Borel probability measures on X form a (possibly empty) weak* compact convex subset K of the dual space $C(X)^*$. We prove the standard result that such a measure μ is ergodic if and only if it is an extreme point of K, under the assumption that the group is discrete and countable. This hypothesis is stronger than necessary, but avoids some technicalities. The proof that extreme points are ergodic measures works in complete generality, in particular, no matter what the group is.

Theorem 2.8. Let X be a compact metric space, let G be a countable discrete group, and let $(g, x) \mapsto gx$ be an action of G on X. Then a G-invariant Borel probability measure μ on X is ergodic if and only if it is an extreme point in the set of all G-invariant Borel probability measures on X.

Proof. First assume that μ is not ergodic. Choose a G-invariant Borel set $F \subset X$ such that $0 < \mu(F) < 1$. Define G-invariant Borel probability measures μ_1 and μ_2 on X by

$$\mu_1(E) = \frac{\mu(E \cap F)}{\mu(F)} \qquad \text{and} \qquad \mu_2(E) = \frac{\mu(E \cap (X \setminus F))}{\mu(X \setminus F)}$$

for every Borel set $E \subset X$. Taking $\alpha = \mu(F)$, we have $\alpha\mu_1 + (1 - \alpha)\mu_2 = \mu$, $\mu_1 \neq \mu_2$, and $\alpha \in (0, 1)$. So μ is not an extreme point.

Now assume that μ is ergodic. Suppose that μ_1 and μ_2 are G-invariant Borel probability measures, that $\alpha \in (0, 1)$, and that $\alpha\mu_1 + (1-\alpha)\mu_2 = \mu$. We prove that $\mu_1 = \mu$. We have $\mu_1 \leq \alpha^{-1}\mu$, so $\mu_1 \ll \mu$. Let $f_0 \colon X \to [0, \infty]$ be a Radon-Nikodym derivative of μ_1 with respect to μ. Since μ_1 and μ are G-invariant, for every $g \in G$ the function $x \mapsto f_0(g^{-1}x)$ is also a Radon-Nikodym derivative of μ_1 with respect to μ, and is therefore equal to $f_0(x)$ almost everywhere with respect to μ. Now define

$$f(x) = \sup_{g \in G} f_0(g^{-1}x).$$

Since G is countable, this function is equal to $f_0(x)$ almost everywhere with respect to μ, so

$$\mu_1(E) = \int_E f \, d\mu$$

for every Borel set $E \subset X$. Also, f is exactly G-invariant.

For $\beta \in [0, \infty)$ set

$$E_\beta = \big\{ x \in X \colon f(x) \geq \beta \big\}.$$

Then E_β is a G-invariant Borel set, so $\mu(E_\beta) \in \{0, 1\}$. Whenever $\beta, \gamma \in [0, \infty)$ satisfy $\gamma \geq \beta$, we have $E_\gamma \subset E_\beta$, so $\mu(E_\beta) \geq \mu(E_\gamma)$. Also, $\mu(E_0) = 1$ and $\mu(E_{\alpha^{-1}+1}) = 0$. Define

$$r = \sup\big(\big\{ \beta \in [0, \infty) \colon \mu(E_\beta) = 1 \big\}\big).$$

If $r = 0$ then $f = 0$ so $\mu_1 = 0$, which is clearly impossible. So there is a strictly increasing sequence $(\beta_n)_{n \in \mathbb{Z}_{>0}}$ in $[0, \infty)$ such that $\lim_{n \to \infty} \beta_n = r$. We have

$$X \setminus E_r = \bigcup_{n=1}^{\infty} (X \setminus E_{\beta_n}),$$

so $\mu(E_r) = 1$. It follows that f is equal to the constant function r almost everywhere with respect to μ. Since $\mu_1(X) = 1$, we get $r = 1$. So $\mu_1 = \mu$, as desired. ☐

Theorem 2.9. Let X be a compact metric space, let G be an amenable locally compact group, and let $(g, x) \mapsto gx$ be an action of G on X. Then there exists a G-invariant Borel probability measure on X.

See the discussion before Theorem 5.50 for more on amenable groups.

Proof. In [100], combine Theorem 3.3.1 and Theorem 2.2.1. ☐

Corollary 2.10. Let X be a compact metric space, let G be an amenable locally compact group, and let $(g, x) \mapsto gx$ be an action of G on X. Then there exists an ergodic G-invariant Borel probability measure μ on X.

Proof. Theorem 2.9 shows that the set of G-invariant Borel probability measures on X is not empty. It is easily seen to be a weak* compact convex subset of the dual space of $C(X)$. Therefore it has an extreme point, by Alaoglu's Theorem. Any extreme point is an ergodic measure by Theorem 2.8. (This direction of the proof of Theorem 2.8 did not need any hypotheses on the group.) ☐

Part of the significance of G-invariant Borel probability measures is that, when G is discrete, they give tracial states (Definition 11.23) on the crossed product C*-algebra. See Example 11.31. Moreover, if the action is free, then sometimes all tracial states on the crossed product arise this way. See Theorem 15.22.

Now we give examples.

Example 2.11. The group G is arbitrary locally compact, the space X consists of just one point, and the action is trivial. This action is minimal, but is as far from being free as possible. It gives the trivial action of G on the C*-algebra $\mathbb{C}$. The full and reduced crossed products are the usual full and reduced group C*-algebras

$C^*(G)$ and $C^*_{\mathrm{r}}(G)$, discussed in Section 5 (when G is discrete) and Section 7. As we will see, this is essentially immediate by comparing definitions. See Example 10.1 below.

More generally, any group has a trivial action on any space.

Example 2.12. The group G is arbitrary locally compact, $X = G$, and the action is given by the group operation: $g \cdot x = gx$. The full and reduced crossed products are both isomorphic to $K(L^2(G))$. We will prove this for the discrete case in Example 10.8 below.

This action is called (left) translation. It is clearly free. It is also minimal, but in a rather trivial way: there are no nontrivial invariant subsets, closed or not. As we will see, in the interesting examples, with more interesting crossed products, the orbits are dense but not equal to the whole space. See the irrational rotations in Example 2.16. Also see Proposition 2.18, Example 2.19, Example 2.21, Definition 2.22, and Example 2.24. Many further examples will appear.

More generally, if $H \subset G$ is a closed subgroup, then G acts continuously on G/H by translation. Example 2.11 is the case $H = G$. See Example 10.11 below for the computation of the crossed product when $G = \mathbb{Z}$ and $H = n\mathbb{Z}$, and for the description of the crossed product in the general case. This action is still minimal (in the same trivial way as before), but for $H \neq \{1\}$ it is no longer free.

Example 2.13. We can generalize left translation in Example 2.12 in a different way. Again let G be an arbitrary locally compact group, set $X = G$, and let $H \subset G$ be a closed subgroup. Then H acts on $X = G$ by left translation. The action is still free, but is now no longer minimal (unless $H = G$).

The crossed product $C^*(H, G)$ turns out to be stably isomorphic to $K(L^2(H)) \otimes C_0(G/H)$. Stably, there is no "twisting", even though G may be a nontrivial bundle over G/H. See Theorem 14 and Corollary 15 in Section 3 of [98].

Example 2.14. Let G be any locally compact group. Then G acts on itself by conjugation: $g \cdot k = gkg^{-1}$ for $g, k \in G$. Unless $G = \{1\}$, this action is neither free nor minimal, since 1 is a fixed point.

There is also a conjugation action of G on any normal subgroup of G.

Example 2.15. Let $G = \mathbb{Z}$ and let $X = \mathbb{Z}^+$, the one point compactification $\mathbb{Z} \cup \{\infty\}$ of $\mathbb{Z}$. Then $\mathbb{Z}$ acts on $\mathbb{Z}^+$ by translation, fixing ∞. This action has a dense orbit (namely $\mathbb{Z}$), but is not minimal (since $\{\infty\}$ is invariant) and not free.

Here are some related examples. The group $\mathbb{Z}$ acts on $\mathbb{Z} \cup \{-\infty, \infty\}$ by translation, fixing $-\infty$ and ∞. Both $\mathbb{R}$ and $\mathbb{Z} \subset \mathbb{R}$ act on both $\mathbb{R}^+ \cong S^1$ and $[-\infty, \infty]$ by translation, fixing the point or points at infinity. None of these actions is either free or minimal.

Example 2.16. Take $X = S^1 = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\}$. Taking $G = S^1$, acting by translation, gives a special case of Example 2.12. But we can also take G to be the finite subgroup of S^1 of order n generated by $\exp(2\pi i/n)$, still acting by translation (in this case, usually called rotation). This is a special case of Example 2.13. The computation of the crossed product for this case is in Example 10.9. Or we can fix $\theta \in \mathbb{R}$, and take $G = \mathbb{Z}$, with $n \in \mathbb{Z}$ acting by $\zeta \mapsto \exp(2\pi i n\theta)\zeta$. (The use of $\exp(2\pi i\theta)$ rather than $\exp(i\theta)$ is standard here.) These are *rational rotations* (for $\theta \in \mathbb{Q}$) or *irrational rotations* (for $\theta \not\in \mathbb{Q}$). The rational rotations are neither free nor minimal. (Their crossed products are discussed in Example 10.16.) The irrational rotations are free (easy) and minimal (intuitively clear but slightly tricky; see Lemma 2.17 and Proposition 2.18 below). Irrational rotations are also uniquely ergodic. (Theorem 1.1 of [90] gives unique ergodicity for a class of homeomorphisms of the circle which contains the irrational rotations.)

Lemma 2.17. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then $\left\{ e^{2\pi i n\theta}\zeta \colon n \in \mathbb{Z} \right\}$ is dense in S^1.

There are a number of ways to prove this lemma. In [292] (see the proof of Lemma 3.29), the basic idea is that all proper closed subgroups of S^1 are finite. One can also get the result from number theory: there is a constant c such that there are pairs (p, q) of integers, with q arbitrarily large, such that $|\theta - p/q| < cq^{-2}$. (See Corollary 1B of [250]. The best general constant is $1/\sqrt{5}$; see Theorem 2F of [250]. We thank Shabnam Akhtari for pointing out this reference.) We give here a proof close to that of [292].

Proof of Lemma 2.17. It suffices to prove that $\mathbb{Z} + \theta\mathbb{Z}$ is dense in $\mathbb{R}$. Suppose not. Let $t = \inf\left(\left\{ x \in \overline{\mathbb{Z} + \theta\mathbb{Z}} \colon x > 0 \right\}\right)$. We will show that $t = 0$. So suppose $t > 0$.

We claim that $\overline{\mathbb{Z} + \theta\mathbb{Z}} = \mathbb{Z}t$. First, $\overline{\mathbb{Z} + \theta\mathbb{Z}}$ is clearly a subgroup of $\mathbb{R}$. So $\mathbb{Z}t \subset \overline{\mathbb{Z} + \theta\mathbb{Z}}$. Suppose the reverse inclusion is false. Then there are $m \in \mathbb{Z}$ and $r \in \overline{\mathbb{Z} + \theta\mathbb{Z}}$ such that $mt < r < (m+1)t$. But then $r - mt \in \overline{\mathbb{Z} + \theta\mathbb{Z}} \cap (0, t)$. This contradiction proves the claim.

It is clear that the only subset of $\mathbb{R}$ whose closure is $\mathbb{Z}t$ is $\mathbb{Z}t$ itself. So $\mathbb{Z}+\theta\mathbb{Z} = \mathbb{Z}t$. Therefore there are $m, n \in \mathbb{Z}$ with $\theta = mt$ and $1 = nt$. So $n \neq 0$ and $\theta = \frac{m}{n} \in \mathbb{Q}$. This contradiction shows that $t = 0$.

Now let $r \in \mathbb{R}$. We claim that $r \in \overline{\mathbb{Z} + \theta\mathbb{Z}}$. Let $\varepsilon > 0$. Choose $s \in \overline{\mathbb{Z} + \theta\mathbb{Z}}$ such that $0 < s < \varepsilon$. Choose $n \in \mathbb{Z}$ such that $ns \leq r < (n+1)s$. Then $ns \in \overline{\mathbb{Z} + \theta\mathbb{Z}}$ and $|r - ns| < s < \varepsilon$. So the closure of $\overline{\mathbb{Z} + \theta\mathbb{Z}}$ contains r. The claim follows. □

Here is a second proof, based on part of a lecture by David Kerr. Again, it suffices to prove that $\mathbb{Z} + \theta\mathbb{Z}$ is dense in $\mathbb{R}$. Suppose this fails. Choose $\lambda_1, \lambda_2 \in \mathbb{R}$ such that (λ_1, λ_2) is a connected component of $\mathbb{R} \setminus \overline{\mathbb{Z} + \theta\mathbb{Z}}$. Let F be the image of $\mathbb{Z} + \theta\mathbb{Z}$ in $\mathbb{R}/\mathbb{Z}$, which we identify with S^1. Since θ is irrational, F is infinite, so that for every $\varepsilon > 0$ there are distinct points in F whose arc length distance is less than ε. Equivalently, for every $\varepsilon > 0$ there are $r, s \in \mathbb{Z}+\theta\mathbb{Z}$ such that $0 < s - r < \varepsilon$. Choose such numbers r and s for $\varepsilon = \lambda_2 - \lambda_1$. We have $\lambda_1 \in \overline{\mathbb{Z} + \theta\mathbb{Z}}$, so there is $t \in \mathbb{Z} + \theta\mathbb{Z}$ such that $|t - \lambda_1| < s - r$. Since $t \notin (\lambda_1, \lambda_2)$ and $s - r < \lambda_2 - \lambda_1$, we have $\lambda_1 - (s - r) < t \leq \lambda_1$. Therefore

$$\lambda_1 < t + (s - r) < t + \lambda_2 - \lambda_1 \leq \lambda_2.$$

It follows that $t + (s - r) \in (\mathbb{Z} + \theta\mathbb{Z}) \cap (\lambda_1, \lambda_2)$, which is a contradiction.

Proposition 2.18. Let G be a topological group, and let $H \subset G$ be a dense subgroup. Then the action of H on G be left translation (in which $h \cdot g$ is just the group product hg for $h \in H$ and $g \in G$) is a free minimal action of H on G.

Proof. That this formula defines an action is obvious, as is freeness. For minimality, let $T \subset G$ be a nonempty closed H-invariant subset. Choose $g_0 \in T$. Then $Hg_0 \subset T$. Moreover, H is dense in G and right multiplication by g_0 is a homeomorphism, so Hg_0 is dense in G. Therefore $T = G$. □

Example 2.19. Let $\gamma \in \mathbb{R}$, let $d \in \mathbb{Z}$, and let $f \colon S^1 \to \mathbb{R}$ be continuous. The associated Furstenberg transformation $h_{\gamma,d,f} \colon S^1 \times S^1 \to S^1 \times S^1$ (introduced in Section 2 of [90]) is defined by

$$h_{\gamma,d,f}(\zeta_1, \zeta_2) = \left(e^{2\pi i \gamma}\zeta_1, \, \exp(2\pi i f(\zeta_1))\zeta_1^d \zeta_2\right)$$

for $\zeta_1, \zeta_2 \in S^1$. The inverse is given by

$$h_{\gamma,d,f}(\zeta_1, \zeta_2) = \left(e^{-2\pi i \gamma}\zeta_1, \, \exp\left(2\pi i [d\gamma - f(e^{-2\pi i \gamma}\zeta_1)]\right)\zeta_1^{-d} \zeta_2\right)$$

for $\zeta_1, \zeta_2 \in S^1$. If $\gamma \notin \mathbb{Q}$ and $d \neq 0$, Furstenberg proved that $h_{\gamma,d,f}$ is minimal. (See the discussion after Theorem 2.1 of [90].) By Theorem 2.1 of [90], if f is in addition smooth (weaker conditions suffice), then $h_{\gamma,d,f}$ is uniquely ergodic. For arbitrary continuous f, Theorem 2 in Section 4 of [121] shows that $h_{\gamma,d,f}$ need not be uniquely ergodic.

These homeomorphisms, and higher dimensional analogs (which also appear in [90]), have attracted significant interest in operator algebras. See, for example, [189], [129], [146], and [237]. The higher dimensional version has the general form

$$(\zeta_1, \zeta_2, \ldots, \zeta_n) \mapsto \left(e^{2\pi i \gamma}\zeta_1, \, g_2(\zeta_1)\zeta_2, \, g_3(\zeta_1, \zeta_2)\zeta_3, \, \ldots, \, g_n(\zeta_1, \zeta_2, \ldots, \zeta_{n-1})\zeta_n\right)$$

for fixed $\gamma \in \mathbb{R}$ and continuous functions

$$g_2 \colon S^1 \to S^1, \qquad g_3 \colon S^1 \times S^1 \to S^1, \qquad \ldots, \qquad g_n \colon (S^1)^{n-1} \to S^1.$$

There are further generalizations, called skew products. Furstenberg transformations and their generalizations have also attracted interest in parts of dynamics not related to C*-algebras; as just two examples, we mention [121] and [248].

Examples 2.35 and 2.36 are related but more complicated. "Noncommutative" Furstenberg transformations (Furstenberg transformations on noncommutative analogs of $S^1 \times S^1$) are given in Example 3.18.

Example 2.20. Take $X = \{0,1\}^{\mathbb{Z}}$, with elements being described as $x = (x_n)_{n \in \mathbb{Z}}$ with $x_n \in \{0,1\}$ for all $n \in \mathbb{Z}$. (This space is homeomorphic to the Cantor set.) Take $G = \mathbb{Z}$, with action generated by the *shift* homeomorphism $h(x)_n = x_{n+1}$ for $x \in X$ and $n \in \mathbb{Z}$. This action is neither free nor minimal; in fact, it has fixed points.

One can replace $\{0,1\}$ by some other compact metric space K. (See Definition 23.9.) Further examples ("subshifts") can be gotten by restricting to closed invariant subsets of X. Some of these are minimal. For example, substitution minimal systems and Toeplitz flows (mentioned after Example 2.21) can be obtained this way, using a general finite set in place of $\{0,1\}$.

Example 2.21. Fix a prime p, and let $X = \mathbb{Z}_p$, the group of p-adic integers. This group can be defined as the completion of $\mathbb{Z}$ in the metric $d(m,n) = p^{-d}$ when p^d is the largest power of p which divides $n - m$. Alternatively, it is $\varprojlim \mathbb{Z}/p^d\mathbb{Z}$. It is a compact topological group, and as a metric space it is homeomorphic to the Cantor set. Let $h \colon X \to X$ be the homeomorphism defined using the group operation in the completion by $h(x) = x + 1$ for $x \in X$. The resulting action is free and minimal by Proposition 2.18.

Next, we consider odometers. They are a generalization of Example 2.21. See Example (i) on page 210 of [258], Section VIII.4 of [52], and the first example in Section 2 of [229]. We refer to these sources for more information, including minimality.

Definition 2.22. Let $d = (d_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $\mathbb{Z}_{>0}$ with $d_n \geq 2$ for all $n \in \mathbb{Z}_{>0}$. The *d-odometer* is the minimal system (X_d, h_d) defined as follows. Set

$$X_d = \prod_{n=1}^{\infty} \{0, 1, 2, \ldots, d_n - 1\},$$

which is homeomorphic to the Cantor set. For $x = (x_n)_{n \in \mathbb{Z}_{>0}} \in X_d$, let

$$n_0 = \inf\big(\{n \in \mathbb{Z}_{>0} \colon x_n \neq d_n - 1\}\big).$$

If $n_0 = \infty$ set $h_d(x) = (0, 0, \ldots)$. Otherwise, $h_d(x) = (h_d(x)_n)_{n \in \mathbb{Z}_{>0}}$ is

$$h_d(x)_n = \begin{cases} 0 & n < n_0 \\ x_n + 1 & n = n_0 \\ x_n & n > n_0. \end{cases}$$

The homeomorphism is "addition of $(1, 0, 0, \ldots)$ with carry to the right". When $n_0 \neq \infty$, we have

$$h(x) = \big(0, 0, \ldots, 0, x_{n_0} + 1, x_{n_0+1}, x_{n_0+2}, \ldots\big).$$

Exercise 2.23. Prove that the odometer homeomorphism of Definition 2.22 is minimal.

See Theorem VIII.4.1 of [52] for the computation of the crossed product by an odometer action.

There are many other classes of interesting minimal homeomorphisms of the Cantor set, such as substitution minimal systems (Section 5 of [258]), Toeplitz flows (Section 6 of [258]), topological versions of interval exchange transformations (the second example in Section 2 of [229]), and restrictions to their minimal sets of Denjoy homeomorphisms, which are nonminimal homeomorphisms of the circle whose rotation numbers are irrational ([234]). The relation of strong orbit equivalence of minimal homeomorphisms of the Cantor set is defined in [94], where it is shown to be equivalent to isomorphism of the transformation group C*-algebras. Sugisaki has shown ([265], [266], and [267]) that all possible values of entropy in $[0, \infty]$ occur in all strong orbit equivalence classes of minimal homeomorphisms of the Cantor set.

One can make various other kinds of examples of free minimal actions using Proposition 2.18. Here is one such example.

Example 2.24. Let $k_1, k_2, \ldots \in \{2, 3, \ldots\}$. Set $X = \prod_{n=1}^{\infty} \mathbb{Z}/k_n\mathbb{Z}$, which is a compact group. Take $G = \bigoplus_{n=1}^{\infty} \mathbb{Z}/k_n\mathbb{Z}$, which is a dense subgroup of X. Give G the discrete topology, so that G becomes a locally compact group. Then the action of G on X by left translation is free and minimal, by Proposition 2.18. The crossed product turns out to be the UHF algebra $\bigotimes_{n=1}^{\infty} M_{k_n}$. See Exercise 10.29.

Example 2.25. The locally compact (but noncompact) Cantor set X is a metrizable totally disconnected locally compact space with no isolated points and which is not compact. This description determines it uniquely up to homeomorphism, by Proposition 2.1 of [51]. Minimal homeomorphisms of X have been studied in [51] and [166]. Section 3 of [51] contains a good sized collection of easy to construct examples, although the construction is slightly more complicated than we want to present here. The comment after Theorem 2.11 of [166] proves the existence of a much larger class of examples.

Example 2.26. For each minimal homeomorphism $h_0 \colon X_0 \to X_0$ of the Cantor set X_0, Gjerde and Johansen construct in [96] a minimal homeomorphism $h \colon X \to X$ of a compact metric space X which has $h_0 \colon X_0 \to X_0$ as a factor (see Definition 2.27 below), and in which some of the connected components of X are points (as for the Cantor set) but some are compact intervals. Among other things, these examples show that if $h \colon X \to X$ is a minimal homeomorphism, then the space X need not be "homogeneous": different points can give different local properties of the space, and, in particular, for $x, y \in X$ there need not be a homeomorphism from any neighborhood of x to any neighborhood of y which sends x to y.

Definition 2.27. Let G be a group, let X and Y be compact Hausdorff spaces, and assume G acts continuously on X and Y. We say that the dynamical system (G, Y) is a *factor* of the dynamical system (G, X) if there is a a surjective continuous map $f \colon X \to Y$ (the *factor map*) such that $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$.

If we take $G = \mathbb{Z}$, then the actions are given by homeomorphisms $h \colon X \to X$ and $k \colon Y \to Y$. Then we are supposed to have a surjective continuous map $f \colon X \to Y$ such that $g \circ h = k \circ g$. That is, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{k} & Y. \end{array}$$

In general, there should be such a diagram for the action of every group element $g \in G$ (always using the same choice of f).

Essentially, (G, Y) is supposed to be a topological quotient of (G, X). Without compactness, presumably one should ask that f be a quotient map of topological spaces.

Example 2.28. Take $X = S^n = \{x \in \mathbb{R}^{n+1} \colon \|x\|_2 = 1\}$. Then the homeomorphism $x \mapsto -x$ has order 2, and so gives an action of $\mathbb{Z}/2\mathbb{Z}$ on S^n. This action is free but is far from minimal. See Example 10.10 below for a description of the crossed product (without proof).

Example 2.29. Take $X = S^1 = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\}$, and consider the order 2 homeomorphism $\zeta \mapsto \overline{\zeta}$. We get an action of $\mathbb{Z}/2\mathbb{Z}$ on S^1. This action is neither free nor minimal. See Example 10.18 below for the computation of the crossed product.

Example 2.30. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $S^1 \times S^1$ as follows. For

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

let n act on $\mathbb{R}^2$ via the usual matrix multiplication. Since n has integer entries, one gets $n\mathbb{Z}^2 \subset \mathbb{Z}^2$, and thus the action is well defined on $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$.

Similarly, $\mathrm{SL}_d(\mathbb{Z})$ acts on $(S^1)^d$.

In fact, the larger group $\mathrm{GL}_2(\mathbb{Z})$ acts on $S^1 \times S^1$ in the same way, and the larger group $\mathrm{GL}_d(\mathbb{Z})$ acts on $(S^1)^d$ in the same way. We have emphasized the action of $\mathrm{SL}_2(\mathbb{Z})$ because it extends much more easily to noncommutative deformations. See Example 3.12.

These actions are neither free nor minimal, because the image of $0 \in \mathbb{R}^2$ (or $\mathbb{R}^d$) is a fixed point.

Example 2.31. Let G be the symmetric group S_n, consisting of all permutations of $\{1, 2, \ldots, n\}$. Let X be any compact metric space. Let S_n act on X^n by permuting the coordinates:

$$\sigma \cdot (x_1, x_2, \cdot, x_n) = \big(x_{\sigma^{-1}(1)}, \, x_{\sigma^{-1}(2)}, \, \ldots, \, x_{\sigma^{-1}(n)}\big).$$

(One must use σ^{-1} in the formula in order to get $\sigma \cdot (\tau \cdot x) = (\sigma\tau) \cdot x$ rather than $(\tau\sigma) \cdot x$.)

These actions are not free. Unless X has only one point, they are also far from minimal.

Example 2.32. The unitary group $U(M_n)$ of the $n \times n$ matrices acts on the unit sphere $S^{2n-1} \subset \mathbb{C}^n$, since S^{2n-1} is invariant under the action of $U(M_n)$ on $\mathbb{C}^n$. This is actually a special case of Example 2.12, gotten by taking $G = U(M_n)$ and $H = U(M_{n-1})$, embedded as a closed subgroup of G via the map $h \mapsto \left(\begin{smallmatrix} h & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The action is thus minimal in a trivial way, but not free.

Restricting to the scalar multiples of the identity, we get an action of S^1 on S^{2n-1}. This action is free but not minimal.

Similarly, $U(M_n)$ and S^1 act on the closed unit ball in $\mathbb{C}^n$.

Example 2.33. Let Z be a compact manifold, or a connected finite complex. (Much weaker conditions on Z suffice, but Z must be path connected.) Let $X = \widetilde{Z}$ be the universal cover of Z, and let $G = \pi_1(Z)$ be the fundamental group of Z. Then there is a standard action of G on X. The space X is locally compact when Z is locally compact, and compact when Z is compact and $\pi_1(Z)$ is finite.

Spaces with finite fundamental groups include real projective spaces (in which case this example is really just Example 2.28) and lens spaces (Example 2.43 of [106]). In Example 1.43 of [106], there is some discussion of spaces with nonabelian finite fundamental groups whose universal covers are spheres, equivalently, free actions of nonabelian finite groups on spheres.

There are also many spaces with interesting infinite fundamental group. Any (discrete) group G is the fundamental group of a two dimensional CW complex X (Corollary 1.28 of [106]), and (as is clear from the proof), if G is finitely presented then X can be taken to be a finite complex.

These actions are all free but are far from minimal.

One can get free minimal actions of $\mathbb{Z}^2$ on compact metric spaces by letting $h_1 \colon X_1 \to X_1$ and $h_2 \colon X_2 \to X_2$ be minimal homeomorphisms of infinite compact metric spaces, setting $X = X_1 \times X_2$, letting one generator of $\mathbb{Z}^2$ act on X via $h_1 \times \mathrm{id}_{X_2}$, and letting the other generator of $\mathbb{Z}^2$ act on X via $\mathrm{id}_{X_1} \times h_2$. A few other examples are known, but examples seem to be hard to find. Here is one, taken from [172].

Example 2.34 (Item 2 on page 311 of [172])**.** Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then the homeomorphisms $h_1, h_2 \colon (S^1)^3 \to (S^1)^3$ (called α_1 and α_2 in [172]) determined by

$$h_1(\zeta_1, \zeta_2, \zeta_3) = (\zeta_1, \, e^{2\pi i\theta}\zeta_2, \, \zeta_1\zeta_3) \qquad \text{and} \qquad h_2(\zeta_1, \zeta_2, \zeta_3) = (e^{2\pi i\theta}\zeta_1, \, \zeta_2, \, \zeta_2\zeta_3)$$

for $\zeta_1, \zeta_2, \zeta_3 \in S^1$, commute and generate a free minimal action of $\mathbb{Z}^2$ on $(S^1)^3$.

The next two examples have some similarity with Example 2.19, but are more complicated.

Example 2.35. Let H be the discrete Heisenberg group, that is,

$$H = \left\{ \begin{pmatrix} 1 & n & k \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} : k, m, n \in \mathbb{Z} \right\}.$$

Equivalently, $H = \mathbb{Z}^3$ as a set, and the group operation is

$$(k_1, m_1, n_1)(k_2, m_2, n_2) = (k_1 + k_2 + n_1 m_2, m_1 + m_2, n_1 + n_2)$$

for $k_1, m_1, n_1, k_2, m_2, n_2 \in \mathbb{Z}$. This formula comes from the assignment

$$(k, m, n) \mapsto \begin{pmatrix} 1 & n & k \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix}$$

for $k, m, n \in \mathbb{Z}$. The proof of Theorem 1 of [172] uses a minimal action of H on $(S^1)^2$ which depends on a parameter $\theta \in \mathbb{R} \setminus \mathbb{Q}$. It is given by

$$(k, m, n) \cdot (\zeta_1, \zeta_2) = \left(e^{-2\pi i n \theta} \zeta_1, \, e^{2\pi i (mn - k)\theta} \zeta_1^{-m} \zeta_2 \right)$$

for $k, m, n \in \mathbb{Z}$ and $\zeta_1, \zeta_2 \in S^1$. This action is not free. For example,

$$(-1, 1, 0) \cdot (e^{2\pi i \theta}, \, 1) = (e^{2\pi i \theta}, \, 1).$$

However, it is essentially free.

Example 2.36. Let H be the discrete Heisenberg group, as in Example 2.35. The proofs of Theorem 2 and Theorem 4 of [173] use free minimal actions of H on $(S^1)^3$ which depend on a parameter $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and (for Theorem 2) on relatively prime integers p and q. The action used in Theorem 2 of [173] is given by

$$(k, m, n) \cdot (\zeta_1, \zeta_2, \zeta_3) = \left(e^{2\pi i q m \theta} \zeta_1, \, e^{2\pi i p n \theta} \zeta_2, \, e^{2\pi i [(p+q)k - qmn]\theta} \zeta_1^{-n} \zeta_2^{m} \zeta_3 \right)$$

for $k, m, n \in \mathbb{Z}$ and $\zeta_1, \zeta_2, \zeta_3 \in S^1$. The action used in Theorem 4 of [173] is given by

$$(k, m, n) \cdot (\zeta_1, \zeta_2, \zeta_3) = \left(e^{2\pi i m \theta} \zeta_1, \, e^{2\pi i (m+n)\theta} \zeta_2, \, e^{2\pi i [2k - mn + m(m-1)/2]\theta} \zeta_1^{-n} \zeta_2^{m} \zeta_3 \right)$$

for $k, m, n \in \mathbb{Z}$ and $\zeta_1, \zeta_2, \zeta_3 \in S^1$.

Example 2.37. Let G be a discrete group. Then the action of G on itself by translation (Example 2.12) extends to an action of G on the Stone-Čech compactification βG of G, and thus to an action of G on the remainder $\beta G \setminus G$.

Example 2.38. Let $n \in \{2, 3, \ldots\}$. The Gromov boundary ∂F_n of F_n consists of all right infinite reduced words in the generators and their inverses, with the topology (given in detail below) in which two words are close if they have the same long finite initial segment. The group F_n acts on it by left translation. We claim that this action is minimal and essentially free, but not free.

Call the standard generators $g_1, g_2, \ldots, g_n$. Set

$$S = \left\{ g_1, g_1^{-1}, g_2, g_2^{-1}, \ldots, g_n, g_n^{-1} \right\},$$

with the discrete topology. Then ∂F_n is the subset of the compact set $S^{\mathbb{Z}_{>0}}$ consisting of those sequences $x = (x_1, x_2, \ldots) \in S^{\mathbb{Z}_{>0}}$ such that $x_{n+1} \neq x_n^{-1}$ for all $n \in \mathbb{Z}_{\geq 0}$. This set is an intersection of closed sets, hence compact.

The element $x = g_1 \cdot g_1 \cdot \cdots$ is a right infinite word such that $g_1 x = x$. Thus the action is not free. (More generally, if $h \in F_n \setminus \{1\}$ is arbitrary, then the reduced form of $h \cdot h \cdot \cdots$ is in ∂F_n and is a fixed point for h.)

We show that the action is minimal. We use Lemma 2.2. Let $x, y \in \partial F_n$. Use sequence notation as above. It suffices to show that for every $n \in \mathbb{Z}_{\geq 0}$ there is $g \in F_n$ such that $(gx)_k = y_k$ for $k = 1, 2, \ldots, n$. Let $g_0 = y_1 y_2 \cdots y_n \in F_n$. Choose $h \in S$ such that $h \notin \{y_n^{-1}, x_1^{-1}\}$. Then $z = (y_1, y_2, \ldots, y_n, h, x_1, x_2, \ldots)$ is a right infinite reduced word which agrees with y in positions $1, 2, \ldots, n$. Moreover, with $g = g_0 h$, we get $gx = z$. This completes the proof of minimality.

It remains to show that the action is essentially free. By Definition 2.3, it suffices to show that if $h \in F_n \setminus \{1\}$ and $x \in \partial F_n$, there is $y \in \partial F_n$ such that $y_j = x_j$ for $j = 1, 2, \ldots, n$ and such that $hy \neq y$. If $hx \neq x$, there is nothing to prove. So suppose $hx = x$. Write h as a reduced word $h = h_1 h_2 \cdots h_l$ with $h_1, h_2, \ldots, h_l \in S$. There is $k \in \{0, 1, \ldots, l\}$ such that, in reduced form, we have

$$hx = \big(h_1,\, h_2,\, \ldots,\, h_k,\, x_{l-k+1},\, x_{l-k+2},\, \ldots\big). \tag{2.1}$$

That is, hx is one of

$$\big(h_1,\, h_2,\, \ldots,\, h_l,\, x_1,\, x_2, \ldots\big), \quad \big(h_1,\, h_2,\, \ldots,\, h_{l-1},\, x_2,\, x_3, \ldots\big),$$
$$\big(h_1,\, h_2,\, \ldots,\, h_{l-2},\, x_3,\, x_4, \ldots\big), \quad \ldots, \quad \big(x_{l+1},\, x_{l+2},\, x_{l+3}, \ldots\big).$$

We claim that $l \neq 2k$. (This means that passing from x to hx actually shifts the sequence x, so that x is eventually periodic.) Suppose that $l = 2k$. The cancellations which occur to make the formula for hx correct imply that

$$h_l = x_1^{-1}, \qquad h_{l-1} = x_2^{-1}, \qquad \ldots, \qquad h_{l-k+1} = x_k^{-1}.$$

Looking at the first k positions of the equation $hx = x$, we get

$$h_1 = x_1, \qquad h_2 = x_2, \qquad \ldots, \qquad h_k = x_k.$$

Combine these (in the opposite order) and use $l - k = k$ to get

$$h_{k+1} = h_k^{-1}, \qquad h_{k+2} = h_{k-1}^{-1}, \qquad \ldots, \qquad h_1 = h_l^{-1}.$$

Therefore $h = 1$. This is a contradiction, and the claim follows.

By the definition of k, we have $(hx)_j = x_{j-2k+l}$ for $j = k+1,\, k+2, \ldots$. Set $m = n + l + 1$. Choose $y_m \in S \setminus \{x_{m-1}^{-1}, x_{m+1}^{-1}, x_m\}$. Then setting $y_j = x_j$ for $j \in \mathbb{Z}_{>0} \setminus \{m\}$ gives a reduced right infinite word $y \in \partial F_n$. Clearly $y_j = x_j$ for $j = 1, 2, \ldots, n$. Since $m > l - k - 1$, we have

$$hy = \big(h_1,\, h_2,\, \ldots,\, h_k,\, y_{l-k+1},\, y_{l-k+2},\, \ldots\big). \tag{2.2}$$

Therefore, using $2k - l \neq 0$ at the first step, (2.1) at the third step, and (2.2) at the fifth step, we get

$$y_{m+2k-l} = x_{m+2k-l} = (hx)_{m+2k-l} = x_m \neq y_m = (hy)_{m+2k-l}.$$

Thus $hy \neq y$.

A related example, in which G is a finite free product of at least two nontrivial cyclic groups (excluding $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$), acting on the Cantor set, is given in Definition 2.1 of [260]. Essential freeness is a consequence of Lemma 3.12 of [260]. Minimality isn't explicitly stated, but it is shown in some cases that the crossed products are simple, by explicitly computing them. (See Example 2.8 and Remark 2.9 of [260].) Actions of a subclass of these groups on the Cantor set are given in Definition 2.1 of [261] and the comment afterwards, and are shown to be minimal and essentially free in Theorem 3.3 of [261]. The crossed products are proved to be Cuntz-Krieger algebras in Theorem 2.2 of [261].

Some further examples of this general nature are given in Section 3 of [149], and some more are in Section 3 of [3].

We mention a few other examples very briefly.

A general construction known as the flow under a ceiling function starts with a homeomorphism h of, say, a compact metric space X, and yields an action of $\mathbb{R}$ on a space that looks like the mapping cylinder of X. One can consider this action or the action of $\mathbb{Z}$ generated by the time t map of this action for a fixed $t \in \mathbb{R}$. The crossed products by some interesting examples of this construction are considered in [122], with X taken to be the Cantor set. We refer to [122] for further details.

Let X be the Cantor set. There are interesting classes of minimal homeomorphisms of $S^1 \times X$ and of $S^1 \times S^1 \times X$. See [154], [155], and [156] for $S^1 \times X$ and [268] for $S^1 \times S^1 \times X$. (The spaces $S^1 \times X$ are locally homeomorphic to those of [122].)

The geodesic flow on a compact Riemannian manifold M is an action of $\mathbb{R}$ on the unit sphere bundle X over M. At $v \in T_x M$ it follows the geodesic starting at x in the direction v at unit speed, carrying v with it. Various dynamical properties of this flow are considered in Chapter 12 of [14]. For example, under suitable conditions on M, it is topologically transitive (Theorem 12.2.10 of [14]). If the Riemannian metric on M is C^3 and the sectional curvatures are all strictly negative, then the geodesic flow is ergodic with respect to the standard measure. See Theorem 5.5 in the appendix to [13].

We would also like to mention several existence theorems for actions.

Theorem 2.39 (Theorem 1.1 of [115])**.** Let G be an infinite countable discrete group. Then there exists a free action of G on the Cantor set which has an invariant Borel probability measure.

By passing to a minimal set for such an action, one obtains:

Corollary 2.40 (Corollary 1.5 of [115])**.** Let G be an infinite countable discrete group. Then there exists a free minimal action of G on the Cantor set.

The action in Corollary 2.40 need not have an invariant Borel probability measure. However, if G is amenable, then, by Theorem 2.9, every action on a compact metric space has an invariant Borel probability measure.

Theorem 2.41 (Theorem 6.11 of [247])**.** Let G be an infinite countable discrete group which is exact but not amenable. Then there exists a free minimal action of G on the Cantor set X such that the transformation group C*-algebra $C^*(G, X)$ is a Kirchberg algebra satisfying the Universal Coefficient Theorem.

The following result is a special case of the combination of Theorem 1 and Theorem 3 of [86]. In [86], freeness of the action of S^1 is weakened to the requirement that the stabilizers of all points be finite and that the action be effective.

Theorem 2.42. Let M be a connected compact C^∞ manifold which admits a free C^∞ action of S^1. Then there exists a uniquely ergodic minimal diffeomorphism of M.

The most obvious examples are spheres S^{2n-1} for $n \in \mathbb{Z}_{>0}$. The free action of S^1 is the one in Example 2.32.

By contrast, there are no minimal homeomorphisms of even spheres. This can be easily proved using the Lefschetz fixed point theorem. The general result is as follows (a special case of Theorem 3 of [89]).

Theorem 2.43. Let X be a finite complex with nonzero Euler characteristic. Then every homeomorphism of X has a periodic point.

The Euler characteristic of an even sphere is 2.

The proof of Theorem 2.42 uses a Baire category argument. For $n > 1$, there is no known explicit formula for even a minimal homeomorphism of S^{2n-1}.

Problem 2.44. Find an explicit formula for a minimal homeomorphism of S^3.

Theorem 2.45 ([293])**.** Let M be a connected compact C^∞ manifold which admits a free C^∞ action of S^1, and let $k \in \mathbb{Z}_{>0}$. Then there exists a minimal diffeomorphism of M which admits exactly k ergodic invariant Borel probability measures.

The following result is a special case of the combination of Theorem 2 and Theorem 4 of [86]. In [86], freeness of the action of S^1 is weakened in the same way as for Theorem 2.42.

Theorem 2.46. Let $n \in \mathbb{Z}_{>0}$, and let M be a connected compact C^∞ manifold which admits a free C^∞ action of $(S^1)^{n+1}$. Then there exists a uniquely ergodic free minimal action of $\mathbb{R}^n$ on M.

By embedding $\mathbb{Z}^d$ in $\mathbb{R}$ as a dense subgroup, one gets:

Corollary 2.47. Let $d \in \mathbb{Z}_{>0}$ with $d \geq 2$, and let M be a connected compact C^∞ manifold which admits a free C^∞ action of $S^1 \times S^1$. Then there exists a uniquely ergodic minimal action of $\mathbb{Z}^d$ on M.

3. Examples of Group Actions on Noncommutative C*-Algebras

In this section, we turn to examples of group actions on noncommutative C*-algebras. Along with a number of miscellaneous examples, we give an assortment of examples from each of several fairly general classes of actions: "gauge type" actions, shifts and other permutations of the factors in various kinds of tensor products and free products, and highly nontrivial actions obtained as direct limits of various much simpler (even inner) actions on smaller C*-algebras. Section 4 contains many more examples of "gauge type" actions.

The most elementary action is the trivial action.

Example 3.1. Let G be a locally compact group, let A be a C*-algebra, and define an action $\alpha\colon G \to \mathrm{Aut}(A)$ by $\alpha_g(a) = a$ for all $g \in G$ and all $a \in A$. This is the trivial action of G on A.

The crossed products turn out to be

$$C^*(G, A, \alpha) = C^*(G) \otimes_{\max} A \qquad \text{and} \qquad C^*_{\mathrm{r}}(G, A, \alpha) = C^*_{\mathrm{r}}(G) \otimes_{\min} A.$$

See Example 10.1.

Before we go farther, the following notation is convenient.

Notation 3.2. Let A be a unital C*-algebra, and let $u \in A$ be unitary. We denote by $\mathrm{Ad}(u)$ the automorphism of A given by $a \mapsto uau^*$. We use the same notation when A is not unital and u is a unitary in its multiplier algebra $M(A)$.

Definition 3.3. Let A be a C*-algebra and let $\alpha \in \mathrm{Aut}(A)$. Then α is *inner* if there is $u \in M(A)$ such that $\alpha = \mathrm{Ad}(u)$. Otherwise, α is *outer*.

Example 3.4. Let G be a locally compact group, let A be a unital C*-algebra, and let $g \mapsto z_g$ be a norm continuous group homomorphism from G to the unitary group $U(A)$ of A. Then the formula $\alpha_g = \mathrm{Ad}(z_g)$, for $g \in G$ and $a \in A$, defines an action of G on A. Actions obtained this way are called *inner actions*.

If A is not unital, let $M(A)$ be its multiplier algebra, and use $U(M(A))$ with the strict topology in place of $U(A)$ with the norm topology.

As a special case, let $g \mapsto u_g$ be a unitary representation of G on a Hilbert space H, which is continuous in the strong operator topology (the conventional topology in this situation; the formal definition is in Definition 5.2 below). Then $g \mapsto \mathrm{Ad}(u_g)$ defines a continuous action of G on the compact operators $K(H)$. (The map $g \mapsto \mathrm{Ad}(u_g)$ is generally not a continuous action, in the C*-algebra sense, of G on the bounded operators $L(H)$.)

The crossed product by an inner action is isomorphic to the crossed product by the trivial action. See Example 10.4 below for the computation of the crossed product when G is discrete.

An action via inner automorphisms is not necessarily an inner action in the sense of Example 3.4. There are no counterexamples with $G = \mathbb{Z}$ (trivial) or when G finite cyclic and A is simple (easy; see Exercise 3.7 below). Here is the smallest counterexample.

Example 3.5. Let $A = M_2$, let $G = (\mathbb{Z}/2\mathbb{Z})^2$ with generators g_1 and g_2, and set

$$\alpha_1 = \mathrm{id}_A, \quad \alpha_{g_1} = \mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad \alpha_{g_2} = \mathrm{Ad}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \quad \text{and} \quad \alpha_{g_1 g_2} = \mathrm{Ad}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right).$$

These define an action $\alpha\colon G \to \mathrm{Aut}(A)$ such that α_g is inner for all $g \in G$, but for which there is no homomorphism $g \mapsto z_g \in U(A)$ such that $\alpha_g = \mathrm{Ad}(z_g)$ for all $g \in G$. The point is that the implementing unitaries for α_{g_1} and α_{g_2} commute up to a scalar, but can't be appropriately modified to commute exactly.

See Exercise 10.18 below for the computation of the crossed product.

Exercise 3.6. Prove the statements made in Example 3.5.

Exercise 3.7. Let A be a simple unital C*-algebra, and let $\alpha\colon \mathbb{Z}/n\mathbb{Z} \to \mathrm{Aut}(A)$ be an action such that each automorphism α_g, for $g \in \mathbb{Z}/n\mathbb{Z}$, is an inner automorphism. Prove that α is an inner action in the sense of Example 3.4.

The result of Exercise 3.7 fails when A is not assumed simple. The following example is due to Jae Hyup Lee.

Example 3.8. Let $A = C(S^1, M_2)$, and define $u \in A$ by

$$u(\zeta) = \frac{1}{2}\begin{pmatrix} \zeta + 1 & i(\zeta - 1) \\ i(\zeta - 1) & -(\zeta + 1) \end{pmatrix}$$

for $\zeta \in S^1$. Then one can check that u is unitary, and that u^2 is the function

$$u(\zeta)^2 = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}, \tag{3.1}$$

which is in the center of A. Therefore $\mathrm{Ad}(u) \in \mathrm{Aut}(A)$ is an automorphism of order 2, and so gives an action α of $\mathbb{Z}/2\mathbb{Z}$ on A.

We claim that this action is not inner in the sense of Example 3.4. That is, there is no unitary $z \in A$ such that $z^2 = 1$ and $\mathrm{Ad}(z) = \mathrm{Ad}(u)$.

Suppose z is such a unitary. Then $\mathrm{Ad}(z^* u) = \mathrm{id}_A$, so $z^* u$ is in the center of A. Thus, there is a continuous function $\lambda \colon S^1 \to S^1$ such that

$$z(\zeta)^* u(\zeta) = \lambda(\zeta) \cdot 1_{M_2}$$

for all $\zeta \in S^1$. We can rearrange this equation to get

$$\lambda(\zeta) z(\zeta) = u(\zeta) \tag{3.2}$$

for all $\zeta \in S^1$. Squaring both sides of (3.2), and using (3.1) and $z^2 = 1$, we get

$$\lambda(\zeta)^2 \cdot 1_{M_2} = u(\zeta)^2 = \zeta \cdot 1_{M_2}$$

for all $\zeta \in S^1$. Thus, $\lambda(\zeta)$ is a continuous square root of ζ on S^1, which is well known not to exist. This contradiction shows that α is not an inner action.

Remark 3.9. There is a generalization of inner actions that should be mentioned. Actions α and β of a locally compact group G on a unital C*-algebra A are called *exterior equivalent* if there is a continuous map $g \mapsto z_g$ from G to the unitary group of A such that $z_{gh} = z_g \alpha_g(z_h)$ and $\beta_g = \mathrm{Ad}(z_g) \circ \alpha_g$ for $g, h \in G$. If A is not unital, use a strictly continuous map to the unitary group of the multiplier algebra. (See 8.11.3 of [198].) An action is inner if and only if it is exterior equivalent to the trivial action, and it turns out that exterior equivalent actions give isomorphic crossed products. See Exercise 10.5 below.

Since they play such a prominent role in our examples, we explicitly recall the rotation algebras.

Example 3.10. Let $\theta \in \mathbb{R}$. The rotation algebra A_θ is the universal C*-algebra generated by two unitaries u and v satisfying the commutation relation $vu = \exp(2\pi i \theta) uv$. (The convention $e^{2\pi i \theta}$ instead of $e^{i\theta}$ has become so standard that it can't be changed.)

The algebra A_θ is often considered to be a noncommutative analog of the torus $S^1 \times S^1$ (more accurately, of $A_0 \cong C(S^1 \times S^1)$). It turns out to be the crossed product by the corresponding rotation $\zeta \mapsto e^{2\pi i \theta}\zeta$ of the circle, the integer action version of Example 2.16.

If $\theta \notin \mathbb{Q}$, then A_θ is known to be simple. This follows from Example 10.25 and Theorem 15.10 below. Thus, one may take *any* C*-algebra generated by two unitaries satisfying the appropriate commutation relation.

If $\theta \in \mathbb{Q}$, then A_θ is the section algebra of a locally trivial bundle over $S^1 \times S^1$ whose fiber is a single matrix algebra. Its structure is determined in [116]. (See Example 8.46 of [292]. Some further discussion is given in Example 10.16.) In the special case $\theta \in \mathbb{Z}$, one just gets $C(S^1 \times S^1)$.

There are also versions with more generators.

Example 3.11. Let $d \in \mathbb{Z}_{>0}$ with $d \geq 2$. Let θ be a skew symmetric real $d \times d$ matrix. Recall ([243]) that the (higher dimensional) noncommutative torus A_θ is the universal C*-algebra generated by unitaries $u_1, u_2, \ldots, u_d$ subject to the relations

$$u_k u_j = \exp(2\pi i \theta_{j,k}) u_j u_k$$

for $j, k = 1, 2, \ldots, d$. Of course, if $\theta_{j,k} \in \mathbb{Z}$ for $j, k = 1, 2, \ldots, d$, it is not really noncommutative.

Some authors use $\theta_{k,j}$ in the commutation relation instead. See for example Section 6 of [143].

The algebra A_θ is simple if and only if θ is nondegenerate, which means that whenever $x \in \mathbb{Z}^d$ satisfies $\exp(2\pi i \langle x, \theta y \rangle) = 1$ for all $y \in \mathbb{Z}^d$, then $x = 0$. That nondegeneracy implies simplicity is Theorem 3.7 of [259]. (Note the standing assumption of nondegeneracy throughout Section 3 of [259].) The converse is essentially 1.8 of [68]; see Theorem 1.9 of [210] for the explicit statement.

It seems worth pointing out that there is a coordinate free way to obtain a higher dimensional noncommutative torus. The algebra A_θ is the universal C*-algebra generated by unitaries u_x, for $x \in \mathbb{Z}^d$, subject to the relations

$$u_y u_x = \exp(\pi i \langle x, \theta(y) \rangle) u_{x+y}$$

for $x, \, y \in \mathbb{Z}^d$. (See the beginning of Section 4 of [242] and the introduction to [244].) It follows that if $b \in \mathrm{GL}_d(\mathbb{Z})$, and if b^{t} denotes the transpose of b, then $A_{b^{\mathrm{t}} \theta b} \cong A_\theta$. That is, A_θ is unchanged if θ is rewritten in terms of some other basis of $\mathbb{Z}^d$.

Example 3.12. Let $\theta \in \mathbb{R}$, and let A_θ be the rotation algebra, as in Example 3.10. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on A_θ by sending the matrix

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$$

to the automorphism determined by

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}} \qquad \text{and} \qquad \alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$

To see that there is such an automorphism, one checks that the intended values of $\alpha_n(u)$ and $\alpha_n(v)$ are unitaries which satisfy the relation

$$\alpha_n(v)\alpha_n(u) = e^{2\pi i \theta} \alpha_n(u)\alpha_n(v).$$

The extra scalar factors in the definition are present in order to get $\alpha_{mn} = \alpha_m \circ \alpha_n$ for $m, n \in \mathrm{SL}_2(\mathbb{Z})$.

If we view A_θ as a noncommutative analog of the torus $S^1 \times S^1$ as in Example 3.10, this action is the analog of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $S^1 \times S^1$ in Example 2.30.

The group $\mathrm{SL}_2(\mathbb{Z})$ has finite subgroups of orders 2, 3, 4, and 6. They can be taken to be generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \ (\text{for } \mathbb{Z}/2\mathbb{Z}), \qquad \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \ (\text{for } \mathbb{Z}/3\mathbb{Z}),$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ (\text{for } \mathbb{Z}/4\mathbb{Z}), \qquad \text{and} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \ (\text{for } \mathbb{Z}/6\mathbb{Z}).$$

Restriction of the action gives actions of these groups on rotation algebras. The crossed products by these actions have been intensively studied. Recently, it has been proved [65] that for $\theta \notin \mathbb{Q}$ they are all AF algebras.

In at least some of these cases, the extra scalar factors are equal to 1. Thus, the action of $\mathbb{Z}/2\mathbb{Z}$ on A_θ is generated by the automorphism determined by

$$u \mapsto u^* \qquad \text{and} \qquad v \mapsto v^*,$$

and the action of $\mathbb{Z}/4\mathbb{Z}$ on A_θ is generated by the automorphism determined by

$$u \mapsto v \qquad \text{and} \qquad v \mapsto u^*.$$

Although we will not prove it here (see [65]), for $\theta \notin \mathbb{Q}$ these actions have the tracial Rokhlin property of Definition 14.1.

It seems to be unknown whether the action of $\mathrm{GL}_2(\mathbb{Z})$ on $S^1 \times S^1$ in Example 2.30 can be deformed to an action on a rotation algebra. This can be done for the subgroup consisting of the diagonal matrices in $\mathrm{GL}_2(\mathbb{Z})$, an order 4 subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Example 3.13. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let A_θ be the rotation algebra, as in Example 3.10. Let $G = \{\mathrm{diag}(\pm 1, \pm 1)\} \subset \mathrm{GL}_2(\mathbb{Z})$. Then there is (Theorem 1.1 of [263]) an action $\beta^{(\theta)} \colon G \to \mathrm{Aut}(A_\theta)$ such that, for $g \in G$, we have $(\beta_g^{(\theta)})_* = g$ on $K_1(A_\theta) \cong \mathbb{Z}^2$.

The K-theory condition matches the action of this subgroup on $K^1(S^1 \times S^1)$. The construction is an existence proof using a direct limit decomposition, and it is not clear how close the action of the diagonal subgroup is to the action of $\mathbb{Z}/2\mathbb{Z}$ in Example 3.12 (although, by Theorem 1.1 of [263], it does have the right fixed point algebra). There is no claim that the actions on the different algebras A_θ can be chosen to vary continuously with θ in a reasonable sense. This is probably not possible. The results of [264] probably imply (although this has not been checked in detail) that for $\theta \in \big[(0,1) \cap \mathbb{Q}\big] \setminus \big\{\frac{1}{2}\big\}$, there is no $\alpha \in \mathrm{Aut}(A_\theta)$ whose induced map on $K_1(A_\theta)$ is in $\mathrm{GL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{Z})$.

Unfortunately, there is in general no action of $\mathrm{SL}_d(\mathbb{Z})$ on the higher dimensional noncommutative torus of Example 3.11 analogous to the action of $\mathrm{SL}_2(\mathbb{Z})$ on A_θ. That is, there is no general noncommutative deformation of the action of $\mathrm{SL}_n(\mathbb{Z})$ on $(S^1)^d$ of Example 2.30.

In Example 3.12, we had a C*-algebra A given in terms of generators and relations, and we defined an action of a discrete group on A by specifying what the group elements are supposed to do to the generators. We want to define actions of not necessarily discrete groups in the same way. We will obviously only do this when the action on the generators is continuous. We need the following lemma to ensure that this method gives an action which is continuous on the entire algebra.

Lemma 3.14. Let X be a topological space, let A be a C*-algebra, and let $x \mapsto \alpha_x$ be a function from X to the endomorphisms of A. Suppose there is a subset $S \subset A$ which generates A as a C*-algebra and such that $x \mapsto \alpha_x(a)$ is continuous for all $a \in S$. Then $x \mapsto \alpha_x(a)$ is continuous for all $a \in A$.

The proof is an $\frac{\varepsilon}{3}$ argument. The key point is that $\sup_{x \in X} \|\alpha_x\|$ is finite. As far as we know, without explicitly including this condition in the hypotheses, there are no analogous results for Banach algebras, even in the situation of group actions.

Proof of Lemma 3.14. Let $A_0 \subset A$ be the complex *-subalgebra of A generated by S. Then $x \mapsto \alpha_x(a)$ is continuous for all $a \in A_0$. Now let $a \in A$ be arbitrary, let $x_0 \in X$, and let $\varepsilon > 0$. We have to find an open set $U \subset X$ with $x_0 \in U$ such that for all $x \in U$, we have $\|\alpha_x(a) - \alpha_{x_0}(a)\| < \varepsilon$. Choose $a_0 \in A_0$ such that $\|a - a_0\| < \frac{\varepsilon}{3}$. Since $x \mapsto \alpha_x(a_0)$ is continuous, there is an open set $U \subset X$ with $x_0 \in U$ such that for all $x \in U$, we have $\|\alpha_x(a_0) - \alpha_{x_0}(a_0)\| < \frac{\varepsilon}{3}$. For $x \in X$, since α_x is a homomorphism of C*-algebras, we have $\|\alpha_x(b)\| \leq \|b\|$ for all $b \in A$. For

$x \in U$, we thus get

$$\begin{aligned}\|\alpha_x(a) - \alpha_{x_0}(a)\| &\leq \|\alpha_x(a) - \alpha_x(a_0)\| + \|\alpha_x(a_0) - \alpha_{x_0}(a_0)\| + \|\alpha_{x_0}(a_0) - \alpha_{x_0}(a)\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.\end{aligned}$$

This completes the proof. □

Example 3.15. Let $\theta \in \mathbb{R}$, and let A_θ be the rotation algebra, as in Example 3.10. For $\zeta_1, \zeta_2 \in S^1$, the unitaries $\zeta_1 u$ and $\zeta_2 v$ satisfy the same commutation relation. Therefore there is an action $\alpha \colon S^1 \times S^1 \to \mathrm{Aut}(A_\theta)$ determined by $\alpha_{(\zeta_1,\zeta_2)}(u) = \zeta_1 u$ and $\alpha_{(\zeta_1,\zeta_2)}(v) = \zeta_2 v$. Continuity of the action follows from Lemma 3.14.

If we fix $\zeta_1, \zeta_2 \in S^1$, then $\alpha_{(\zeta_1,\zeta_2)}$ generates an action of $\mathbb{Z}$. The crossed product by this action turns out to be a three dimensional noncommutative torus as in Example 3.11, namely the universal C*-algebra generated by unitaries u, v, w such that

$$vu = \exp(2\pi i\theta)uv, \qquad wu = \zeta_1 uw, \qquad \text{and} \qquad wv = \zeta_2 ww.$$

Repeating the construction, one realizes an arbitrary higher dimensional noncommutative torus as an iterated crossed product. See Example 4.1 below.

If both ζ_1 and ζ_2 have finite order, we get an action of a finite cyclic group. For example, there is an action of $\mathbb{Z}/n\mathbb{Z}$ generated by the automorphism which sends u to $\exp(2\pi i/n)u$ and v to v.

Problem 3.16. Find examples of actions of finite groups on higher dimensional noncommutative tori with interesting crossed products. For this purpose, the actions one gets from the higher dimensional version of Example 3.15 are not very interesting, because the crossed product is closely related to another higher dimensional noncommutative torus. The only known general example that is interesting in this sense is the "flip" action of $\mathbb{Z}/2\mathbb{Z}$, generated by $u_k \mapsto u_k^*$ for $1 \leq k \leq d$. Whenever the higher dimensional noncommutative torus is simple, the crossed product by this action is known to be AF [65].

There is recent work in this direction in [126], and some further work has been done. Also see [92] for some related work.

The following example gives the one related general family of finite group actions that we know of. It isn't on quite the same algebras as in Problem 3.16, but more examples like this one would also be interesting.

Example 3.17. Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let $A_\theta^{6,4}$ be the universal unital C*-algebra generated by unitaries u, v, w, x, y satisfying the following commutation relations (the relations (CR) at the beginning of Section 2 of [172]):

$$\begin{gathered} uv = xvu, \qquad uw = wu, \qquad ux = xu, \qquad uy = e^{2\pi i\theta}yu, \qquad vw = ywv, \\ vx = xv, \qquad vy = yv, \qquad wx = e^{-2\pi i\theta}xw, \qquad wy = yw, \qquad xy = yx. \end{gathered}$$

This algebra is simple, and in fact it is the crossed product of the action of the discrete Heisenberg group on $S^1 \times S^1$ in Example 2.35. (See Theorem 1 of [172]. The motivation is that $A_\theta^{6,4}$ is a simple quotient of a discrete cocompact subgroup of a particular nilpotent Lie group.) Then there is an automorphism $\alpha \in \mathrm{Aut}\big(A_\theta^{6,4}\big)$ of order 4, given by

$$\alpha(y) = x, \qquad \alpha(w) = u, \qquad \alpha(u) = w^*, \qquad \alpha(x) = y^*, \qquad \text{and} \qquad \alpha(v) = v^*.$$

See Remark 2 on page 312 of [172].

As far as we know, nothing is known about the crossed products by the actions of $\mathbb{Z}/4\mathbb{Z}$ in Example 3.17. We hope, for example, that this action has the tracial Rokhlin property (Definition 14.1), and that this can be used to help identify the crossed product, perhaps by methods similar to those of [65].

The following example is a noncommutative version of Example 2.19. The automorphisms in this example were introduced in Definition 1.1 of [186]. Several special cases were considered earlier, in [190] and [191].

Example 3.18. Let $\theta \in \mathbb{R}$, and let A_θ be the rotation algebra, as in Example 3.10. Let $\gamma \in \mathbb{R}$, let $d \in \mathbb{Z}$, and let $f \colon S^1 \to \mathbb{R}$ be a continuous function. The Furstenberg transformation on A_θ determined by (θ, γ, d, f) is the automorphism $\alpha_{\theta,\gamma,d,f}$ of A_θ such that

$$\alpha_{\theta,\gamma,d,f}(u) = e^{2\pi i\gamma} u \qquad \text{and} \qquad \alpha_{\theta,\gamma,d,f}(v) = \exp(2\pi i f(u)) u^d v.$$

The parameter θ does not appear in the formulas; its only role is to specify the algebra on which the automorphism acts. When $\theta = 0$, we get the action determined by the homeomorphism of Example 2.19.

When $\theta \notin \mathbb{Q}$, the automorphism $\alpha_{\theta,\gamma,d,f}$ is the most general automorphism α of A_θ for which $\alpha(u)$ is a scalar multiple of u. (See Proposition 1.6 of [186].)

Exercise 3.19 (Lemma 1.2 of [186])**.** Prove that the formula for $\alpha_{\theta,\gamma,d,f}$ in Example 3.18 does in fact define an automorphism of A_θ.

Example 3.20. Let $n \in \mathbb{Z}_{>0}$ satisfy $n \geq 2$. Recall that the Cuntz algebra $\mathcal{O}_n$ is the universal unital C*-algebra on generators $s_1, s_2, \ldots, s_n$, subject to the relations $s_j^* s_j = 1$ for $1 \leq j \leq n$ and $\sum_{j=1}^{n} s_j s_j^* = 1$. (It is in fact simple, so any C*-algebra generated by elements satisfying these relations is isomorphic to $\mathcal{O}_n$.)

There is an action of $(S^1)^n$ on $\mathcal{O}_n$ such that $\alpha_{(\zeta_1,\zeta_2,\ldots,\zeta_n)}(s_j) = \zeta_j s_j$ for $1 \leq j \leq n$. (Check that the elements $\zeta_j s_j$ satisfy the required relations.) The restriction to the diagonal elements of $(S^1)^n$ gives an action of S^1 on $\mathcal{O}_n$, sometimes called the gauge action.

In fact, regarding $(S^1)^n$ as the diagonal unitary matrices, this action extends to an action of the unitary group $U(M_n)$ on $\mathcal{O}_n$, defined as follows. If $u = (u_{j,k})_{j,k=1}^n \in M_n$ is unitary, then define an automorphism α_u of $\mathcal{O}_n$ by the following action on the generating isometries $s_1, s_2, \ldots, s_n$:

$$\alpha_u(s_j) = \sum_{k=1}^{n} u_{k,j} s_k.$$

The assignment $u \mapsto \alpha_u$ determines a continuous action of the compact group $U(M_n)$ on $\mathcal{O}_n$. (This action is described, in a different form, in Section 2 of [79].)

Any individual automorphism from this action gives an action of $\mathbb{Z}$ on $\mathcal{O}_n$. More generally, if G is a topological group, and $\rho \colon G \to U(M_n)$ is a continuous homomorphism (equivalently, a unitary representation of G on $\mathbb{C}^n$), then the composition $\alpha \circ \rho$ is an action of G on $\mathcal{O}_n$. Such actions are called *quasifree actions*.

Several specific quasifree actions are used for counterexamples in the discussion after Theorem 15.26.

The action of $U(M_n)$ on $\mathcal{O}_n$ in Example 3.20 is actually a special case of a much more general (and natural looking) construction. See Example 4.8 and Exercise 4.10.

Exercise 3.21. Verify that the formula given in Example 3.20 does in fact define a continuous action of $U(M_n)$ on $\mathcal{O}_n$.

The actions of $(S^1)^2$ in Example 3.15 and of $U(M_n)$ in Example 3.20 are examples of what we think of as "gauge type" actions. (The actions usually called gauge actions are the restrictions of these to S^1, embedded diagonally. Thus, in Example 3.20, this is the action $\beta_\zeta(s_j) = \zeta s_j$ for $\zeta \in S^1$ and $j = 1, 2, \ldots, n$.) There are many more actions of this same general type, and we give a collection of such actions in Section 4. Here we mention only the dual action on a crossed product by an abelian group.

Example 3.22. Let A be a C*-algebra, and let $\alpha \in \mathrm{Aut}(A)$ be an automorphism. Then the dual action of $\widehat{\mathbb{Z}} = S^1$ is a continuous action of S^1 on the crossed product $C^*(\mathbb{Z}, A, \alpha)$. We will describe this action in Remark 9.25 below, after we have given the construction of crossed products.

Example 3.23. More generally, let G be any locally compact group, let A be a C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be a continuous action of G on A. Then there is a dual action $\widehat{\alpha} \colon \widehat{G} \to \mathrm{Aut}\big(C^*(G, A, \alpha)\big)$. Again, we will describe this action in Remark 9.25 below, after we have given the construction of crossed products.

Although we will not give any details here, there are several kinds of more general dual actions. Crossed products by partial automorphisms, and more generally by partial actions of groups, are defined in [82]. When the group G which acts partially is abelian, such a crossed product has an action of $\widehat{G}$. In a somewhat different direction, there are coactions of (not necessarily abelian) locally compact groups on C*-algebras, and (full and reduced) crossed products by coactions are defined. The full and reduced crossed products by a coaction of a locally compact group G have a dual action, which is an action of the (not necessarily abelian) group G.

The following result, giving actions on direct limits of equivariant direct systems, is useful for the next several examples. We state it in general, but in most of its applications, the directed set I is $\mathbb{Z}_{>0}$ or $\mathbb{Z}_{\geq 0}$ with its usual order, and the maps of the direct system are all injective. Then we can think of $\varinjlim A_n$ as being made by arranging to have $A_1 \subset A_2 \subset \cdots$ and taking $\overline{\bigcup_{n=1}^{\infty} A_n}$. Equivariance is then the condition that the restriction to A_n of the action on A_{n+1} is the action on A_n. The action on $\bigcup_{n=1}^{\infty} A_n$ is then defined in the obvious way, and is extended to $\overline{\bigcup_{n=1}^{\infty} A_n}$ by continuity.

Proposition 3.24. Let G be a locally compact group. Let

$$\big((G, A_i, \alpha^{(i)})_{i \in I}, \, (\varphi_{j,i})_{i \leq j}\big)$$

be a direct system of G-algebras. Let $A = \varinjlim A_i$. Then there exists a unique action $\alpha \colon G \to \mathrm{Aut}(A)$ such that $\alpha_g = \varinjlim \alpha_g^{(i)}$ for all $g \in G$.

Proof. Existence of the automorphisms α_g for $g \in G$, and their algebraic properties, is easily obtained from the universal property of the direct limit. Continuity of the action follows from Lemma 3.14. □

Example 3.25. Let $k_1, k_2, \ldots$ be integers with $k_n \geq 2$ for all $n \in \mathbb{Z}_{>0}$. Consider the UHF algebra A of type $\prod_{n=1}^{\infty} k_n$. We construct it as $\bigotimes_{n=1}^{\infty} M_{k_n}$, or, in more detail, as $\varinjlim A_n$ with $A_n = M_{k_1} \otimes M_{k_2} \otimes \cdots \otimes M_{k_n}$. Thus $A_n = A_{n-1} \otimes M_{k_n}$, and the map $\varphi_n \colon A_{n-1} \to A_n$ is given by $a \mapsto a \otimes 1_{M_{k_n}}$.

Let G be a locally compact group, and for $n \in \mathbb{Z}_{>0}$ let $\beta^{(n)} \colon G \to \mathrm{Aut}(M_{k_n})$ be an action of G on M_{k_n}. (The easiest way to get such an action is to use an inner action as in Example 3.4. That is, choose a unitary representation $g \mapsto u_n(g)$ on $\mathbb{C}^{k_n}$, and set $\beta_g^{(n)}(a) = u_n(g) a u_n(g)^*$ for $g \in G$ and $a \in M_{k_n}$.) Then there is a unique action $\alpha^{(n)} \colon G \to \mathrm{Aut}(A_n)$ such that

$$\alpha_g^{(n)}(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = \beta_g^{(1)}(a_1) \otimes \beta_g^{(2)}(a_2) \otimes \cdots \otimes \beta_g^{(n)}(a_n)$$

for

$$a_1 \in M_{k_1}, \qquad a_2 \in M_{k_2}, \qquad \ldots, \qquad a_n \in M_{k_n}, \qquad \text{and} \qquad g \in G.$$

One checks immediately that $\varphi_n \circ \alpha_g^{(n-1)} = \alpha_g^{(n)} \circ \varphi_n$ for all $n \in \mathbb{Z}_{>0}$ and $g \in G$, so, by Proposition 3.24, there is a direct limit action $g \mapsto \alpha_g$ of G on $A = \varinjlim A_n$. It is written $\alpha_g = \bigotimes_{n=1}^{\infty} \beta_g^{(n)}$.

We call such actions *infinite tensor product actions*. If each $\beta^{(n)}$ is inner, the resulting action was originally called a *product type action*. The general case of such actions was first seriously investigated in [103] and [104].

As a specific example, take $G = \mathbb{Z}/2\mathbb{Z}$, and for every n take $k_n = 2$ and take $\beta^{(n)}$ to be generated by $\mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. For another specific example, take $G = S^1$, and for every n take $k_n = 2$ and for $\zeta \in S^1$ take

$$\beta_\zeta^{(n)} = \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix} \qquad \text{or} \qquad \beta_\zeta^{(n)} = \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & \zeta^{2^{n-1}} \end{pmatrix}.$$

The second choice gives

$$\beta_\zeta^{(n)} = \mathrm{Ad}\big(\mathrm{diag}\big(1, \zeta, \zeta^2, \ldots, \zeta^{2^n - 1}\big)\big)$$

for $n \in \mathbb{Z}_{>0}$ and $\zeta \in S^1$.

In Example 3.25, even if all the actions $\beta^{(n)}$ (and hence also the actions $\alpha^{(n)}$) in the construction are inner, one does not expect the action α to be inner. It is often easy to compute the crossed product (see Example 10.22 for an illustration of the method), and the result is often not the same as the crossed product by an inner action. Here, though, we prove that the action is not inner in one case for which a direct proof is easy.

Lemma 3.26. In Example 3.25, assume that $k_n \geq 2$ for all $n \in \mathbb{Z}_{>0}$, take $G = \mathbb{Z}/2\mathbb{Z}$, for $n \in \mathbb{Z}_{>0}$ choose $r_n, s_n \in \mathbb{Z}_{>0}$ such that $r_n + s_n = k_n$, set $z_n = \mathrm{diag}(1_{r_n}, -1_{s_n}) \in M_{k_n}$, and let $\beta^{(n)} \colon G \to \mathrm{Aut}(M_{k_n})$ be the action generated by $\mathrm{Ad}(z_n)$. Let A and $\alpha \colon G \to \mathrm{Aut}(A)$ be as in the construction of Example 3.25. Then α is not an inner action.

Proof. Let $\gamma \in \mathrm{Aut}(A)$ be the automorphism given by the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$. Assume that there is $v \in U(A)$ such that $\gamma = \mathrm{Ad}(v)$. Choose $n \in \mathbb{Z}_{>0}$ and $c \in A_n \subset A$ such that $\|c - v\| < \frac{1}{2}$. Define projections $e_0, e_1 \in M_{k_{n+1}}$ by

$$e_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad e_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then e_0 and e_1 are orthogonal projections which are exchanged by the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ under the action $\beta^{(n+1)}$. Therefore

$$p_0 = 1_{A_n} \otimes e_0 \qquad \text{and} \qquad p_1 = 1_{A_n} \otimes e_1,$$

regarded as elements of A, are orthogonal projections such that $\gamma(p_0) = p_1$ and $\gamma(p_1) = p_0$. Also c commutes with p_0. Therefore, using $vp_0v^* = \gamma(p_0) = p_1$ at the third step,

$$1 = \|p_0 - p_1\| = \|(p_0 - p_1)v\| = \|p_0v - vp_0\| \leq \|p_0c - cp_0\| + 2\|c - v\| = 2\|c - v\| < 1.$$

This is a contradiction, and we have proved that γ is not inner. □

The following example is taken from the beginning of Section 4 of [218], and is a special case of the adaptation to C*-algebras of the construction of Proposition 1.6 of [41], where an analogous example is constructed on the hyperfinite factor of type II_1. We give the formulas for the action and the beginning of the proof that it is an action, but we refer to the proof of Proposition 1.6 of [41] for details.

Example 3.27. Let $D = \bigotimes_{m=1}^{\infty} M_d$ be the d^{∞} UHF algebra. We describe an action $\alpha\colon \mathbb{Z}/d^2\mathbb{Z} \to \mathrm{Aut}(D)$ such that, writing $\mathbb{Z}/d^2\mathbb{Z} = \{0, 1, 2, \ldots, d^2 - 1\}$, the automorphism α_d is inner, but every unitary v such that $\alpha_d(v) = vav^*$ for all $a \in D$ satisfies $\alpha(v) = \exp(2\pi i/d)v$. Thus the image γ of α_1 in the outer automorphism group $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$ (the quotient of $\mathrm{Aut}(A)$ by the inner automorphisms) has order d, but γ can't be lifted to an order d element of $\mathrm{Aut}(A)$.

We identify D as the closed linear span of all elements of the form

$$a = a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes 1 \otimes 1 \otimes \cdots \tag{3.3}$$

with $n \in \mathbb{Z}_{\geq 0}$ and $a_1, a_2, \ldots, a_n \in M_d$. For $n \in \mathbb{Z}_{\geq 0}$, set $D_n = \bigotimes_{k=1}^{n} M_d$, and let $\psi_n\colon D_n \to D_{n+1}$ be the unique homomorphism such that $\psi_n(a) = a \otimes 1_{M_d}$ for all $a \in D_n$. Thus $D = \varinjlim_n D_n$. For $n \in \mathbb{Z}_{\geq 0}$ let $\gamma_n\colon D_n \to D$ be the map obtained from the direct limit.

For $n \in \mathbb{Z}_{>0}$ let $\pi_n\colon M_d \to D$ be the embedding of M_d as the tensor factor in position n. Thus, $\pi_n(a) = \gamma_n(1_{D_{n-1}} \otimes a)$. Equivalently,

$$\pi_n(x) = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes a \otimes 1 \otimes 1 \otimes \cdots,$$

with a in position n. Let $\lambda\colon D \to D$ be the shift endomorphism of D, that is, using the notation (3.3), the endomorphism given by $\lambda(a) = 1 \otimes a$ for $a \in D$. Then $\lambda \circ \pi_n = \pi_{n+1}$ for all $n \in \mathbb{Z}_{>0}$. Let $(e_{j,k})_{j,k=1,2,\ldots,d}$ be the standard system of matrix units for M_d. Define unitaries $v, u \in D$ by

$$v = \pi_1\left(\sum_{j=1}^{d} e^{2\pi i j/d} e_{j,j}\right) \qquad \text{and} \qquad u = \pi_1(e_{d,1})\lambda(v^*) + \sum_{j=1}^{d-1} \pi_1(e_{j,j+1}).$$

Then define $\alpha_n \in \mathrm{Aut}(D)$ by $\alpha_n = \mathrm{Ad}\big(u\lambda(u)\lambda^2(u)\cdots\lambda^{n-1}(u)\big)$.

We claim that there is $\alpha \in \mathrm{Aut}(D)$ such that $\alpha(a) = \lim_{n\to\infty} \alpha_n(a)$ for all $a \in D$. Moreover, we claim that $\alpha^d = \mathrm{Ad}(v)$, that $\alpha(v) = e^{2\pi i/d}v$, and that α^l is an outer automorphism of D for $l = 1, 2, \ldots, d-1$. Finally, we claim that for every unitary $w \in D$, there is a unitary $z \in D$ such that $(\mathrm{Ad}(w) \circ \alpha)^d = \mathrm{Ad}(z)$, and that for every such z we have $(\mathrm{Ad}(w) \circ \alpha)^d(z) = e^{2\pi i/d}z$. We prove only the first part of this, and refer to the calculations in the proof of Proposition 1.6 of [41] for the rest.

We start by proving the existence of a homomorphism $\alpha\colon D \to D$ such that $\alpha(a) = \lim_{n\to\infty} \alpha_n(a)$ for all $a \in D$. By a standard $\frac{\varepsilon}{3}$ argument, it suffices to prove that $\lim_{n\to\infty} \alpha_n(a)$ exists for every a in a dense subset $S \subset D$. Our choice for S is $S = \bigcup_{n=0}^{\infty} \gamma_n(D_n)$. For every $m \in \mathbb{Z}_{>0}$, every element of $\gamma_m(D_m)$ commutes with every element in the range of λ_m, and in particular with $\lambda^n(u)$ for every $n \geq m$. If $a \in \Gamma_m(D_m)$, it therefore follows that $\alpha_n(a) = \alpha_m(a)$ for all $n \geq m$, so that $\lim_{n\to\infty} \alpha_n(a)$ certainly exists.

Since D is simple, α is injective.

The next step is to prove that $\alpha^d(a) = \mathrm{Ad}(v)(a)$ for every $a \in D$. It then follows that α is surjective. Thus $\alpha \in \mathrm{Aut}(D)$. We omit the rest of the proof.

Example 3.28. Let $d \in \{2, 3, \ldots\}$. Let $D = \bigotimes_{m=1}^{\infty} M_d$ be the d^{∞} UHF algebra. We describe an action of $\mathbb{Z}$ on $K \otimes D$ which scales the trace on D, by describing its generating automorphism α.

We will identify D as the closed linear span of all elements of the form

$$a = a_1 \otimes a_2 \otimes \cdots \otimes a_m \otimes 1 \otimes 1 \otimes \cdots$$

with $m \in \mathbb{Z}_{\geq 0}$ and $a_1, a_2, \ldots, a_m \in M_d$. To help keep the notation straight, we use the isomorphism $\mu \colon M_d \otimes D \to D$ given by, for a as above and $x \in M_d$,

$$\mu(x \otimes a) = x \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_m \otimes 1 \otimes 1 \otimes \cdots .$$

To be explicit, on the right hand side x is in the first tensor factor of M_d in $D = \bigotimes_{m=1}^{\infty} M_d$, the factor a_1, which previously was in the first tensor factor, is now in the second, etc.

Set $C_n = \big(\bigotimes_{k=-n}^{0} M_d\big) \otimes D$ for $n \in \mathbb{Z}_{\geq 0}$. (The indexing is chosen so that we can think of C_n as $\bigotimes_{m=-n}^{\infty} M_d$.) Let $(e_{j,k})_{j,k=1,2,\ldots,d}$ be the standard system of matrix units for M_d. For $n \in \mathbb{Z}_{\geq 0}$, there are homomorphisms

$$\psi_n \colon C_n \to C_{n+1} \qquad \text{and} \qquad \alpha_n \colon C_n \to C_n$$

such that, for $x_{-n}, x_{-n+1}, \ldots, x_0 \in M_d$ and $a \in D$, we have

$$\psi_n\big(x_{-n} \otimes x_{-n+1} \otimes \cdots \otimes x_0 \otimes a\big) = e_{1,1} \otimes x_{-n} \otimes x_{-n+1} \otimes \cdots \otimes x_0 \otimes a$$

and

$$\alpha_n\big(x_{-n} \otimes x_{-n+1} \otimes \cdots \otimes x_0 \otimes a\big) = e_{1,1} \otimes x_{-n} \otimes x_{-n+1} \otimes \cdots \otimes \mu(x_0 \otimes a).$$

For all $n \in \mathbb{Z}_{\geq 0}$, one checks that the diagram

$$\begin{array}{ccc}
\bigotimes_{k=-n}^{0} M_d \otimes D & \xrightarrow{\alpha_n} & \bigotimes_{k=-n}^{0} M_d \otimes D \\
\Big\downarrow{\scriptstyle \psi_n} & & \Big\downarrow{\scriptstyle \psi_n} \\
\bigotimes_{k=-n-1}^{0} M_d \otimes D & \xrightarrow[\alpha_{n+1}]{} & \bigotimes_{k=-n-1}^{0} M_d \otimes D
\end{array}$$

commutes. Indeed, both possible maps from the top left to the bottom right are given by

$$x_{-n} \otimes x_{-n+1} \otimes \cdots \otimes x_0 \otimes a \mapsto e_{1,1} \otimes e_{1,1} \otimes x_{-n} \otimes x_{-n+1} \otimes \cdots \otimes \mu(x_0 \otimes a)$$

for $x_{-n}, x_{-n+1}, \ldots, x_0 \in M_d$ and $a \in D$. Set $C = \varinjlim_n C_n$, using the maps $\psi_n \colon C_n \to C_{n+1}$, and for $n \in \mathbb{Z}_{\geq 0}$ let $\gamma_n \colon C_n \to C$ be the associated map. Then there is a homomorphism $\alpha \colon C \to C$ such that $\alpha \circ \gamma_n = \gamma_n \circ \alpha_n$ for all $n \in \mathbb{Z}_{\geq 0}$. The map α is injective because C is simple. Also, for $n \in \mathbb{Z}_{\geq 0}$, $\alpha(C)$ contains $\gamma_n(C_n)$, since $\psi_n(C_n) = \alpha_{n+1}(C_{n+1})$. Thus α is an automorphism.

It is easy to see that $C \cong K \otimes D$. The crossed product $C^*(\mathbb{Z}, K \otimes D, \alpha)$ turns out to be the stabilized Cuntz algebra $K \otimes \mathcal{O}_d$. See Section 2.1 of [45].

We can't quite use Proposition 3.24 here, because $\alpha_n \colon C_n \to C_n$ is not surjective.

Many stable Kirchberg algebras A satisfying the Universal Coefficient Theorem can be realized as crossed products by actions of $\mathbb{Z}$ on stable AF algebras of the same general type as in Example 3.28. The group $K_1(A)$ must be torsion free; then see Corollary 4.6 of [246]. (The statement there is for unital algebras obtained as crossed products by "corner endomorphisms". See Proposition 2.1 of [246] for the relation to our construction.) For general $K_1(A)$, suitable actions on AT algebras are given in Theorem 3.6 of [246]. It isn't proved there that the crossed products are Kirchberg algebras. However, they are certainly nuclear and satisfy the Universal Coefficient Theorem. It is presumably easy to show that they are purely infinite and simple, and it would follow from the classification theorem that they are Kirchberg algebras satisfying the Universal Coefficient Theorem.

We now give several examples of direct limit actions on AH algebras in which homeomorphisms of the spaces in the construction are used to define the actions.

Example 3.29. In [22], Blackadar gives an action α of $\mathbb{Z}/2\mathbb{Z}$ on the 2^∞ UHF algebra D such that $C^*(\mathbb{Z}/2\mathbb{Z}, D, \alpha)$ is not an AF algebra. We refer to that paper for the details, which require a fair amount of description. The action is obtained by realizing D as a direct limit $D = \varinjlim_n C(S^1, M_{4^n})$, with the maps $\varphi_n \colon C(S^1, M_{4^{n-1}}) \to C(S^1, M_{4^n})$ of the system being described as follows. Choose a unitary path $t \mapsto s_t \in M_2$, for $t \in [0,1]$, such that

$$s_0 = 1 \qquad \text{and} \qquad s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Define (justification afterwards) $\psi \colon C(S^1) \to C(S^1, M_2)$ by

$$\psi(f)(e^{2\pi i\theta}) = s_t \begin{pmatrix} f(e^{\pi i t}) & 0 \\ 0 & f(e^{\pi i(t+1)}) \end{pmatrix} s_t^*$$

for $t \in [0,1]$ and $f \in C(S^1)$. The only point requiring justification is that the values at $t = 0$ and at $t = 1$ (both corresponding to the point $1 \in S^1$) are equal, and this is easily checked. (This kind of map will implicitly reappear in the computations in Example 10.9.)

The map $\varphi_0 \colon C(S^1) \to C(S^1, M_4)$ is then given by

$$\varphi_0(f)(\zeta) = \operatorname{diag}\big(\psi(f)(\zeta),\, \psi(f)(\zeta^{-1})\big)$$

for $f \in C(S^1)$ and $\zeta \in S^1$, and φ_n is obtained by tensoring φ_0 with $\mathrm{id}_{M_{4^n}}$. Of course, one must prove that the resulting direct limit is in fact the 2^∞ UHF algebra. These days, the isomorphism is an immediate consequence of standard classification theorems. (At the time this example was constructed, no applicable classification theorems were known.)

The action of Example 3.29 is also an ingredient in the construction of the action in Example 12.5.

The following example is adapted from [91].

Example 3.30. Let G be a compact metrizable group. Let $(k_n)_{n \in \mathbb{Z}_{\geq 0}}$ be a sequence in G such that $\{k_n \colon n \geq N\}$ is dense in G for all $N \in \mathbb{Z}_{>0}$. (The only use of density in the construction is to ensure that the algebra we get at the end is simple. Everything else works for an arbitrary sequence $(k_n)_{n \in \mathbb{Z}_{\geq 0}}$.) For $n \in \mathbb{Z}_{\geq 0}$, define $\varphi_n \colon C(G, M_{2^{n-1}}) \to C(G, M_{2^n})$ by $\varphi_n(a)(g) = \operatorname{diag}(a(g),\, a(gk_n))$ for

$a \in C(G, M_{2^n})$ and $g \in G$. Define an action $\alpha^{(n)} \colon G \to \mathrm{Aut}(C(G, M_{2^n}))$ by $\alpha_g^{(n)}(a)(h) = a(g^{-1}h)$ for $a \in C(G, M_{2^n})$ and $g, h \in G$. It is easy to check that $\varphi_n \circ \alpha_g^{(n-1)} = \alpha_g^{(n)} \circ \varphi_n$ for all $n \in \mathbb{Z}_{>0}$ and $g \in G$, so, by Proposition 3.24, there is a direct limit action $g \mapsto \alpha_g$ of G on $A = \varinjlim C(G, M_{2^n})$. The direct limit A is a simple AH algebra. (Use Proposition 2.1 of [50].) The resulting direct limit action of G (Proposition 3.24) turns out to have the Rokhlin property for actions of compact groups, as in Definition 3.2 of [113].

When $G = S^1$, one gets an action of S^1 on a simple AT algebra with the Rokhlin property.

Actions of compact groups with the Rokhlin property seem to be hard to find.

Exercise 3.31. Prove the statements made in Example 3.30.

Several further examples of this general type are found in Exercise 10.23 and Exercise 10.24.

Example 3.32. Let A be a C*-algebra. The *tensor flip* is the automorphism $\varphi \in \mathrm{Aut}(A \otimes_{\max} A)$ of order 2 determined by the formula $\varphi(a \otimes b) = b \otimes a$ for $a, b \in A$. To prove the existence of such an automorphism in the unital case, use the universal property of $A \otimes_{\max} A$. Reduce the nonunital case to the unital case. This gives an action of $\mathbb{Z}/2\mathbb{Z}$ on $A \otimes_{\max} A$.

The same formula also defines a tensor flip action of $\mathbb{Z}/2\mathbb{Z}$ on $A \otimes_{\min} A$. To prove the existence of such an automorphism, choose an injective representation $\pi \colon A \to L(H)$, and consider $\pi \otimes \pi$ as a representation of $A \otimes_{\min} A$ on $H \otimes H$. Let $u \in L(H \otimes H)$ be the unitary which exchanges the two tensor factors. Then the required automorphism is given by conjugation by u.

In a similar manner, the symmetric group S_n acts on the n-fold maximal and minimal tensor products of A with itself. This is a noncommutative generalization of Example 2.31.

Example 3.33. The Jiang-Su algebra Z, introduced in [130], is an infinite dimensional simple separable nuclear C*-algebra with no nontrivial projections whose K-theory is the same as that of $\mathbb{C}$, and such that $Z \otimes Z \cong Z$. Thus, the tensor flip action of $\mathbb{Z}/2\mathbb{Z}$ on $Z \otimes Z$, as in Example 3.32, gives an action of $\mathbb{Z}/2\mathbb{Z}$ on Z. Similarly, tensor permutation as in Example 3.32 gives an action of the symmetric group S_n on Z.

The Jiang-Su algebra plays a key role in classification theory, but will appear in only a few places in these notes.

Example 3.34. Let A be a unital C*-algebra. Let $B = \bigotimes_{n \in \mathbb{Z}} A$ be the infinite minimal tensor product of copies of A. We define the *minimal shift* on B as follows. Set $B_n = A^{\otimes (2n)}$, the (minimal) tensor product of $2n$ copies of A. (Take $B_0 = \mathbb{C}$.) For $n \in \mathbb{Z}_{\geq 0}$, define $\varphi_n \colon B_n \to B_{n+1}$ by $\varphi_n(a) = 1_A \otimes a \otimes 1_A$ for $a \in B_n$. Identify B with $\varinjlim B_n$, using the maps φ_n in the direct system. Then take $\sigma \colon B \to B$ to be the direct limit of the maps $\sigma_n \colon B_n \to B_{n+1}$ defined by $\sigma_n(a) = 1_A \otimes 1_A \otimes a$ for $a \in B_n$.

We define the *maximal shift* on the infinite maximal tensor product in the same manner.

These are called *tensor shifts* or *Bernoulli shifts* over $\mathbb{Z}$. There are Bernoulli shifts over any discrete group G.

Example 3.34 is the noncommutative analog of Example 2.20. Indeed, using the notation there, if $A = \mathbb{C}^2$, then $\bigotimes_{n \in \mathbb{Z}} A \cong C(X)$, and the tensor shift is the automorphism induced by the shift on X.

Example 3.35. Let A be a C*-algebra. The *free flip* on the (full) free product $A \star A$ is the automorphism $\varphi \in \mathrm{Aut}(A \star A)$ of order 2 given as follows. Let $\iota_1, \iota_2 \colon A \to A \star A$ be the inclusions of the two free factors. Then φ is determined by the formula $\varphi(\iota_1(a)) = \iota_2(a)$ and $\varphi(\iota_2(a)) = \iota_1(a)$ for $a \in A$. (To see that it exists, use the universal property of $A \star A$.) This gives an action of $\mathbb{Z}/2\mathbb{Z}$ on $A \star A$.

The same formula also defines a free flip action of $\mathbb{Z}/2\mathbb{Z}$ on the reduced free product $A \star_{\mathrm{r}} A$, taken with respect to the same state on both copies of A. One also gets a flip action of $\mathbb{Z}/2\mathbb{Z}$ on the amalgamated free product $A \star_B A$ over a subalgebra $B \subset A$, taking the same inclusion of B into both copies of A. If A is unital, one important choice is $B = \mathbb{C} \cdot 1_A$, giving a unital amalgamated free product. One can also used reduced amalgamated free products.

In a similar manner, the symmetric group S_n acts on the n-fold full and reduced (amalgamated) free products of A with itself. This is a different noncommutative generalization of Example 2.31.

There are (reduced or amalgamated) free Bernoulli shifts on free products of copies of A indexed by $\mathbb{Z}$ (the free analog of Example 3.34), free Bernoulli shifts over other discrete groups, and more general versions of the same kind of construction.

Free Bernoulli shifts are used in Section 2 of [217] to give (initially surprising) examples of actions of noncompact groups which are equivariantly semiprojective.

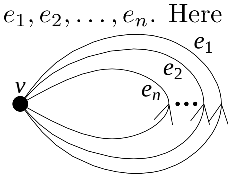
Our next example involves graph algebras. We take [235] as our main reference. However, we warn that there are two conflicting conventions, both in common use, for the relation between the direction of the arrows in the graph and the definition of its C*-algebra. (For example, the papers [148] and [262], cited below, use the opposite convention from [235].) In the definition below, the other convention exchanges s_e and s_e^*. When reading papers about graph algebras, one must therefore always check which convention is being used. We also warn that the graph terminology commonly used in this subject conflicts with graph terminology used in some other parts of graph theory.

The following definition is from the beginning of Chapter 5 of [235]. See Proposition 1.21 of [235] for the case of a row-finite graph. We emphasize that graphs are allowed to have parallel edges and edges which begin and end at the same vertex, and that the edges are oriented.

Definition 3.36. Let $E = \big(E^{(0)}, E^{(1)}, r, s\big)$ be a directed graph, with vertex set $E^{(0)}$, edge set $E^{(1)}$, and range and source maps $r, s \colon E^{(1)} \to E^{(0)}$. That is, if $e \in E^{(1)}$ is an edge, then e begins at $s(e)$ and ends at $r(e)$. The graph C*-algebra $C^*(E)$ is the universal C*-algebra on generators p_v for $v \in E^{(0)}$ and s_e for $e \in E^{(1)}$, subject to the following relations:

(1) The elements p_v for $v \in E^{(0)}$ are mutually orthogonal projections.
(2) The elements s_e for $e \in E^{(1)}$ are partial isometries.
(3) $s_e^* s_e = p_{s(e)}$ for all $e \in E^{(1)}$.
(4) $p_{r(e)} s_e s_e^* = s_e s_e^*$ for all $e \in E^{(1)}$.
(5) For every $v \in E^{(0)}$ for which $r^{-1}(v) = \big\{e \in E^{(1)} \colon r(e) = v\big\}$ is finite but not empty, we have $\sum_{e \in r^{-1}(v)} s_e s_e^* = p_v$.

We give brief descriptions of some examples. For $n \in \mathbb{Z}_{\geq 0}$, let E_n be the graph with one vertex v and n edges $e_1, e_2, \ldots, e_n$. Here is the picture:



The relations guarantee that $C^*(E_n)$ is unital, with identity p_v. The graph E_0 has no edges, so $C^*(E_0)$ has no other generators, and is isomorphic to $\mathbb{C}$. The graph E_1 gives one additional generator, namely an element s_e such that $s_e^* s_e = s_e s_e^* = p_v = 1$. Thus $C^*(E_1) \cong C(S^1)$. For the graph E_n, with $n \geq 2$, the additional generators are $s_{e_1}, e_{e_2}, \ldots, s_{e_n}$, and the relations are $s_{e_j}^* s_{e_j} = 1$ for $j = 1, 2, \ldots, n$ and $\sum_{j=1}^n s_{e_j} s_{e_j}^* = 1$. Under the identification

$$s_1 = s_{e_1}, \quad s_2 = s_{e_2}, \quad \ldots, \quad s_n = s_{e_n},$$

these obviously generate the Cuntz algebra $\mathcal{O}_n$ which was used in Example 3.20.

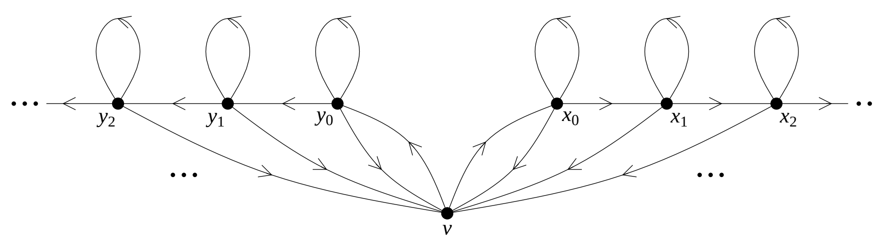
The following well known C*-algebras are also isomorphic to C*-algebras of suitable graphs: the Toeplitz C*-algebra (Example 1.23 of [235]), Cuntz-Krieger algebras (Remark 2.8 of [235]), M_n (this is essentially contained in Proposition 1.18 of [235]), and many AF algebras (Proposition 2.12 and Remark 2.13 of [235]).

Automorphisms of graphs give automorphisms of the corresponding graph algebras. This is essentially immediate from Definition 3.36. See the discussion before Lemma 3.1 and before Example 3.2 in [148]. Here are some specific examples.

Example 3.37. For $n \in \mathbb{Z}_{>0}$ with $n \geq 2$, let E_n be the graph above (with one vertex and n edges). Then the permutation group S_n acts on E_n by permuting the edges. The corresponding action $\alpha \colon S_n \to \mathrm{Aut}(\mathcal{O}_n)$ is given on the generators $s_1, s_2, \ldots, s_n$ by $\alpha_\sigma(s_j) = s_{\sigma(j)}$ for $j = 1, 2, \ldots, n$.

This action is a special case of the quasifree actions in Example 3.20, obtained by restricting from the unitary group $U(M_n)$ to the permutation matrices.

Example 3.38. Consider the following graph Q:



It is taken from the proof of Theorem 2.2 of [262]. We have reversed the arrows, because the convention used in [262] is the opposite to that of Definition 3.36. We have also used different names for the vertices. It is shown in [262] that $C^*(Q)$ (called $\mathcal{O}(Q)$ in the notation of [262]) is the nonunital Kirchberg algebra satisfying the Universal Coefficient Theorem, $K_0(C^*(Q)) = 0$, and $K_1(C^*(Q)) \cong \mathbb{Z}$. (The algebra is nonunital since the graph has infinitely many vertices.)

We derive the computation of $K_1(C^*(Q))$ from Theorem 6.1 of [15]. (The corresponding formula in [262], in Equation (2.2) there, has a misprint: in the formula for $K_1(C^*(E))$, the first condition on $f(x)$ there should be required to hold for all $x \in E^{(0)}$, not just the vertices x which emit a nonzero finite number of edges.)

Accordingly, $K_1(C^*(Q))$ can be identified with the set of functions

$$f\colon \{x_0, x_1, x_2, \ldots\} \cup \{y_0, y_1, y_2, \ldots\} \to \mathbb{Z}$$

which have finite support, such that

$$f(x_0) + f(y_0) = 0 \tag{3.4}$$

and

$$f(x_j) - [f(x_j) + f(x_{j+1})] = 0 \qquad \text{and} \qquad f(y_j) - [f(y_j) + f(y_{j+1})] = 0$$

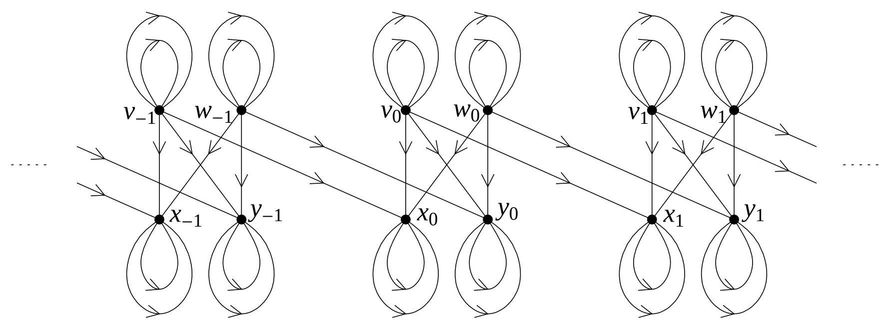
for $j = 0, 1, 2, \ldots$. These simplify to

$$f(x_j) = f(y_j) = 0 \qquad \text{for } j = 1, 2, \ldots. \tag{3.5}$$

One checks immediately that there is an injective homomorphism $\lambda\colon \mathbb{Z} \to K_1(C^*(Q))$ defined by $\lambda(n)(x_0) = n$, $\lambda(n)(y_0) = -n$, and $\lambda(n)(x_j) = \lambda(n)(y_j) = 0$ for $j = 1, 2, \ldots$. It follows easily from (3.4) and (3.5) that λ is surjective.

There is a unique automorphism α of Q of order 2 such that $\alpha(x_j) = y_j$ and $\alpha(y_j) = x_j$ for $j = 0, 1, 2, \ldots$, and $\alpha(v) = v$. It gives rise to an automorphism of $C^*(Q)$ of order 2, which we also call α, such that $\alpha_*\colon K_1(C^*(Q)) \to K_1(C^*(Q))$ is multiplication by -1. This is an example of the conclusion of Corollary 3.41 below. (The paper [262] is a predecessor of [135]. Its Corollary 2.3 is Corollary 3.41 below when n is prime. The method is to construct a suitable automorphism of a suitable graph.)

Example 3.39. Consider the following graph F:



(This graph appears as an example in [194], in the discussion after Example 4.12 of [194]. Its C*-algebra is a nonsimple purely infinite C*-algebra with a composition series whose subquotients have finite primitive ideal spaces.) There is an automorphism $h\colon F \to F$ of order 2 which acts on the vertices by

$$h(v_n) = w_n, \qquad h(w_n) = v_n, \qquad h(x_n) = y_n, \qquad \text{and} \qquad h(y_n) = x_n$$

for $n \in \mathbb{Z}$, and which sends the inner loop at each vertex z to the inner loop at $h(z)$ and the outer loop at z to the outer loop at $h(z)$. This automorphism induces an automorphism φ of $C^*(F)$ of order 2, which was used as an example for a theorem in [194]. The corresponding action of $\mathbb{Z}_2$ on F is free. Free actions on graphs are the subject of a very nice result, Theorem 1.1 of [148], according to which the reduced crossed product is stably isomorphic to the C*-algebra of the quotient graph.

One can easily write down many other examples of actions of finite or infinite groups on this graph, or on others.

We also give some theorems on the existence of actions.

Theorem 3.40 (Theorem 3.5 of [135]). Let G be a finite group such that every Sylow subgroup of G is cyclic. Let A be a Kirchberg algebra (separable nuclear purely infinite simple C*-algebra; Definition 1.1) which satisfies the Universal Coefficient Theorem. Let $\sigma \colon G \to \mathrm{Aut}(K_*(A))$ be an action of G on the K-theory of A. If A is unital, also assume that $\sigma_g([1_A]) = [1_A]$ for all $g \in G$. Then there exists an action $\alpha \colon G \to \mathrm{Aut}(A)$ such that $(\alpha_g)_* = \sigma_g$ for all $g \in G$.

Corollary 3.41 (Corollary 3.6 of [135]). Let A be a Kirchberg algebra which satisfies the Universal Coefficient Theorem. Let $n \in \mathbb{Z}_{>0}$, and let $\sigma \in \mathrm{Aut}(K_*(A))$ be an automorphism such that $\sigma^n = \mathrm{id}_{K_*(A)}$. If A is unital, also assume that $\sigma([1_A]) = [1_A]$. Then there exists an automorphism $\alpha \in \mathrm{Aut}(A)$ such that $\alpha_* = \sigma$ and $\alpha^n = \mathrm{id}_A$.

Theorem 3.42 (Theorem 4.8(3) of [124]). Let Γ_0 and Γ_1 be countable abelian groups which are uniquely 2-divisible. Then there exists an action $\alpha \colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathcal{O}_2)$ such that $B = C^*(\mathbb{Z}/2\mathbb{Z}, \mathcal{O}_2, \alpha)$ satisfies the Universal Coefficient Theorem, $K_0(B) \cong \Gamma_0$, and $K_1(B) \cong \Gamma_1$.

4. Additional Examples of Generalized Gauge Actions

In this section, we give further examples of what we think of as "gauge type" actions. Example 3.15 (on the rotation algebras), Example 3.20 (on Cuntz algebras), Example 3.22, Example 3.23, and the actions in the discussion after Example 3.23 (dual actions), are all of this type.

In many of the examples, there is an action on the C*-algebra which is conventionally referred to as a gauge action. Usually this is an action of S^1. (For the C*-algebras of rank k graphs, discussed in Example 4.7, it is an action of $(S^1)^k$.) In most cases, we give actions of a larger group G, but which is still usually compact. (For $\mathcal{O}_\infty$ (Example 4.5) and Cuntz-Pimsner algebras (Example 4.8), our larger group G is not even locally compact.) There is usually an obvious embedding of S^1 in G as a diagonal in some sense, and the action usually called the gauge action is the restriction to this subgroup.

If $\alpha \colon G \to \mathrm{Aut}(A)$ is an action of a compact group G on a C*-algebra A, then the fixed point algebra A^G and the crossed product $C^*(G, A, \alpha)$ are, in suitable senses, not more complicated than A. Often they are in fact less complicated; indeed, for some of the applications of gauge actions, this is an important feature. Since the main thrust of the later part of these notes is situations in which the crossed products are more complicated than the original algebra, these examples are thus less relevant than some of the others. However, one can often get more relevant examples by considering actions of other groups which factor through a gauge action or an action of one of the larger groups in the examples of this section. As a very elementary example, let $\alpha \colon S^1 \to \mathrm{Aut}(C(S^1))$ be the rotation action (Example 2.12 with $G = S^1$). This action is the dual action from the identification of $C(S^1)$ as $C^*(\mathbb{Z}, \mathbb{C})$ using the trivial action of $\mathbb{Z}$ on $\mathbb{C}$. It is also the gauge action of S^1 obtained from Example 4.6 using the realization of $C(S^1)$ as the C*-algebra of the graph with one vertex and one edge, as in the discussion after Definition 3.36. The fixed point algebra is clearly $\mathbb{C}$. The crossed product is $K(L^2(S^1))$. (See the discussion at the beginning of Example 10.8.) However, for $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the irrational rotation action of $\mathbb{Z}$ (see Example 2.16) is the composition of α with the homomorphism $\mathbb{Z} \to S^1$ given by $n \mapsto \exp(2\pi i n)$ for $n \in \mathbb{Z}$. By Example 10.25, the crossed product is the well known irrational rotation algebra of Example 3.10. This algebra is a simple infinite dimensional C*-algebra not of type I.

The action of Example 3.15 generalizes to an arbitrary higher dimensional noncommutative torus.

Example 4.1. Let $d \in \mathbb{Z}_{>0}$ with $d \geq 2$. Let θ be a skew symmetric real $d \times d$ matrix. Let A_θ be the (higher dimensional) noncommutative torus of Example 3.11. By similar reasoning as in Example 3.15, there is an action $\alpha \colon (S^1)^d \to \mathrm{Aut}(A_\theta)$ determined by $\alpha_{(\zeta_1, \zeta_2, \ldots, \zeta_d)}(u_j) = \zeta_j u_j$ for $j = 1, 2, \ldots, d$.

Also, as in Example 3.15, each individual element $(\zeta_1, \zeta_2, \ldots, \zeta_d) \in (S^1)^d$ gives an automorphism of A_θ, and hence an action of $\mathbb{Z}$ on A_θ. As mentioned in Example 3.15, using these automorphisms, it is possible to realize an arbitrary higher dimensional noncommutative torus as an iterated crossed product by $\mathbb{Z}$, starting with a rotation algebra.

Finite subgroups of $(S^1)^d$ give actions of finite abelian groups on A_θ. Although we will not prove it in these notes, their crossed products turn out to be strongly Morita equivalent to other higher dimensional noncommutative tori.

Example 4.2. Recall that the *unilateral shift* is the operator s on $l^2(\mathbb{Z}_{\geq 0})$ which sends a sequence $\xi = (\xi_0, \xi_1, \xi_2, \ldots)$ to the sequence $s\xi = (0, \xi_0, \xi_1, \xi_2, \ldots)$. One checks that

$$s^*(\xi_0, \xi_1, \xi_2, \ldots) = (\xi_1, \xi_2, \xi_3, \ldots).$$

(This operator is called the backward shift.) The C*-subalgebra $T \subset L(H)$ generated by s is called the *Toeplitz algebra*. We recall that there is an exact sequence

$$0 \longrightarrow K(l^2(\mathbb{Z}_{\geq 0})) \longrightarrow T \longrightarrow C(S^1) \longrightarrow 0,$$

in which the map $T \to C(S^1)$ sends s to the function $f(\zeta) = \zeta$ for $\zeta \in S^1$.

The algebra T can also be obtained as the universal C*-algebra generated by an isometry (which is s). (See Example 1.3(e)(6) of [20].)

There is a unique action $\gamma \colon S^1 \to \mathrm{Aut}(T)$ such that $\alpha_\zeta(s) = \zeta s$ for all $\zeta \in S^1$.

Uniqueness follows from the fact that s generates T.

Existence is immediate from the description of T as a universal C*-algebra, but we can also give a direct proof using the description as a subalgebra of $L(l^2(\mathbb{Z}_{\geq 0}))$. For $\zeta \in S^1$, define a unitary $u_\zeta \in L(l^2(\mathbb{Z}_{\geq 0}))$ by

$$u_\zeta(\xi_0, \xi_1, \xi_2, \ldots) = (\xi_0, \zeta\xi_1, \zeta^2\xi_2, \ldots).$$

Then one checks that $u_\zeta s u_\zeta^* = \zeta s$. It follows that $u_\zeta T u_\zeta^* \subset T$. Moreover, since ζs generates T just as well as s does, we get $u_\zeta T u_\zeta^* = T$. Since $u_{\zeta_1} u_{\zeta_2} = u_{\zeta_1 \zeta_2}$ for $\zeta_1, \zeta_2 \in S^1$, it follows that the formula $\alpha_\zeta(a) = u_\zeta a u_\zeta^*$ defines a homomorphism from S^1 to $\mathrm{Aut}(T)$.

Continuity of this action follows from Lemma 3.14.

Example 4.3. Let $n \in \mathbb{Z}_{>0}$. Recall that the extended Cuntz algebra E_n is the universal unital C*-algebra on generators $s_1, s_2, \ldots, s_n$, subject to the relations stating that $s_j^* s_j = 1$ for $1 \leq j \leq n$ and $s_1 s_1^*$, $s_2 s_2^*$, $\ldots$, $s_n s_n^*$ are orthogonal projections. (The difference from the relations in Example 3.20 is that we no longer require that $\sum_{j=1}^n s_j s_j^* = 1$. It follows that $\mathcal{O}_n$ is a quotient of E_n. The kernel is K.)

The same formula as in Example 3.20 defines an action of $U(M_n)$ on E_n. That is, if $u = (u_{j,k})_{j,k=1}^n \in M_n$ is unitary, then there is an automorphism β_u of E_n such that

$$\beta_u(s_j) = \sum_{k=1}^{n} u_{k,j} s_k$$

for $j = 1, 2, \ldots, n$.

The restriction to S^1, realized as the scalar multiples of the identity in $U(M_n)$, is the gauge action on E_n.

The case $n = 1$ makes sense. The algebra is then the Toeplitz algebra of Example 4.2, and the action is the same action as in Example 4.2.

Example 4.4. Recall that the Cuntz algebra $\mathcal{O}_\infty$ is the universal unital C*-algebra on generators $s_1, s_2, \ldots$, subject to the relations stating that $s_j^* s_j = 1$ for $j \in \mathbb{Z}_{>0}$ and $s_1 s_1^*$, $s_2 s_2^*$, ... are orthogonal projections. (Like $\mathcal{O}_n$, it is in fact simple, so any C*-algebra generated by elements satisfying these relations is isomorphic to $\mathcal{O}_\infty$.)

Now let $u \in L(l^2(\mathbb{Z}_{>0}))$ be unitary. Write u in infinite matrix form, as $u = (u_{j,k})_{j,k=1}^n$. Then there is an automorphism α_u of $\mathcal{O}_\infty$ such that

$$\alpha_u(s_j) = \sum_{k=1}^{\infty} u_{k,j} s_k$$

for $j \in \mathbb{Z}_{>0}$. By Exercise 4.5 below, $u \mapsto \alpha_u$ a continuous action of the unitary group $U(l^2(\mathbb{Z}_{>0}))$ on $\mathcal{O}_\infty$. Its restriction to S^1, realized as the scalar multiples of the identity in $U(l^2(\mathbb{Z}_{>0}))$, is the gauge action on $\mathcal{O}_\infty$.

The restriction of this action to S^1 is used for a counterexample in the discussion after Theorem 15.26.

Exercise 4.5. Verify that the formula given in Example 4.4 does in fact define a continuous action of $U(l^2(\mathbb{Z}_{>0}))$ on $\mathcal{O}_\infty$. (Among other things, one must show that the series in the definition of $\alpha_u(s_j)$ actually converges.)

Example 4.6. Recall from Definition 3.36 that the C*-algebra $C^*(E)$ of a directed graph $E = \left(E^{(0)}, E^{(1)}\right)$ is generated by projections p_v for $v \in E^{(0)}$ and partial isometries s_e for $e \in E^{(1)}$. There is a gauge action α of S^1 on $C^*(E)$, defined by $\alpha_\zeta(p_v) = p_v$ for $v \in E^{(0)}$ and $\alpha_\zeta(s_e) = \zeta s_e$ for $e \in E^{(1)}$. See Proposition 2.1 of [235] for the case of a row-finite graph. The gauge action plays a fundamental role in the theory of graph C*-algebras, as can be seen from [235].

This action generalizes the gauge actions in Example 3.20, Example 4.2, and Example 4.3.

The action extends to an action β of $G = \prod_{e \in E^{(1)}} S^1$. For $\zeta = (\zeta_e)_{e \in E^{(1)}}$, we take $\beta_\zeta(p_v) = p_v$ for $v \in E^{(0)}$ and $\beta_\zeta(s_e) = \zeta_e s_e$ for $e \in E^{(1)}$.

Example 4.7. Higher rank graphs and their C*-algebras are a generalization of graph C*-algebras. They are described in Chapter 10 of [235], the C*-algebra being defined under the assumption that the graph is row finite and has no sources. (Weaker conditions are also considered.) We do not repeat the definitions of higher rank graphs and their C*-algebras here, but we give some of the ideas. A graph of rank k has edges of k colors, and there are specific conditions relating edges of different colors. The C*-algebra $C^*(E)$ of a row finite higher rank graph E with no sources is generated by a family of projections, one for each vertex, and a family of partial isometries, one for each finite path in the graph. A finite path in a rank k graph E has a degree $n = (n_1, n_2, \ldots, n_k) \in (\mathbb{Z}_{\geq 0})^k$, in which n_j is the number of edges in the path of color j. There is a gauge action $\alpha\colon (S^1)^k \to \mathrm{Aut}(C^*(E))$, described after Corollary 10.13 [235]. For $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_k) \in (S^1)^k$, the automorphism α_ζ fixes the projections corresponding to the vertices, and multiplies the partial isometry corresponding to a finite path of degree n by $\zeta_1^{n_1} \zeta_2^{n_2} \cdots \zeta_k^{n_k}$. This action generalizes the gauge action of S^1 on a graph C*-algebra in Example 4.6. It plays a role in the theory of C*-algebras of higher rank graphs similar to the role of a gauge action of S^1 in the theory of ordinary directed graphs.

Example 4.8. The algebras now known as Cuntz-Pimsner algebras were introduced in [220]. Also see Chapter 8 of [235]. We don't give details here, but we give a brief outline. One starts with a C*-algebra A and a Hilbert bimodule E over A, that is, a right Hilbert module E over A with a homomorphism from A to the algebra $L(E)$ of adjointable right A-module homomorphisms of E. (The perhaps more descriptive term "correspondence" is used instead of "Hilbert bimodule" in [235]. See the discussion after Example 8.4 of [235].) One constructs a Toeplitz algebra $\mathcal{T}_E$, which is described in Definition 1.1 of [220] and after Proposition 8.8 of [235]. It is generated by creation and annihilation operators on the Fock space made from E. There is further a Cuntz-Pimsner algebra $\mathcal{O}_E$, given in Definition 1.1 of [220] and after Proposition 8.11 of [235]. It is a suitable quotient of $\mathcal{T}_E$.

The gauge action $\lambda\colon S^1 \to \mathrm{Aut}(\mathcal{T}_E)$ is described on page 198 of [220]. (The algebra $\mathcal{P}_E$ which appears there is described at the beginning of Section 3 [220].) The associated $\mathbb{Z}$-grading is given in Proposition 8.9 of [235]. The automorphism λ_ζ multiplies the creation operator coming from an element of $E^{\otimes n}$ by ζ^n. This action descends to a gauge action of S^1 on $\mathcal{O}_E$.

As described in the Examples starting on page 192 of [220], Cuntz-Pimsner algebras generalize Cuntz algebras, Cuntz-Krieger algebras, crossed products by actions of $\mathbb{Z}$, and crossed products by partial actions of $\mathbb{Z}$. The corresponding gauge actions of S^1 turn out to be the usual gauge actions on the Cuntz algebras (Example 3.20) and Cuntz-Krieger algebras and the dual actions on the crossed products (Example 3.22 for an action of $\mathbb{Z}$). Graph C*-algebras (Definition 3.36) are special cases of Cuntz-Pimsner algebras (Example 8.13 of [235]), and this example generalizes Example 4.6.

Example 4.9. In the situation of Example 4.8, as with various other examples of gauge actions, there is in fact an action of a much bigger group. Again let A be a C*-algebra, let E be a Hilbert bimodule (or correspondence) over A, and let $\mathcal{T}_E$ and $\mathcal{O}_E$ be the associated Toeplitz and Cuntz-Pimsner algebras. As described in Remark 4.10(2) of [220], the whole automorphism group $\mathrm{Aut}(E)$ of E acts on $\mathcal{T}_E$ and $\mathcal{O}_E$. In fact, consider the group $\mathrm{Aut}(A, E)$ of automorphisms of the pair (A, E), that is, pairs (α, σ) consisting of an automorphism $\alpha \in \mathrm{Aut}(A)$ and an automorphism of E as a Banach space which is compatible with α in a suitable sense. Then $\mathrm{Aut}(A, E)$ acts on $\mathcal{T}_E$ and $\mathcal{O}_E$.

Exercise 4.10. In Example 4.9, take $A = \mathbb{C}$ and $E = \mathbb{C}^n$. Then $\mathcal{T}_E$ is the extended Cuntz algebra E_n of Example 4.3 and $\mathcal{O}_E$ is the Cuntz algebra $\mathcal{O}_n$ as in Example 3.20.

Prove that the action of $\mathrm{Aut}(E)$ on $\mathcal{O}_n$ can be identified with the action of $U(M_n)$ on $\mathcal{O}_n$ given in Example 3.20, and that action of $\mathrm{Aut}(E)$ on E_n can be identified with the action of $U(M_n)$ on E_n given in Example 4.3.

Examples 4.11, 4.12, and 4.14 use full and reduced group C*-algebras of discrete groups, which are formally introduced in Section 5, and Exercise 4.13 and part of Example 4.14 use full crossed products (Section 8) and reduced crossed products (Section 9).

Example 4.11. Let F_n be the free group on n generators. Then $C^*(F_n)$ is the universal C*-algebra generated by n unitaries $u_1, u_2, \ldots, u_n$, with no other relations. (Use the description of the group C*-algebra in Exercise 5.19.) It follows that for $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in (S^1)^n$, there is a homomorphism $\alpha_\zeta \colon C^*(F_n) \to C^*(F_n)$ such that $\alpha_\zeta(u_k) = \zeta_k u_k$ for $k = 1, 2, \ldots, n$. Lemma 3.14 implies that these homomorphisms define a continuous action $\alpha \colon (S^1)^n \to \mathrm{Aut}(C^*(F_n))$.

An analogous procedure works for the full C*-algebra of the free group on countably many generators, or even on an arbitrary set of generators.

Example 4.12. The action α of $(S^1)^n$ on $C^*(F_n)$ in Example 4.11 descends to an action of $(S^1)^n$ on the reduced group C*-algebra $C^*_{\mathrm{r}}(F_n)$. (See Definition 5.20.) That is, letting $\pi \colon C^*(F_n) \to C^*_{\mathrm{r}}(F_n)$ be the quotient map, there is an action $\beta \colon (S^1)^n \to \mathrm{Aut}(C^*_{\mathrm{r}}(F_n))$ such that for every $\zeta \in (S^1)^n$, we have $\pi \circ \alpha_\zeta = \beta_\zeta \circ \pi$.

We prove this by exhibiting unitaries in $L(l^2(F_n))$ which implement the action β. For $g \in F_n$, let δ_g denote the corresponding element of the standard Hilbert basis for $l^2(F_n)$. Let $g_1, g_2, \ldots, g_n$ denote the standard generators of F_n. Then the unitaries $u_1, u_2, \ldots, u_n$ of Example 4.11 are the standard unitaries $u_{g_1}, u_{g_2}, \ldots, u_{g_n}$ of the group C*-algebra. Let $\gamma_k \colon F_n \to \mathbb{Z}$ be the homomorphism determined by $\gamma_k(g_k) = 1$ and $\gamma_k(g_j) = 0$ for $j \neq k$. For $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in (S^1)^n$, define a unitary $v_\zeta \in L(l^2(F_n))$ by

$$v_\zeta \delta_g = \zeta_1^{\gamma_1(g)} \zeta_2^{\gamma_2(g)} \cdots \zeta_n^{\gamma_n(g)} \delta_g$$

for $g \in F_n$. Then one can check that

$$v_\zeta \pi(u_k) v_\zeta^* = \pi(\alpha_\zeta(u_k))$$

for $k = 1, 2, \ldots, n$ and all $\zeta \in (S^1)^n$. This proves the existence of β.

Continuity follows easily from continuity of α.

Exercise 4.13. Show that the constructions in Examples 4.11 and 4.12 work not just for the full and reduced C*-algebras of F_n, but for full and reduced crossed products by F_n. For the full crossed product, use the description of the crossed product in Theorem 8.21. For the reduced crossed product, see Definition 9.4.

There are other groups for which there is a construction similar to that of Example 4.11, Example 4.12, and Exercise 4.13. Here is one such example.

Example 4.14. Recall (see the discussion before Corollary 1.27 of [106]) that the fundamental group Γ_n of a compact orientable surface of genus n is generated by $2n$ elements

$$g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_n$$

subject to the single relation (with $[g, h]$ denoting the group commutator $[g, h] = ghg^{-1}h^{-1}$)

$$[g_1, h_1][g_2, h_2] \cdots [g_n, h_n] = 1.$$

It follows that $C^*(\Gamma_n)$ is the universal C*-algebra generated by unitaries

$$u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$$

(with, following Notation 5.21 below, $u_j = u_{g_j}$ and $v_j = u_{h_j}$ for $j = 1, 2, \ldots, n$), subject to the single additional relation

$$(u_1 v_1 u_1^* v_1^*)(u_2 v_2 u_2^* v_2^*) \cdots (u_n v_n u_n^* v_n^*) = 1.$$

If

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in (S^1)^n \qquad \text{and} \qquad \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in (S^1)^n,$$

then the elements

$$\lambda_1 u_1,\ \lambda_2 u_2,\ \ldots,\ \lambda_n u_n,\ \zeta_1 v_1,\ \zeta_2 v_2,\ \ldots,\ \zeta_n v_n \in C^*(\Gamma_n)$$

are also unitaries satisfying the same additional relation. Therefore there is a unique endomorphism $\alpha_{\lambda,\zeta} \colon C^*(\Gamma_n) \to C^*(\Gamma_n)$ such that

$$\alpha_{\lambda,\zeta}(u_1) = \lambda_1 u_1, \qquad \alpha_{\lambda,\zeta}(u_2) = \lambda_2 u_2, \qquad \ldots, \qquad \alpha_{\lambda,\zeta}(u_n) = \lambda_n u_n,$$

and

$$\alpha_{\lambda,\zeta}(v_1) = \zeta_1 v_1, \qquad \alpha_{\lambda,\zeta}(v_2) = \zeta_2 v_2, \qquad \ldots, \qquad \alpha_{\lambda,\zeta}(v_n) = \zeta_n v_n.$$

Lemma 3.14 implies that these endomorphisms actually form a continuous action $\alpha \colon (S^1)^{2n} \to \mathrm{Aut}(C^*(\Gamma_n))$.

An argument similar to that in Example 4.12 shows that the action α descends to an action $\beta \colon (S^1)^{2n} \to \mathrm{Aut}(C^*_{\mathrm{r}}(\Gamma_n))$. For $g \in \Gamma_n$, let δ_g denote the corresponding element of the standard Hilbert basis for $l^2(\Gamma_n)$. For $k = 1, 2, \ldots, n$, there is a unique group homomorphism $\gamma_k \colon \Gamma_n \to \mathbb{Z}$ such that $\gamma_k(g_k) = 1$, $\gamma_k(g_j) = 0$ for $j \neq k$, and $\gamma_k(h_j) = 0$ for $j = 1, 2, \ldots, n$, and there is a unique group homomorphism $\rho_k \colon \Gamma_n \to \mathbb{Z}$ such that $\rho_k(g_j) = 0$ for $j = 1, 2, \ldots, n$, $\rho_k(h_k) = 1$, and $\rho_k(h_j) = 0$ for $j \neq k$. For

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in (S^1)^n \qquad \text{and} \qquad \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in (S^1)^n,$$

define a unitary $v_{\lambda,\zeta} \in L(l^2(F_n))$ by

$$v_{\lambda,\zeta} \delta_g = \lambda_1^{\gamma_1(g)} \lambda_2^{\gamma_2(g)} \cdots \lambda_n^{\gamma_n(g)} \zeta_1^{\rho_1(g)} \zeta_2^{\rho_2(g)} \cdots \zeta_n^{\rho_n(g)} \delta_g$$

for $g \in \Gamma_n$. Then one can check that

$$v_{\lambda,\zeta} \pi(u_k) v_{\lambda,\zeta}^* = \pi(\alpha_{\lambda,\zeta}(u_k)) \qquad \text{and} \qquad v_{\lambda,\zeta} \pi(v_k) v_{\lambda,\zeta}^* = \pi(\alpha_{\lambda,\zeta}(v_k))$$

for $k = 1, 2, \ldots, n$ and all $\lambda, \zeta \in (S^1)^n$. This proves the existence of β.

One also checks, in the same way as for Exercise 4.13, that the same thing works for full and reduced crossed products by Γ_n. We omit the details.

Example 4.15. The C*-algebra U_n^{nc} is defined to be the universal unital C*-algebra generated by elements $u_{j,k}$, for $1 \leq j, k \leq n$, subject to the relation that the matrix

$$u = \begin{pmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,n} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n,1} & u_{n,2} & \cdots & u_{n,n} \end{pmatrix} \in M_n(U_n^{\mathrm{nc}})$$

is unitary. This amounts to $2n^2$ relations on the generators $u_{j,k}$, namely

$$\sum_{k=1}^{n} u_{j,k} u_{l,k}^* = \delta_{j,l} \qquad \text{and} \qquad \sum_{k=1}^{n} u_{k,j}^* u_{k,l} = \delta_{j,l}$$

for $1 \leq j, k \leq n$. (This C*-algebra was introduced in (2b) in Section 3 of [33].)

There is an action α of the unitary group $U(M_n)$ on U_n^{nc}, defined as follows. Let

$$g = \begin{pmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n,1} & g_{n,2} & \cdots & g_{n,n} \end{pmatrix} \in U(M_n).$$

Regard g as an element of $M_n(U_n^{\mathrm{nc}})$ via the unital inclusion of $\mathbb{C}$ in U_n^{nc}. Then the product gu is defined and is unitary. Thus its entries

$$(gu)_{j,l} = \sum_{k=1}^{n} g_{j,k} u_{k,l}$$

form an $n \times n$ unitary matrix. So there exists a unique unital homomorphism $\alpha_g \colon U_n^{\mathrm{nc}} \to U_n^{\mathrm{nc}}$ such that $\alpha_g(u_{j,l}) = (gu)_{j,l}$ for $j, l = 1, 2, \ldots, n$. The proof that this gives a continuous action is requested in Exercise 4.16.

Any unitary representation of a group G in M_n therefore also gives an action of G on U_n^{nc}. Here is a special case, coming from the representation

$$\zeta \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$$

of S^1 on $\mathbb{C}^2$. For $\zeta \in S^1$, we take $\sigma_\zeta \in \mathrm{Aut}(U_n^{\mathrm{nc}})$ to be the automorphism determined by

$$\sigma_\zeta(u_{1,1}) = u_{1,1}, \quad \sigma_\zeta(u_{1,2}) = u_{1,2}, \quad \sigma_\zeta(u_{2,1}) = \zeta u_{2,1}, \quad \text{and} \quad \sigma_\zeta(u_{2,2}) = \zeta u_{2,2}.$$

There is a second action β of $U(M_n)$ on U_n^{nc}, determined by $\beta_g(u_{j,l}) = (ug^*)_{j,l}$ for $g \in U(M_n)$ and $j, l = 1, 2, \ldots, n$. These actions are different, as can be checked with $n = 2$ and

$$u = \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$$

(as above): one now gets

$$u_{1,1} \mapsto u_{1,1}, \quad u_{1,2} \mapsto u_{1,2}, \quad u_{2,1} \mapsto \zeta u_{2,1}, \quad \text{and} \quad u_{2,2} \mapsto \zeta u_{2,2}.$$

A third action comes from letting $U(M_n)$ act on M_n by conjugation. The same matrix u as above now gives the automorphism determined by

$$u_{1,1} \mapsto u_{1,1}, \quad u_{1,2} \mapsto \zeta^{-1} u_{1,2}, \quad u_{2,1} \mapsto \zeta u_{2,1}, \quad \text{and} \quad u_{2,2} \mapsto u_{2,2}.$$

Exercise 4.16. Prove that the definition of the action of $U(M_n)$ on U_n^{nc} given in Example 4.15 actually gives a continuous action.

Example 4.17. Let U_n^{nc}, its generators $u_{j,k}$ for $1 \leq j, k \leq n$, and the unitary matrix

$$u = \begin{pmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,n} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n,1} & u_{n,2} & \cdots & u_{n,n} \end{pmatrix} \in M_n(U_n^{\mathrm{nc}}),$$

be as in Example 4.15. Let $h \colon S^1 \to S^1$ be a continuous map. Then functional calculus gives an element $h(u) \in M_n(U_n^{\mathrm{nc}})$, so that there is an endomorphism $\alpha_h \colon U_n^{\mathrm{nc}} \to U_n^{\mathrm{nc}}$ such that $\alpha_h(u_{j,k}) = h(u)_{j,k}$ for $1 \leq j, k \leq n$. This endomorphism is uniquely determined by the relation $(\mathrm{id}_{M_n} \otimes \alpha_h)(u) = h(u)$.

Suppose $h_1, h_2 \colon S^1 \to S^1$ are continuous. We prove that $\alpha_{h_1 \circ h_2} = \alpha_{h_2} \circ \alpha_{h_1}$, that is, that $h \mapsto \alpha_h$ is an antihomomorphism from the semigroup of continuous maps $S^1 \to S^1$ to the semigroup of endomorphisms of U_n^{nc}. To prove the claim, first observe that for any C*-algebras A and B, any unital homomorphism $\varphi \colon A \to B$, and any unitary $v \in A$, we have $\varphi(h_2(v)) = h_2(\varphi(v))$. Apply this fact with $\varphi = \mathrm{id}_{M_n} \otimes \alpha_{h_1}$ and $v = u$ at the third step in the following calculation:

$$\begin{aligned}(\mathrm{id}_{M_n} \otimes \alpha_{h_1 \circ h_2})(u) &= h_1(h_2(u)) = h_1\big((\mathrm{id}_{M_n} \otimes \alpha_{h_2})(u)\big) \\ &= (\mathrm{id}_{M_n} \otimes \alpha_{h_2})(h_1(u)) = \big[(\mathrm{id}_{M_n} \otimes \alpha_{h_2}) \circ (\mathrm{id}_{M_n} \otimes \alpha_{h_1})\big](u).\end{aligned}$$

The claim follows.

The claim implies that, in particular, $h \mapsto \alpha_{h^{-1}}$ is a well defined action of the group of homeomorphisms of S^1 on U_n^{nc}.

Some special cases: take the rotations by all $\zeta \in S^1$ to get an action of S^1; take a rotation by a fixed $\zeta \in S^1$ to get an action of $\mathbb{Z}$ which is a noncommutative analog of a rational or irrational rotation; take a rotation by $e^{2\pi i l/m}$ to get an action of $\mathbb{Z}/m\mathbb{Z}$. In general, if $h \colon S^1 \to S^1$ is any fixed homeomorphism, then $n \mapsto \alpha_h^n$ is an action of $\mathbb{Z}$ on U_n^{nc}.

There is a reduced version $U_{n,\mathrm{red}}^{\mathrm{nc}}$ of the algebra U_n^{nc} used in Examples 4.15 and 4.17, and presumably there are reduced versions of some of the actions above. The algebra is defined in the discussion after Proposition 3.1 of [168]. To describe it, start with the fact (Proposition 2.2 of [168]) that U_n^{nc} can be identified with the relative commutant of M_n in the amalgamated free product $M_n \star_{\mathbb{C}} C(S^1)$, the amalgamation identifying the subalgebras $\mathbb{C} \cdot 1$ in both factors. The isomorphism

$$\varphi \colon U_n^{\mathrm{nc}} \to M_n' \cap \big(M_n \star_{\mathbb{C}} C(S^1)\big)$$

is defined as follows. We let $e_{j,k} \in M_n$ be the standard matrix units, and we let $z \in C(S^1)$ be the function $z(\zeta) = \zeta$ for $\zeta \in S^1$. Then

$$\varphi(u_{j,k}) = \sum_{l=1}^{n} e_{l,j} z e_{k,l}.$$

for $j, k = 1, 2, \ldots, n$. Now take $U_{n,\mathrm{red}}^{\mathrm{nc}}$ to be the relative commutant of M_n in the reduced amalgamated free product $M_n \star_{\mathbb{C},\mathrm{r}} C(S^1)$ with respect to the unique tracial state on M_n and Lebesgue measure on S^1, the amalgamation identifying the subalgebras $\mathbb{C} \cdot 1$ in both factors as above. It is thus a quotient of the algebra U_n^{nc}.

The actions of Example 4.15 presumably descend to actions on $U_{n,\mathrm{red}}^{\mathrm{nc}}$. Similarly, the automorphism α_h of Example 4.17 presumably descends to an automorphism of $U_{n,\mathrm{red}}^{\mathrm{nc}}$ provided h preserves Lebesgue measure on S^1. In particular, the rotation action of S^1, the rational and irrational rotations, and the rotation actions of $\mathbb{Z}/m\mathbb{Z}$ presumably all descend to actions on $U_{n,\mathrm{red}}^{\mathrm{nc}}$. As far as we know, nobody has checked that any of these presumed actions really exists.

Example 4.18. Example 4.15 can be generalized as follows. Let $m, n \in \mathbb{Z}_{>0}$. The C*-algebra $U_{m,n}^{\mathrm{nc}}$, introduced in Section 2 of [169], is defined to be the universal unital C*-algebra generated by elements $u_{j,k}$, for $1 \leq j \leq m$ and $1 \leq k \leq n$, subject to the relation that the $m \times n$ matrix

$$u = \begin{pmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,n} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m,1} & u_{m,2} & \cdots & u_{m,n} \end{pmatrix}$$

is unitary, that is, $u^* u$ is the identity matrix in $M_n(U_{m,n}^{\mathrm{nc}})$ and uu^* is the identity matrix in $M_m(U_{m,n}^{\mathrm{nc}})$.

The constructions of Example 4.15 now give actions of $U(M_m)$ and $U(M_n)$ on $U_{m,n}^{\mathrm{nc}}$.

If $m = 1$, then the relations are exactly those for the Cuntz algebra $\mathcal{O}_n$, and the action of $U(M_n)$ generalizes the action of $U(M_n)$ on $\mathcal{O}_n$ of Example 3.20.

A free product description of $U_{m,n}^{\mathrm{nc}}$ is given in Section 2 of [169], but it is of a different form from the free product description after Example 4.17 for the case $m = n$. As far as we know, no reduced version of $U_{m,n}^{\mathrm{nc}}$ (analogous to the algebra $U_{n,\mathrm{red}}^{\mathrm{nc}}$ discussed after Example 4.17) has been proposed. One does not expect automorphisms or actions like those of Example 4.17, because the matrix u here, not being square, isn't an element of a C*-algebra.

Part 2. **Group C*-algebras and Crossed Products**

5. C*-Algebras of Discrete Groups

The main focus of these notes is the structure of certain kinds of crossed products. The C*-algebra of a group is a special case of a crossed product—it comes from the trivial action of the group on $\mathbb{C}$—but not one of the ones we are mainly concerned with. We devote this section and Section 7 to group C*-algebras anyway, in order to provide an introduction to crossed products in a simpler case, and because understanding the group C*-algebra is helpful, at least at a heuristic level, for understanding more general crossed products. Section 8 treats crossed product C*-algebras and Section 9 treats reduced crossed product C*-algebras. In Section 10 we give a number of explicit computations of crossed product C*-algebras. The brief Section 6 contains a proof that the reduced C*-algebra of a finitely generated nonabelian free group is simple.

We recall that, by convention, all topological groups will be assumed to be Hausdorff.

We start with discrete groups (groups with the discrete topology), because this case avoids many technicalities. Moreover, in the later part of these notes, almost all groups will be discrete. (The term "discrete" could be considered redundant. We routinely include it anyway for clarity.) C*-algebras of locally compact groups will be discussed in Section 7, but in less detail and without full proofs. However, some of the elementary definitions in this section, and some theorems (in particular, the summary of duality and the Fourier transform for locally compact abelian groups), are given for general locally compact groups, to avoid later repetition.

The notation we use is chosen to avoid conflicts with later notation for crossed products and other C*-algebras. The letters most commonly used for unitary representations of locally compact groups are π and σ (which we use for representations of C*-algebras) and u (we use u_g for the image of the group element g in the group ring and various C*-algebras made from it). Our notation for group rings is designed to be compatible with commonly used notation for crossed product C*-algebras, and is not the same as the notation usually used in algebra. The common notation λ for the left regular representation of a locally compact group also conflicts with notation we use elsewhere.

The construction of $C^*(G)$ is designed so that the representations of $C^*(G)$ are the "same" as the unitary representations of G.

Notation 5.1. Let H be a Hilbert space We denote by $U(H)$ the unitary group of H.

We repeat for reference the standard definition of a unitary representation. Our main reason is to emphasize the topology in which continuity is required.

Definition 5.2. Let G be a topological group and let H be a nonzero Hilbert space. A *unitary representation* of G on H is a group homomorphism $w \colon G \to U(H)$ which is continuous in the strong operator topology on $L(H)$, that is, such that for every $\xi \in H$, the function $g \mapsto w(g)\xi$ is a continuous function from G to H with the norm topology on H.

Norm continuity of representations is much too strong a condition to be useful. For example, it follows from Exercise 7.3 that the left regular representation (Definition 5.3 below) of a locally compact group which is not discrete is never norm continuous. Of course, if G is discrete, the main subject of this section, there is no difference.

Since representations of groups are not the main subject of these notes, we won't give a list of examples. But we want to mention at least two: the one dimensional trivial representation, which sends every group element to the identity operator on a one dimensional Hilbert space, and the left regular representation.

Definition 5.3. Let G be a discrete group. The *left regular representation* of G is the representation $v \colon G \to U(l^2(G))$ given by $(v(g)\xi)(h) = \xi(g^{-1}h)$ for $g, h \in G$ and $\xi \in l^2(G)$.

Exercise 5.4. Prove that the formula of Definition 5.3 gives a unitary representation $v \colon G \to U(l^2(G))$.

The main point of this exercise is to see why g^{-1} appears in the formula.

Here is an alternative description of the left regular representation. For $h \in G$, let $\delta_h \in l^2(G)$ be the standard basis vector corresponding to h. Then v is determined by $v(g)\delta_h = \delta_{gh}$ for $g, h \in G$.

There is also a right regular representation $w \colon G \to U(l^2(G))$, given by $(w(g)\xi)(h) = \xi(hg)$ for $g, h \in G$ and $\xi \in l^2(G)$. It is determined by $w(g)\delta_h = \delta_{hg^{-1}}$ for $g, h \in G$.

Remark 5.5. The elementary theory of unitary representations of topological groups is very much like the elementary theory of representations of C*-algebras. Unitary equivalence, invariant subspaces, irreducible representations, subrepresentations, direct sums (not necessarily finite) of representations, and cyclic vectors and cyclic representations, are all defined just as for representations of C*-algebras. The same proofs as for C*-algebras show that the orthogonal complement of an invariant subspace is again invariant, so that every subrepresentation is a direct summand, and that every representation is a direct sum of cyclic representations. All of this can be found in Section 3.1 of [87].

Exercise 5.6. Supply the definitions and prove the statements in Remark 5.5.

Remark 5.7. There is one significant construction for unitary representations of topological groups which does not make sense for representations of general C*-algebras, namely the tensor product of two representations. Let G be a topological group, let H_1 and H_2 be Hilbert spaces, and let $w_1 \colon G \to U(H_1)$ and $w_2 \colon G \to U(H_2)$ be unitary representations. Then there is a unitary representation

$$w_1 \otimes w_2 \colon G \to U(H_1 \otimes H_2)$$

(using the Hilbert space tensor product) such that $(w_1 \otimes w_2)(g) = w_1(g) \otimes w_2(g)$ for all $g \in G$.

The construction can be found in Section 7.3 of [87], which starts with the construction of the Hilbert space tensor product of Hilbert spaces. Our $w_1 \otimes w_2$ is what is called the inner tensor product before Theorem 7.20 of [87]. (Section 7.3 of [87] is mainly about the tensor product of representations of two groups as a representation of the product of the groups, a construction for which there is an analog for representations of general C*-algebras.)

We make only a little use of tensor products of representations, because there is no analog in the context of crossed products. However, some parts of the representation theory of compact groups are primarily concerned with how a tensor product of two irreducible representations decomposes as a direct sum of other irreducible representations.

One consequence of the properties of the C*-algebra of a locally compact group is that the elementary representation theory of locally compact groups is a special case of the elementary representation theory of C*-algebras.

We start with a purely algebraic construction, the group ring.

Definition 5.8. A **-algebra* over the complex numbers is a complex algebra A with an adjoint operation $a \mapsto a^*$ satisfying the following properties:

(1) $(a+b)^* = a^* + b^*$ for all $a, b \in A$.
(2) $(\lambda a)^* = \overline{\lambda} a^*$ for all $a \in A$ and $\lambda \in \mathbb{C}$.
(3) $(ab)^* = b^* a^*$ for all $a, b \in A$.
(4) $a^{**} = a$ for all $a \in A$.

If A and B are complex *-algebras, then a **-homomorphism* from A to B is an algebra homomorphism $\varphi \colon A \to B$ such that $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$.

That is, a *-algebra has all the structure of a Banach *-algebra or a C*-algebra except for the norm.

Definition 5.9. Let G be a discrete group. We define its (complex) *group ring* $\mathbb{C}[G]$ to be the set of formal linear combinations of elements of G with coefficients in $\mathbb{C}$. We write u_g for the element of $\mathbb{C}[G]$ corresponding to $g \in G$. Thus, for every $b \in \mathbb{C}[G]$ there is a unique family $(b_g)_{g \in G}$ of complex numbers such that $b_g = 0$ for all but finitely many $g \in G$ and such that $b = \sum_{g \in G} b_g u_g$. Multiplication is determined by specifying that $u_g u_h = u_{gh}$ for all $g, h \in G$, and extending linearly. Justified by Exercise 5.12 below, we make $\mathbb{C}[G]$ into a *-algebra by

$$\left(\sum\nolimits_{g \in G} b_g u_g \right)^* = \sum\nolimits_{g \in G} \overline{b_g} \cdot u_{g^{-1}}. \tag{5.1}$$

Remark 5.10. The product in $\mathbb{C}[G]$ as defined above can be written in the following equivalent ways:

$$\left(\sum\nolimits_{g \in G} a_g u_g \right) \left(\sum\nolimits_{g \in G} b_g u_g \right) = \sum\nolimits_{g, h \in G} a_g b_h u_{gh}$$

or

$$\left(\sum_{g\in G} a_g u_g\right)\left(\sum_{g\in G} b_g u_g\right) = \sum_{g\in G}\left(\sum_{h\in G} a_h b_{h^{-1}g}\right) u_g. \tag{5.2}$$

Remark 5.11. The convention in algebra seems to be that our $\sum_{g\in G} b_g u_g$ is just written $\sum_{g\in G} b_g \cdot g$. Also, algebraists have no reason to always choose $\mathbb{C}$ as the coefficients. Indeed, for any field K they routinely construct $K[G]$ in the same way, except that usually there is nothing quite like the adjoint operation we defined above. More generally, for any ring R, one can form $R[G]$ in the same way, and we will do this when we consider crossed products by discrete groups. (See Remark 8.6.) One needs an adjoint on R in order to get an adjoint on $R[G]$.

Exercise 5.12. Let G be a discrete group. Prove that the product given in Definition 5.9 makes $\mathbb{C}[G]$ into a unital algebra over $\mathbb{C}$. Further prove that the operation (5.1) makes $\mathbb{C}[G]$ a *-algebra as in Definition 5.8.

Definition 5.13. Let G be a discrete group, let H be a Hilbert space, and let $v\colon G\to U(H)$ be a unitary representation of G. We define $\rho_v\colon \mathbb{C}[G]\to L(H)$ as follows. For a family $(b_g)_{g\in G}$ of complex numbers such that $b_g = 0$ for all but finitely many $g\in G$, we set

$$\rho_v\left(\sum_{g\in G} b_g u_g\right) = \sum_{g\in G} b_g v(g). \tag{5.3}$$

Proposition 5.14. Let G be a discrete group, and let H be a Hilbert space. For any unital *-representation π of $\mathbb{C}[G]$ on H, we define a unitary representation $w_\pi\colon G\to U(H)$ by $w_\pi(g) = \pi(u_g)$. Then $\pi\mapsto w_\pi$ is a bijection from unital representations of $\mathbb{C}[G]$ on H to unitary representations of G on H. The inverse is given by $v\mapsto \rho_v$ as in Definition 5.13.

Exercise 5.15. Prove Proposition 5.14.

To demonstrate that this really is easy, we prove that if $\pi\colon \mathbb{C}[G]\to L(H)$ is a unital *-representation, then w_π is a group homomorphism. Let $g,h\in G$. Then, using $u_{gh} = u_g u_h$ at the second step and the fact that π is a homomorphism at the third step, we have

$$w_\pi(gh) = \pi(u_{gh}) = \pi(u_g u_h) = \pi(u_g)\pi(u_h) = w_\pi(g)w_\pi(h).$$

One must also prove that ρ_v is in fact a unital *-homomorphism; this is just algebra.

We recall the universal representation of a discrete group G. The construction is essentially the same as that of the universal representation of a C*-algebra. We would like it to be a representation z such that every unitary representation is unitarily equivalent to a subrepresentation of z, and the obvious way to do this is to take z to be the direct sum of all possible unitary representations of G. Unfortunately, there are set theoretic problems with this definition. First, there are representations on arbitrarily large Hilbert spaces, and there is no set whose elements include sets with arbitrarily large cardinality. Second, even the collection of all one dimensional Hilbert spaces is not a set. We therefore proceed as follows.

Definition 5.16. Let G be a discrete group. Choose a fixed Hilbert space M with dimension (cardinality of an orthonormal basis) equal to $\mathrm{card}(G)$. Let z be the unitary representation obtained as the direct sum of all possible unitary representations of G on closed subspaces of M. We call it the *universal representation* of G.

We really need only make sure that $\dim(M) \geq \operatorname{card}(G)$. But then the unitary equivalence class of our choice of universal representation would depend $\dim(M)$. This dependence would not matter in any essential way, but would be annoying.

Remark 5.17. With the construction of Definition 5.16, the universal representation z is unique up to unitary equivalence. It has the property that every unitary representation with a cyclic vector is unitarily equivalent to a subrepresentation of z. Since every representation is a direct sum of cyclic subrepresentations, it follows that every unitary representation of G is a direct sum of subrepresentations which are unitarily equivalent to subrepresentations of z.

We can now define two standard C*-algebras associated to a discrete group.

Definition 5.18. Let G be a discrete group. Let $z \colon G \to U(M)$ be the universal unitary representation of G, as in Definition 5.16. Using the notation of Definition 5.13, we define the *group C*-algebra* $C^*(G)$ to be the closure $\overline{\rho_z(\mathbb{C}[G])}$ of $\rho_z(\mathbb{C}[G]) \subset L(M)$ in the norm topology on $L(M)$.

When z is the universal representation of G, we write ρ_z for both the map $\mathbb{C}[G] \to L(M)$ and for the same map with restricted codomain $C^*(G)$.

Equivalently, $C^*(G) = \overline{\operatorname{span}}\big(\{z(g) \colon g \in G\}\big)$.

The C*-algebra $C^*(G)$ has the following description in terms of generators and relations.

Exercise 5.19. Let G be a discrete group. Prove that $C^*(G)$ is the universal unital C*-algebra with generators u_g for $g \in G$ and relations $u_g u_g^* = u_g^* u_g = 1$ for $g \in G$ and $u_g u_h = u_{gh}$ for $g, h \in G$.

Definition 5.20. Let G be a discrete group. Let $v \colon G \to U(l^2(G))$ be the left regular representation (Definition 5.3). Using the notation of Definition 5.13, we define the *reduced group C*-algebra* $C^*_{\mathrm{r}}(G)$ to be the closure $\overline{\rho_v(\mathbb{C}[G])}$ of $\rho_v(\mathbb{C}[G]) \subset L(l^2(G))$ in the norm topology on $L(l^2(G))$.

When v is the left regular representation of G, we write ρ_v for both the map $\mathbb{C}[G] \to L(l^2(G))$ and for the same map with restricted codomain $C^*_{\mathrm{r}}(G)$.

Notation 5.21. Let G be a discrete group. For $g \in G$, we also write u_g for the images of $u_g \in \mathbb{C}[G]$ in both $C^*(G)$ and $C^*_{\mathrm{r}}(G)$. (No confusion should arise. In effect, in Exercise 5.19, we already used this notation in $C^*(G)$.)

The algebra $C^*(G)$ is sometimes called the full C*-algebra of G. Sometimes the notation $C^*_{\mathrm{max}}(G)$ is used, and correspondingly $C^*_{\mathrm{min}}(G)$ for $C^*_{\mathrm{r}}(G)$. The reduced C*-algebra is also sometimes written $C^*_{\lambda}(G)$, based on the traditional notation λ for the left regular representation.

Besides the full and reduced group C*-algebras, there are "exotic" group C*-algebras, completions of $\mathbb{C}[G]$ with convolution multiplication in norms which lie between those giving the full and reduced C*-algebras. The first systematic study of such algebras seems to be the recent paper [36]. Further work on such algebras appears in [178] (where uncountably many such algebras are given for nonabelian free groups), [249], and [291]. We don't discuss these algebras in these notes.

The next theorem shows that the full C*-algebra of a group plays a role for unitary representations analogous to the role of the group ring for representations in the purely algebraic situation.

Theorem 5.22. Let G be a discrete group, and let H be a Hilbert space. For any unital representation π of $C^*(G)$ on H, we define a unitary representation $w_\pi \colon G \to U(H)$ by $w_\pi(g) = \pi(u_g)$. Then $\pi \mapsto w_\pi$ is a bijection from unital representations of $C^*(G)$ on H to unitary representations of G on H. In addition, if $v \colon G \to U(H)$ is a unitary representation of G on a Hilbert space H, if ρ_v is as in Definition 5.13, and if $\pi \colon C^*(G) \to L(H)$ is the corresponding representation of $C^*(G)$, then:

(1) Let z be the universal unitary representation of G. Then π is uniquely determined by the relation $\rho_v(a) = \pi(\rho_z(a))$ for all $a \in C^*(G)$.
(2) $\pi(C^*(G)) = C^*(v(G)) = \overline{\operatorname{span}}(v(G)) = \overline{\rho_v(\mathbb{C}[G])}$.

We have used the same notation $\pi \mapsto w_\pi$ as in Proposition 5.14. We don't quite get the formula (5.3) of Definition 5.13 for the inverse correspondence. The sums in (5.3) are finite, and one might hope that one could simply replace them with convergent series, and proceed in the obvious way. However, not all elements of $C^*(G)$ can be represented by convergent series which directly generalize the finite sums in (5.3). See Remark 5.60(3), Remark 5.60(4), and Remark 5.61 for further discussion. Given a unitary representation v, the best we can do is to extend ρ_v by continuity, which is what part (1) of the theorem amounts to.

Proof of Theorem 5.22. If π is a unital representation of $C^*(G)$ on H, it is easy to check that w_π is a unitary representation of G on H. (The proof is the same as the proof of the corresponding part of Proposition 5.14.)

Suppose π and σ are unital representations of $C^*(G)$ on H, and that $w_\pi = w_\sigma$. The definition immediately implies that $\pi(u_g) = \sigma(u_g)$ for all $g \in G$. Since $\{u_g \colon g \in G\}$ spans a dense subset of $C^*(G)$, it follows that $\pi = \sigma$.

Now let v be any unitary representation of G on H. Then there are an index set I and orthogonal invariant subspaces $H_i \subset H$ for $i \in I$ such that $H = \bigoplus_{i \in I} H_i$ and such that for $i \in I$ the restriction v_i of v to H_i is a cyclic representation.

Let $z \colon G \to U(M)$ be the universal representation of G, on the Hilbert space M, as described in Definition 5.16. By construction, for every $i \in I$ there is a direct summand $M_i \subset M$ such that the restriction z_i of z to M_i is unitarily equivalent to v_i. That is, there is a unitary $c_i \in L(M_i, H_i)$ such that $c_i z_i(g) c_i^* = v_i(g)$ for all $g \in G$. Now define a unital representation $\pi_i \colon C^*(G) \to L(H_i)$ by $\pi(a) = c_i (a|_{M_i}) c_i^*$. Define a unital representation $\pi \colon C^*(G) \to L(H)$ by $\pi = \bigoplus_{i \in I} \pi_i$. It is immediate that $w_\pi = v$.

We prove (1). Let v be given. Since $w_\pi = v$, we have $\pi(u_g) = v(g)$ for all $g \in G$. It follows from linearity that $\rho_v(a) = \pi(\rho_z(a))$ for all $a \in \mathbb{C}[G]$. By definition, $\rho_z(\mathbb{C}[G])$ is dense in $C^*(G)$, so this equation determines π uniquely.

For (2), the equality $C^*(v(G)) = \overline{\rho_v(\mathbb{C}[G])}$ follows from the fact that $\rho_v(\mathbb{C}[G])$ is a *-subalgebra of $L(H)$. The equality $\overline{\operatorname{span}}(v(G)) = \overline{\rho_v(\mathbb{C}[G])}$ follows from the fact that $\rho_v(\mathbb{C}[G]) = \operatorname{span}(v(G))$. The relation $C^*(v(G)) \subset \pi(C^*(G))$ holds because $\pi(C^*(G))$ is closed and $v(g) = \pi(u_g) \in \pi(C^*(G))$ for all $g \in G$. The relation $\pi(C^*(G)) \subset \overline{\rho_v(\mathbb{C}[G])}$ follows from $\rho_v(a) = \pi(\rho_z(a))$ for $a \in \mathbb{C}[G]$ and density of $\rho_z(\mathbb{C}[G])$ in $C^*(G)$. $\square$

Corollary 5.23. Let G be a discrete group. Then there is a unique surjective homomorphism $\kappa \colon C^*(G) \to C^*_{\mathrm{r}}(G)$ determined (following Notation 5.21) by $u_g \mapsto u_g$ for $g \in G$.

Proof. It follows from Theorem 5.22 that $u_g \mapsto u_g$ determines a unique homomorphism $\kappa_0 \colon C^*(G) \to L(l^2(G))$, and from Theorem 5.22(2) and Definition 5.3 that $\kappa_0(C^*(G)) = C^*_{\mathrm{r}}(G)$. $\square$

The map $\kappa \colon C^*(G) \to C^*_{\mathrm{r}}(G)$ of Corollary 5.23 need not be injective. In fact, κ is injective if and only if G is amenable (Theorem 5.51, for which we do not give a proof, and Theorem 5.50). Amenability is an important property, which we mostly do not treat in these notes; we refer to the discussion before Theorem 5.50 for more information. We do include enough in these notes to see that κ is not injective when G is a countable nonabelian free group. Indeed, we show in Theorem 6.6 that $C^*_{\mathrm{r}}(G)$ is simple. However, $C^*(G)$ is never simple unless G has only one element. To see this, let H be a one dimensional Hilbert space, and let $v \colon G \to U(H)$ be the trivial representation, that is, $v(g) = 1$ for all $g \in G$. Applying Theorem 5.22(2) to this representation, we obtain a nonzero homomorphism $\pi \colon C^*(G) \to L(H) = \mathbb{C}$. It follows from Corollary 5.25 below that π is not injective, so $\mathrm{Ker}(\pi)$ is a nontrivial ideal in $C^*(G)$.

Proposition 5.24. Let G be a discrete group. Then the map $\rho_v \colon \mathbb{C}[G] \to C^*_{\mathrm{r}}(G)$ of Definition 5.20 is injective.

Proof. As in Definition 5.20, let $v \colon G \to U(l^2(G))$ be the left regular representation. Let $b \in \mathbb{C}[G]$. Then there is a family $(b_g)_{g \in G}$ of complex numbers such that $b_g = 0$ for all but finitely many $g \in G$ and such that $b = \sum_{g \in G} b_g u_g$. For $g \in G$, let $\delta_g \in l^2(G)$ be the standard basis vector corresponding to g. Then $\rho_v(b)\delta_1 = \sum_{g \in G} b_g \delta_g$. If $b \neq 0$, then there is $g \in G$ such that $b_g \neq 0$, so $\langle \rho_v(b)\delta_1, \delta_g \rangle = b_g \neq 0$. Thus $\rho_v(b) \neq 0$. $\square$

Corollary 5.25. Let G be a discrete group. Then the map $\rho_z \colon \mathbb{C}[G] \to C^*(G)$ of Definition 5.18 is injective.

Proof. This follows from Proposition 5.24 and Corollary 5.23. $\square$

We are primarily interested in crossed products, and the sort of functoriality we are most interested in is what happens for a suitable homomorphism between algebras on which a fixed group G acts. But functoriality of group C*-algebras is a sufficiently obvious question that we should at least describe what happens.

Exercise 5.26. Let G_1 and G_2 be discrete groups, and let $\varphi \colon G_1 \to G_2$ be a homomorphism. Prove that there is a unique homomorphism $C^*(\varphi) \colon C^*(G_1) \to C^*(G_2)$ such that $C^*(\varphi)(u_g) = u_{\varphi(g)}$ for all $g \in G_1$. Prove that, with this definition of the action on morphisms, $G \mapsto C^*(G)$ is a functor from the category of discrete group and group homomorphisms to the category of unital C*-algebras and unital homomorphisms.

The main point is that if w is a unitary representation of G_2, then $w \circ \varphi$ is a unitary representation of G_1.

Exercise 5.27. Let G_1 and G_2 be discrete groups, and let $\varphi \colon G_1 \to G_2$ be an injective homomorphism. Prove that there is a unique homomorphism $C^*_{\mathrm{r}}(\varphi) \colon C^*_{\mathrm{r}}(G_1) \to C^*_{\mathrm{r}}(G_2)$ such that $C^*_{\mathrm{r}}(\varphi)(u_g) = u_{\varphi(g)}$ for all $g \in G_1$. Prove that, with this definition of the action on morphisms, $G \mapsto C^*_{\mathrm{r}}(G)$ is a functor from the category of discrete group and injective group homomorphisms to the category of unital C*-algebras and unital homomorphisms.

The main point here is that if v is the regular representation of G_2, then $v \circ \varphi$ is a direct sum of copies of the regular representation of G_1. (The number of copies is the cardinality of the coset space $G_2/\varphi(G_1)$.)

Without injectivity of φ, there might be no nonzero homomorphism from $C^*_{\mathrm{r}}(G_1)$ to $C^*_{\mathrm{r}}(G_2)$.

As an example, let $n \in \{2, 3, \ldots, \infty\}$, and let F_n be the free group on n generators. Theorem 6.6 implies that $C^*_{\mathrm{r}}(F_n)$ is simple. Therefore there is no nonzero homomorphism from $C^*_{\mathrm{r}}(G_1)$ to $\mathbb{C}$ to go with the homomorphism from F_n to the group with one element.

We warn that when the groups are not discrete, there is much less functoriality. See the discussion after Proposition 7.23.

We so far haven't given any justification for the use of $C^*_{\mathrm{r}}(G)$. Here is one reason for its importance.

Recall that a state ω on a C*-algebra A is said to be *faithful* if whenever $a \in A$ satisfies $\omega(a^*a) = 0$, then $a = 0$. A state ω on a C*-algebra A is *tracial* if $\omega(ab) = \omega(ba)$ for all $a, b \in A$. (We state this formally as Definition 11.23 below.)

Theorem 5.28. Let G be a discrete group. Then there is a unique continuous linear functional $\tau \colon C^*_{\mathrm{r}}(G) \to \mathbb{C}$ such that $\tau(u_1) = 1$ and $\tau(u_g) = 0$ for $g \in G \setminus \{1\}$. Moreover, τ is a faithful tracial state.

The condition on τ means that if $(b_g)_{g \in G}$ is a family of complex numbers such that $b_g = 0$ for all but finitely many $g \in G$, then

$$\tau\left(\sum\nolimits_{g \in G} b_g u_g\right) = b_1. \tag{5.4}$$

Our main application of Theorem 5.28 will be to the existence of "coefficients" for elements of $C^*_{\mathrm{r}}(G)$. See Proposition 5.58 and Proposition 5.59, and see Remark 5.60 for warnings about the use of these coefficients. Remark 5.61 explains one thing which goes wrong in $C^*(G)$ when $C^*(G) \neq C^*_{\mathrm{r}}(G)$.

Proof of Theorem 5.28. Since $\mathbb{C}[G]$ is dense in $C^*_{\mathrm{r}}(G)$, there can be at most one such continuous linear functional.

We now prove existence. As before, for $g \in G$, let $\delta_g \in l^2(G)$ be the standard basis vector corresponding to g. Define $\tau \colon C^*_{\mathrm{r}}(G) \to \mathbb{C}$ by $\tau(a) = \langle a\delta_1, \delta_1 \rangle$. We immediately check that $\tau(u_1) = \langle \delta_1, \delta_1 \rangle = 1$ and that if $g \in G \setminus \{1\}$ then $\tau(u_g) = \langle \delta_g, \delta_1 \rangle = 0$.

It is obvious that τ is a state on $C^*_{\mathrm{r}}(G)$. To prove that τ is tracial, by linearity and continuity it suffices to prove that $\tau(u_g u_h) = \tau(u_h u_g)$ for all $g, h \in G$. This reduces immediately to the fact that $gh \neq 1$ if and only if $hg \neq 1$.

It remains to show that τ is faithful. Identify $\mathbb{C}[G]$ with its image in $C^*_{\mathrm{r}}(G)$. We first claim that $\mathbb{C}[G]\delta_1$ is dense in $l^2(G)$. It suffices to show that if $(b_g)_{g \in G}$ is a family of complex numbers such that $b_g = 0$ for all but finitely many $g \in G$, then $\sum_{g \in G} b_g \delta_g \in \mathbb{C}[G]\delta_1$. Set $b = \sum_{g \in G} b_g u_g$, which is in $\mathbb{C}[G]$, and observe that $\sum_{g \in G} b_g \delta_g = b\delta_1 \in \mathbb{C}[G]\delta_1$. This proves the claim.

Now let $a \in C^*_{\mathrm{r}}(G)$ satisfy $\tau(a^*a) = 0$. Let $b, c \in \mathbb{C}[G]\delta_1$. Using the Cauchy-Schwarz inequality at the fourth step, we have

$$|\langle ab\delta_1, c\delta_1 \rangle| = |\langle c^*ab\delta_1, \delta_1 \rangle| = |\tau(c^*ab)| = |\tau(bc^*a)| \leq \tau(a^*a)^{1/2} \tau(bc^*cb^*)^{1/2} = 0.$$

So $\langle ab\delta_1, c\delta_1 \rangle = 0$. Since $b\delta_1$ and $c\delta_1$ are arbitrary elements of a dense subset of $l^2(G)$, it follows that $a = 0$. $\square$

We now look at two easy classes of examples: finite groups and discrete abelian groups.

Example 5.29. Let G be a finite group. Then $\mathbb{C}[G]$ is finite dimensional, hence already complete in any norm. Therefore $C^*(G) = C^*_{\mathrm{r}}(G) = \mathbb{C}[G]$ is a finite dimensional C*-algebra, with dimension equal to $\mathrm{card}(G)$. So there are $m \in \mathbb{Z}_{>0}$ and $r(1) \leq r(2) \leq \cdots \leq r(m)$ such that $C^*(G) \cong \bigoplus_{j=1}^{m} M_{r(j)}$ and $\sum_{j=1}^{m} r(j)^2 = \mathrm{card}(G)$. The numbers $r(1), r(2), \ldots, r(m)$ are the dimensions of the distinct equivalence classes of irreducible representations of $C^*(G)$, equivalently, of G. Since the one dimensional trivial representation of G is irreducible, we must have $r(1) = 1$.

A standard theorem from algebra (Theorem 7 in Section 2.5 of [254]) asserts that the number of distinct equivalence classes of irreducible representations of G is equal to the number of conjugacy classes in G.

The book [254] contains much more information about the representation theory of finite groups.

We turn to discrete abelian groups. We will need Pontryagin duality and various related results, which we state without proof. To avoid later repetition, we give the statements for the case of locally compact abelian groups. A discussion of the very beginnings of this subject (including the identification of the dual $\widehat{G}$ with the maximal ideal space of the Banach algebra $L^1(G)$, but not including the Pontryagin duality theorem) appears in Section 1.4 of [292]. There is a more extensive discussion in Chapter 4 of [87], and there are thorough presentations in Chapter 6 of [109] and Chapter 8 of [108].

Definition 5.30 (Definition 1.74 of [292]; beginning of Section 4.1 of [87]; Definition 23.3 of [109])**.** Let G be a locally compact abelian group. Its *Pontryagin dual* (or just dual) $\widehat{G}$ is the set of continuous homomorphisms $\chi \colon G \to S^1$, with the topology of uniform convergence on compact sets.

There are two motivations for this definition. One is the duality theorem (Theorem 5.34), in condensed form

$$\widehat{\widehat{G}} = G.$$

The other is that $\widehat{G}$ is essentially the set of one dimensional representations of G (see Proposition 5.33 below), and that the irreducible representations are exactly the one dimensional representations. For this, we recall Schur's Lemma for unitary representations of topological groups. The proofs of Schur's Lemma and the corollary are essentially the same as that of the analogous statements for C*-algebras. If G is a topological group and $v_1 \colon G \to U(H_1)$ and $v_2 \colon G \to U(H_2)$ are unitary representations of G on Hilbert spaces H_1 and H_2, then we let $C(v_1, v_2)$ be the set of intertwining operators, that is,

$$C(v_1, v_2) = \big\{ a \in L(H_1, H_2) \colon a v_1(g) = v_2(g) a \text{ for all } g \in G \big\}.$$

Theorem 5.31 (3.5(b) in [87])**.** Let G be a topological group and let $v_1 \colon G \to U(H_1)$ and $v_2 \colon G \to U(H_2)$ be irreducible unitary representations of G on Hilbert spaces H_1 and H_2. If v_1 and v_2 are unitarily equivalent, then there is a unitary $u \in L(H_1, H_2)$ such that $C(v_1, v_2) = \mathbb{C} u$. Otherwise, $C(v_1, v_2) = 0$.

Corollary 5.32 (Corollary 3.6 of [87])**.** Let G be an abelian topological group. Then every irreducible unitary representation of G is one dimensional.

Proof. Let $v\colon G \to U(H)$ be an irreducible unitary representation of G on a Hilbert space H. It follows from Theorem 5.31 that $C(v, v)$ is one dimensional. Since $1 \in C(v, v)$, we get $C(v, v) = \mathbb{C} \cdot 1$.

Since G is abelian, for every $h \in G$ we have $v(h) \in C(v, v)$. Therefore $v(h) \in \mathbb{C}\cdot 1$. It follows that every closed subspace of H is invariant. Since v is irreducible, this is only possible if $\dim(H) = 1$. □

The set $\widehat{G}$ exactly parametrizes the one dimensional representations of G:

Proposition 5.33. Let G be a locally compact abelian group. Identify S^1 with the unitary group of the one dimensional Hilbert space $\mathbb{C}$ in the obvious way. Then:

(1) Every one dimensional representation of G is unitarily equivalent to some element of $\widehat{G}$.
(2) If $\chi_1, \chi_2 \in \widehat{G}$ are unitarily equivalent, then $\chi_1 = \chi_2$.

Proof. Both parts are immediate. □

In the following theorem, local compactness is Corollary 1.79 of [292], the discussion after Theorem 4.2 in [87], or Theorem 23.13 of [109]. Duality (the statement that ε_G is an isomorphism) is Theorem 4.31 of [87], or Theorem 24.8 of [109]. The fact that $G \mapsto \widehat{G}$ is a contravariant functor is clear (and is in Theorem 24.38 of [109]), naturality of ε_G is obvious, and that $G \mapsto \widehat{G}$ is a category equivalence follows from duality and naturality of ε_G.

Theorem 5.34. Let G be a locally compact abelian group. Then $\widehat{G}$ is a locally compact abelian group. The assignment $G \mapsto \widehat{G}$ is the map on objects of a contravariant category equivalence from the category of locally compact abelian groups and continuous group homomorphisms to itself, for which the map on morphisms assigns to a continuous group homomorphism $\varphi\colon G \to H$ the homomorphism $\chi \mapsto \chi \circ \varphi$ from $\widehat{H}$ to $\widehat{G}$. There is a natural isomorphism of locally compact abelian groups

$$\varepsilon_G \colon G \to \widehat{\widehat{G}}$$

(Pontryagin duality), given by $\varepsilon_G(g)(\chi) = \chi(g)$ for $g \in G$ and $\chi \in \widehat{G}$.

In the following collection of examples, the one we care most about is $\widehat{\mathbb{Z}} = S^1$.

Example 5.35. We give the examples of dual groups which are most important for our purposes.

(1) Let G be a finite abelian group. Then there is a (noncanonical) isomorphism $\widehat{G} \cong G$. See Corollary 4.7 of [87], or 23.27(d) of [109].
(2) For $\zeta \in S^1$, define $\chi_\zeta \in \widehat{\mathbb{Z}}$ by $\chi_\zeta(n) = \zeta^n$ for $n \in \mathbb{Z}$. Then $\zeta \mapsto \chi_\zeta$ defines an isomorphism $S^1 \to \widehat{\mathbb{Z}}$. See Theorem 4.5(c) of [87], or 23.27(b) of [109].
(3) For $n \in \mathbb{Z}$, define $\chi_n \in \widehat{\mathbb{Z}}$ by $\chi_n(\zeta) = \zeta^n$ for $\zeta \in S^1$. Then $n \mapsto \chi_n$ defines an isomorphism $\mathbb{Z} \to \widehat{S^1}$. See Theorem 4.5(b) of [87], or 23.27(a) of [109].
(4) For $t \in \mathbb{R}$, define $\chi_t \in \widehat{\mathbb{R}}$ by $\chi_t(x) = \exp(ixt)$ for $x \in \mathbb{R}$. Then $t \mapsto \chi_t$ defines an isomorphism $\mathbb{R} \to \widehat{\mathbb{R}}$. See Theorem 4.5(a) of [87], or 23.27(e) of [109]. (In Theorem 4.5(a) of [87], the slightly different formula $\chi_t(x) = \exp(2\pi ixt)$ is used, but clearly one formula gives an isomorphism if and only if the other does. The difference shows up in formulas for Fourier transforms and related objects.)

(5) Let $G_1, G_2, \ldots, G_n$ be locally compact abelian groups, and let $G = \prod_{k=1}^n G_k$. Then $\widehat{G} \cong \prod_{k=1}^n \widehat{G_k}$. The isomorphism sends $(\chi_1, \chi_2, \ldots, \chi_n) \in \prod_{k=1}^n \widehat{G_k}$ to the function

$$(g_1, g_2, \ldots, g_n) \mapsto \chi_1(g_1)\chi_2(g_2)\cdots\chi_n(g_n)$$

for $(g_1, g_2, \ldots, g_n) \in \prod_{k=1}^n G_k$. See Proposition 4.6 of [87], or Theorem 23.18 of [109].

(6) Let I be an index set, and for $i \in I$ let G_i be a compact abelian group. Let $G = \prod_{i \in I} G_i$. Then $\widehat{G} \cong \bigoplus \widehat{G_i}$. (The direct sum is the algebraic direct sum of the discrete abelian groups $\widehat{G_i}$.) The map is the obvious generalization of that of (5); the product is well defined because the factors commute and all but finitely many of them are equal to 1. See Proposition 4.8 of [87], or Theorem 23.21 of [109].

For many further results about the relations between G and $\widehat{G}$, we refer to Chapter 4 of [87] and particularly to Chapter 6 of [109]. Here we point out just a few facts.

Theorem 5.36. Let G be a locally compact abelian group. Then:

(1) G is discrete if and only if $\widehat{G}$ is compact.
(2) G is compact if and only if $\widehat{G}$ is discrete.

Proof. The forward implication in each of the two parts is in Proposition 4.4 of [87] or Theorem 23.17 of [109]. The reverse direction in each part follows from the forward implication in the other part by duality (Theorem 5.34). ∎

Further statements of this general nature can be found in Theorems 24.23, 24.25, 24.26, and 24.28 of [109]. In the statements of all these results, $\mathsf{X} = \widehat{G}$, and additional related theorems can be obtained by using duality (Theorem 5.34) to exchange G and $\widehat{G}$.

The first part of the following result is known as Plancherel's Theorem. The element $y\xi$ is a generalized Fourier transform, and is often written $\widehat{\xi}$.

Theorem 5.37. Let G be a locally compact abelian group. For any choice of Haar measure on G, there is a choice of Haar measure on $\widehat{G}$ such that there is a unitary $y \in L\big(L^2(G),\, L^2(\widehat{G})\big)$ such that

$$(y\xi)(\chi) = \int_G \overline{\chi(g)}\xi(g)\, d\mu(g)$$

for all $\chi \in \widehat{G}$ and $\xi \in L^1(G) \cap L^2(G)$.

Further, let v be the left regular representation of G on $L^2(G)$, and let w be the unitary representation of G on $L^2(\widehat{G})$ defined by $(w(g)\eta)(\chi) = \overline{\chi(g)}\eta(\chi)$ for $g \in G$, $\chi \in \widehat{G}$, and $\eta \in L^2(\widehat{G})$. Then y intertwines v and w, that is, $yv(g)y^* = w(g)$ for all $g \in G$.

The first part is Theorem 4.25 of [87], or Theorem 31.18 of [108].

Given the first part of the theorem, the second part is easily justified. For $g \in G$, $\chi \in \widehat{G}$, and $\xi \in L^1(G) \cap L^2(G)$, we have

$$\begin{aligned}(yv(g)\xi)(g) &= \int_G \overline{\chi(h)}(v(g)\xi)(h)\,d\mu(h) = \int_G \overline{\chi(h)}\xi(g^{-1}h)\,d\mu(h) \\ &= \int_G \overline{\chi(gh)}\xi(h)\,d\mu(h) = \overline{\chi(g)}\widehat{\xi}(\chi) = (w(g)y\xi)(\chi).\end{aligned}$$

Since $L^1(G) \cap L^2(G)$ is dense in $L^2(G)$, the second part follows.

We are now ready to calculate the C*-algebra of a discrete abelian group. The answer is essentially the same without discreteness: $C^*(G) \cong C_0(\widehat{G})$ for every locally compact abelian group G. We also point out that, according to some presentations of the theory, what we are doing here is backwards: $\widehat{G}$ is (almost) defined as the maximal ideal space of $C^*(G)$. (The common version of this approach is to define $\widehat{G}$ to be the maximal ideal space of $L^1(G)$.)

Theorem 5.38. Let G be a discrete abelian group. Then there is an isomorphism $\gamma\colon C^*(G) \to C(\widehat{G})$ determined by the following formula. If $(b_g)_{g\in G}$ is a family of complex numbers such that $b_g = 0$ for all but finitely many $g \in G$, then

$$\gamma\left(\sum\nolimits_{g\in G} b_g u_g\right)(\chi) = \sum\nolimits_{g\in G} \chi(g) b_g$$

for all $\chi \in \widehat{G}$.

Proof. Since $C^*(G)$ is a commutative unital C*-algebra, we can let X be its maximal ideal space $\mathrm{Max}(C^*(G))$, which we think of as the set of unital homomorphisms from $C^*(G)$ to $\mathbb{C}$. Then there is a canonical isomorphism $\varphi\colon C^*(G) \to C(X)$.

Proposition 5.33 identifies $\widehat{G}$ with the set of representations of G on the one dimensional Hilbert space $\mathbb{C}$, and Theorem 5.22 provides a bijection from such representations to the unital homomorphisms from $C^*(G)$ to $\mathbb{C}$. Combining them, we obtain a bijection $h\colon \widehat{G} \to X$ such that $h(\chi)(u_g) = \chi(g)$ for all $\chi \in \widehat{G}$ and $g \in G$.

We claim that h is continuous. Let $(\chi_i)_{i\in I}$ be a net in $\widehat{G}$ which converges uniformly on compact sets to $\chi \in \widehat{G}$. Then for all $g \in G$ we have

$$\lim_{i\in I} h(\chi_i)(u_g) = \lim_{i\in I} \chi_i(g) = \chi(g) = h(\chi)(u_g).$$

It follows that $\lim_{i\in I} h(\chi_i)(a) = h(\chi)(a)$ for all $a \in \mathrm{span}(\{u_g\colon g \in G\}) \subset C^*(G)$. It now follows from an $\frac{\varepsilon}{3}$ argument that $\lim_{i\in I} h(\chi_i)(a) = h(\chi)(a)$ for all $a \in \overline{\mathrm{span}}(\{u_g\colon g \in G\}) = C^*(G)$. By the definition of the topology on $\mathrm{Max}(C^*(G))$, this means that $\lim_{i\in I} h(\chi_i) = h(\chi)$. Continuity of h follows.

We now know that h is a continuous bijection of compact Hausdorff spaces. Therefore h is a homeomorphism. So h determines an isomorphism $\mathrm{Max}(C^*(G)) \to \widehat{G}$. The theorem follows. □

The following theorem holds in much greater generality (for arbitrary amenable locally compact groups—see Theorem 5.50 and Theorem 9.7 below), but this special case has an easy proof, which we give here.

Theorem 5.39. Let G be a discrete abelian group. Then the canonical homomorphism $\kappa\colon C^*(G) \to C^*_{\mathrm{r}}(G)$ is an isomorphism.

Proof. For any unitary representation w of G on a Hilbert space H, let $\rho_w \colon C^*(G) \to L(H)$ be the corresponding representation of $C^*(G)$ as in Theorem 5.22.

We have to prove that $\|\kappa(b)\| = \|b\|$ for all $b \in C^*(G)$. As in Theorem 5.37, let v be the left regular representation of G on $L^2(G)$, and let $w \colon G \to U\big(L^2\big(\widehat{G}\big)\big)$ be $(w(g)\eta)(\chi) = \overline{\chi(g)}\eta(\chi)$ for $g \in G$, $\chi \in \widehat{G}$, and $\eta \in L^2\big(\widehat{G}\big)$. Also let $y \in L\big(L^2(G), L^2\big(\widehat{G}\big)\big)$ be as in Theorem 5.37. By definition, $\kappa = \rho_v$. Since y intertwines v and w, it is immediate that y intertwines ρ_v and ρ_w. Therefore $\|\kappa(b)\| = \|\rho_w(b)\|$.

Let $\gamma \colon C^*(G) \to C\big(\widehat{G}\big)$ be as in Theorem 5.38. For $g \in G$, the operator $\rho_w(u_g)$ is multiplication by the function $\chi \mapsto \overline{\chi(g)} = \gamma(u_g)(\chi^{-1})$. Therefore $\rho_w(b)$ is multiplication by the function $\chi \mapsto \gamma(b)(\chi^{-1})$. Since Haar measure on $\widehat{G}$ has full support, we get $\|\rho_w(b)\| = \|\gamma(b)\|$. Combining this with the result of the previous paragraph, and with $\|\gamma(b)\| = \|b\|$ (from Theorem 5.38), we get $\|\kappa(b)\| = \|b\|$. $\square$

The following remark and problem are not directly related to the main topic of these notes, but they seem interesting enough to include.

Remark 5.40. Neither $C^*(G)$ nor $C^*_{\mathrm{r}}(G)$ determines G, not even for G discrete abelian. One example that is easy to get from what has already been done is that the full and reduced C*-algebras of all second countable infinite compact groups are the same, namely $C_0(S)$ for a countable infinite set S. Any two finite abelian groups with the same cardinality have isomorphic C*-algebras, since if $\mathrm{card}(G) = n$ then $\mathrm{card}\big(\widehat{G}\big) = n$ and $C^*(G) = C^*_{\mathrm{r}}(G) \cong \mathbb{C}^n$. Among nonabelian groups, the simplest example is that both the nonabelian groups of order 8 have both full and reduced C*-algebras isomorphic to $\mathbb{C}^4 \oplus M_2$.

However, the following problem, from the introduction to [120], seems to be open. (We are grateful to Narutaka Ozawa for this reference.)

Problem 5.41. Let G and H be countable torsion free groups such that $C^*_{\mathrm{r}}(G) \cong C^*_{\mathrm{r}}(H)$. Does it follow that $G \cong H$?

As discussed in the introduction to [120], the answer is yes if G and H are abelian.

In much of what we have done, one can use the algebra $l^1(G)$ in place of $\mathbb{C}[G]$.

Definition 5.42. Let G be a discrete group. We write elements of $l^1(G)$ as functions $a \colon G \to \mathbb{C}$ (such that $\sum_{g \in G} |a(g)| < \infty$). We make $l^1(G)$ into a Banach *-algebra as follows. The Banach space structure is as usual. Multiplication is given by convolution: for $a, b \in l^1(G)$,

$$(ab)(g) = \sum_{h \in G} a(h) b(h^{-1}g).$$

The adjoint is

$$a^*(g) = \overline{a(g^{-1})}$$

for $a \in l^1(G)$. For $g \in G$, we define $u_g \in l^1(G)$ by $u_g(g) = 1$ and $u_g(h) = 0$ for $h \neq g$.

We give the properties of $l^1(G)$ as a series of easy exercises.

Exercise 5.43. Let G be a discrete group. Prove that the operations in Definition 5.42 make $l^1(G)$ into a unital Banach *-algebra whose identity is u_1.

The following result justifies the use of the notation u_g for elements of both $\mathbb{C}[G]$ and $l^1(G)$. Using it, we normally regard $\mathbb{C}[G]$ as a dense subalgebra of $l^1(G)$.

Exercise 5.44. Let G be a discrete group. Prove that there is a unique algebra homomorphism $\iota\colon \mathbb{C}[G] \to l^1(G)$ such that the image of the element $u_g \in \mathbb{C}[G]$ of Definition 5.9 is the element $u_g \in l^1(G)$ of Definition 5.42. Prove that ι is injective, preserves the adjoint operation, and has dense range.

Definition 5.45. Let G be a discrete group, let H be a Hilbert space, and let $w\colon G \to U(H)$ be a unitary representation of G. We define $\overline{\rho}_w\colon l^1(G) \to L(H)$ by

$$\overline{\rho}_w(b) = \sum_{g \in G} b(g) w(g) \tag{5.5}$$

for $b \in l^1(G)$.

Exercise 5.46. Let G be a discrete group, let H be a Hilbert space, and let $w\colon G \to U(H)$ be a unitary representation of G. Prove that the map $\overline{\rho}_w$ of Definition 5.45 is a well defined unital *-homomorphism from $l^1(G) \to L(H)$. Prove that the representation ρ_w of Definition 5.13 and the map ι of Exercise 5.44 satisfy $\overline{\rho}_w \circ \iota = \rho_w$.

Exercise 5.47. Let G be a discrete group, let H be a Hilbert space, and let $\pi\colon l^1(G) \to L(H)$ be a unital *-homomorphism (no continuity is assumed). Prove that $\|\pi(b)\| \leq \|b\|$ for all $b \in l^1(G)$.

Exercise 5.48. Let G be a discrete group, and let H be a Hilbert space. Prove that the assignment $w \mapsto \overline{\rho}_w$ of Definition 5.45 defines a bijection from unitary representations $w\colon G \to U(H)$ to unital *-homomorphisms $l^1(G) \to L(H)$.

Exercise 5.49. Let G be a discrete group. Prove that the map which for $g \in G$ sends $u_g \in l^1(G)$ to $u_g \in C^*(G)$ extends to a contractive *-homomorphism $\lambda\colon l^1(G) \to C^*(G)$ with dense range. Further prove that if $w\colon G \to U(H)$ is a unitary representation of G on a Hilbert space H, $\overline{\rho}_w$ is as in Definition 5.45, and $\pi\colon C^*(G) \to L(H)$ is the representation of $C^*(G)$ corresponding to w (as in Theorem 5.22), then $\pi \circ \lambda = \overline{\rho}_w$.

We state three important theorems about $C^*(G)$ and $C^*_{\mathrm{r}}(G)$. In the first and second, we consider arbitrary locally compact groups; their full and reduced C*-algebras are discussed in Section 7. We give a proof only for the first. We restrict here to the case of a discrete group, in which the ideas are exposed with less distraction, but the proof of the crossed product generalization (Theorem 9.7 below) includes the case of a general locally compact group in Theorem 5.50.

All three involve amenability of a group. For information on amenable groups, including many equivalent conditions for amenability, we refer to [100] or to Section A.2 of [292]. We will use the Følner set criterion. A discrete group G is amenable if and only if for every finite set $F \subset G$ and every $\varepsilon > 0$ there is a nonempty finite set $S \subset G$ such that for all $g \in F$ the symmetric difference $gS \triangle S$ satisfies $\mathrm{card}(gS \triangle S) < \varepsilon \mathrm{card}(S)$. (See Theorem 3.6.1 of [100].) When G is locally compact, one uses Haar measure instead of cardinality: if μ is a left Haar measure on G, then G is amenable if and only if for every compact set $F \subset G$ and every $\varepsilon > 0$ there is a compact set $S \subset G$ such that $\mu(S) > 0$ and $\mu(gS \triangle S) < \varepsilon\mu(S)$ for all $g \in F$. (See Theorem 3.6.2 of [100].) It is easy to show that the condition for a discrete group is equivalent if the conclusion is rewritten to require that

$\mathrm{card}(FS \triangle S) < \varepsilon \mathrm{card}(S)$. (This will be implicit in the proof below of the discrete case of Theorem 5.50 below.) It is true, but nontrivial to prove, that the condition for a locally compact group is equivalent if the conclusion is rewritten to require that $\mu(FS \triangle S) < \varepsilon\mu(S)$. The equivalence is in neither [100] nor Section A.2 of [292], but it is the main result of [76]. (Also see Theorem 3.1.1 there.)

Locally compact abelian groups are amenable. (Combine Theorems 1.2.1 and 2.2.1 of [100].) From the conditions involving invariant means, it is obvious that compact groups (in particular, finite groups) are amenable. The class of amenable locally compact groups is closed under passage to closed subgroups (Theorem 2.3.2 of [100]), quotients by closed normal subgroups (Theorem 2.3.1 of [100]), extensions (Theorem 2.3.3 of [100]), and increasing unions (Theorem 2.3.4 of [100]). In particular, all solvable locally compact groups are amenable, and direct limits of discrete amenable groups are amenable.

Theorem 5.50 (One direction of Theorem A.18 of [292] and Theorem 7.3.9 of [198])**.** Let G be a an amenable locally compact group. Then the map $\kappa \colon C^*(G) \to C^*_{\mathrm{r}}(G)$ (in Corollary 5.23 for discrete groups; in Proposition 7.23 for general locally compact groups) is an isomorphism.

We will give a direct proof for discrete groups from the Følner set criterion described above. The proof for the locally compact case is very similar. In fact, essentially the same proof shows that for an amenable group, the map from a full crossed product to the corresponding reduced crossed product is an isomorphism. See Theorem 9.7 below, for which we do give a full proof. Our proof does not use the machinery of positive definite functions. This machinery is very important, but doing without it has the advantage that one sees the role of amenability very clearly in the proof. It is instructive to specialize our proof to the case of a finite group, in which ε is not needed and the finite subsets F and S in the proof can both be taken to be G.

Proof of Theorem 5.50 for discrete groups. For any unitary representation w of G on a Hilbert space H, let $\rho_w \colon C^*(G) \to L(H)$ be the corresponding representation of $C^*(G)$ as in Theorem 5.22. We have to prove that, for any unitary representation w of G on a Hilbert space H, and any $b \in C^*(G)$, we have $\|\rho_w(b)\| \leq \|\kappa(b)\|$. Let v be the left regular representation of G on $l^2(G)$. We can rewrite the relation to be proved as $\|\rho_w(b)\| \leq \|\rho_v(b)\|$.

The main tool is the tensor product representation $v \otimes w$ as in Remark 5.7. It acts on the Hilbert space $l^2(G) \otimes H$. Throughout the proof, we identify $l^2(G) \otimes H$ with the space $l^2(G, H)$ of l^2 functions from G to H.

We first claim that $v \otimes w$ is unitarily equivalent to the tensor product of v and the trivial representation of G on H. Let $z \in U(l^2(G, H))$ be the unitary determined by $(z\xi)(g) = w_g^*(\xi(g))$ for $\xi \in l^2(G, H)$ and $g \in G$. Now let $\xi \in l^2(G, H)$ and let $g, h \in G$. Then

$$\begin{aligned}\big(z(v_h \otimes w_h)\xi\big)(g) &= w_g^*\big([(v_h \otimes w_h)\xi](g)\big) = w_g^*\big(w_h(\xi(h^{-1}g))\big)\\ &= w_{g^{-1}h}(\xi(h^{-1}g)) = (z\xi)(h^{-1}g) = \big((v_h \otimes 1)z\xi\big)(g),\end{aligned}$$

which is the statement of the claim.

It follows that $z\rho_{v\otimes w}(b)z^* = \rho_v(b) \otimes 1$, so

$$\|\rho_{v\otimes w}(b)\| = \|\rho_v(b) \otimes 1\| = \|\rho_v(b)\|.$$

It remains to prove that $\|\rho_w(b)\| \leq \|\rho_{v\otimes w}(b)\|$. It suffices to prove this for b in the subalgebra $\mathbb{C}[G]$. Thus, there is a finite set $S \subset G$ and a family $(b_g)_{g\in S}$ of complex numbers such that $b = \sum_{g\in S} b_g u_g$.

Let $\varepsilon > 0$. We prove that $\|\rho_w(b)\| - \varepsilon < \|\rho_{v\otimes w}(b)\|$. Without loss of generality $\rho_w(b) \neq 0$ and $\varepsilon < \|\rho_w(b)\|$. Choose $\xi_0 \in H$ such that

$$\|\xi_0\| = 1 \qquad \text{and} \qquad \|\rho_w(b)\xi_0\| > \|\rho_w(b)\| - \frac{\varepsilon}{2}.$$

Set

$$\delta = \frac{1}{\mathrm{card}(S)}\left(1 - \left(\frac{\|\rho_w(b)\| - \varepsilon}{\|\rho_w(b)\| - \frac{\varepsilon}{2}}\right)^2\right).$$

Then $\delta > 0$. The Følner set condition for amenability (Theorem 3.6.1 of [100]) provides a nonempty finite subset $K \subset G$ such that

$$\mathrm{card}\big(gK \triangle K\big) < \delta\mathrm{card}(K)$$

for all $g \in S$. Define $\xi \in l^2(G, H)$ by

$$\xi(g) = \begin{cases} \xi_0 & g \in K \\ 0 & g \notin K. \end{cases}$$

Then $\|\xi\| = \mathrm{card}(K)^{1/2}$.

We estimate $\big\|\rho_{v\otimes w}(b)\xi\big\|$. Set

$$E = \big\{ g \in K \colon h^{-1}g \in K \text{ for all } h \in S \big\}.$$

Then

$$\mathrm{card}(K \setminus E) \leq \sum_{h\in S} \mathrm{card}(K \setminus hK) < \mathrm{card}(S)\delta\mathrm{card}(K).$$

So $\mathrm{card}(E) > \big(1 - \mathrm{card}(S)\delta\big)\mathrm{card}(K)$. Moreover, for $g \in E$ we have, using the definition of E at the third step,

$$\begin{aligned} \big(\rho_{v\otimes w}(b)\xi\big)(g) &= \sum_{h\in S} b_h\big((v_h \otimes w_h)\xi\big)(g) \\ &= \sum_{h\in S} b_h w_h(\xi(h^{-1}g)) \\ &= \sum_{h\in S} b_h w_h \xi_0 = \rho_w(b)\xi_0. \end{aligned}$$

Therefore

$$\big\|\rho_{v\otimes w}(b)\xi\big\| \geq \mathrm{card}(E)^{1/2}\|\rho_w(b)\xi_0\| > \mathrm{card}(E)^{1/2}\left(\|\rho_w(b)\| - \frac{\varepsilon}{2}\right),$$

from which it follows that

$$\begin{aligned} \big\|\rho_{v\otimes w}(b)\big\| &\geq \frac{\mathrm{card}(E)^{1/2}\left(\|\rho_w(b)\| - \frac{\varepsilon}{2}\right)}{\mathrm{card}(K)^{1/2}} \\ &> \big(1 - \mathrm{card}(S)\delta\big)^{1/2}\left(\|\rho_w(b)\| - \frac{\varepsilon}{2}\right) = \|\rho_w(b)\| - \varepsilon, \end{aligned}$$

as desired. ∎

Theorem 5.51 (The other direction of Theorem A.18 of [292] and Theorem 7.3.9 of [198])**.** Let G be a locally compact group. If the standard homomorphism $C^*(G) \to C^*_{\mathrm{r}}(G)$ is an isomorphism, then G is amenable.

Theorem 5.52. Let G be a discrete group. Then the following are equivalent:

(1) G is amenable.
(2) $C^*_{\mathrm{r}}(G)$ is nuclear.
(3) $C^*(G)$ is nuclear.

The equivalence of the first two conditions (and many others) is contained in Theorem 2.6.8 of [37]. (The definition of amenability used there is existence of an invariant mean. See Definition 2.6.1 of [37].) If G is amenable, then $C^*(G)$ is nuclear because $C^*(G) \cong C^*_{\mathrm{r}}(G)$. If $C^*(G)$ is nuclear, then $C^*_{\mathrm{r}}(G)$ is nuclear because it is a quotient of $C^*(G)$.

Theorem 5.52 does not hold without discreteness. Even the full group C*-algebras of connected semisimple Lie groups are not only type I but even CCR: the image of every irreducible representation is exactly the compact operators. This fact follows from Theorem 5 on page 248 of [105]. Not only are most such groups not amenable; many even have Kazhdan's Property (T). Example: $\mathrm{SL}_3(\mathbb{R})$. Theorem 2 on page 47 of [228] describes exactly when the full group C*-algebra of a connected simply connected Lie group is CCR, and Theorem 1 on page 39 of [228] gives some conditions under which the full group C*-algebra of a connected simply connected Lie group has type I.

Although we say very little about von Neumann algebras in these notes, we want to at least mention the group von Neumann algebra.

Definition 5.53. Let G be a discrete group. Regard $C^*_{\mathrm{r}}(G)$ as a subalgebra of $L(l^2(G))$, as in Definition 5.20. We define the *group von Neumann algebra* $W^*_{\mathrm{r}}(G)$ to be the closure of $C^*_{\mathrm{r}}(G)$ in the weak operator topology on $L(l^2(G))$.

Equivalently, using the notation of Definition 5.20 and taking v to be the left regular representation of G, the algebra $W^*_{\mathrm{r}}(G)$ is the closure of $\rho_v(\mathbb{C}[G])$ in the weak operator topology on $L(l^2(G))$. (This is the definition given in the introduction to Section VII.3 of [277]. Also see Definition V.7.4 of [276].) We can also write $W^*_{\mathrm{r}}(G) = \rho_v(\mathbb{C}[G])''$.

The notation follows a suggestion of Simon Wassermann. It was previously common to write $W^*(G)$, which unfortunately suggests a relation with $C^*(G)$ instead of with $C^*_{\mathrm{r}}(G)$. These days, the notation $L(G)$ (or $\mathcal{L}(G)$) is much more common.

The group von Neumann algebra carries much less information about the group than its full or reduced C*-algebra. For example, although we will not prove this here, it is not difficult to show that if G is discrete abelian, then $W^*_{\mathrm{r}}(G) \cong L^\infty\big(\widehat{G}\big)$. In particular, these algebras are the same for every countable infinite discrete abelian group. This is much worse than the situation for group C*-algebras, as described in Remark 5.40.

We can give some description of the elements of the reduced C*-algebra and von Neumann algebra of a discrete group. The term in the following definition is motivated by the case $G = \mathbb{Z}$, and the ideas are based on a lecture of Nate Brown. We think of elements of $L(l^2(\mathbb{Z}))$ as being given by infinite matrices $a = (a_{j,k})_{j,k\in\mathbb{Z}}$. The main diagonal consists of the elements $a_{j,j}$ for $j \in \mathbb{Z}$, and the other diagonals are gotten by fixing $m \in \mathbb{Z}$ and taking the elements $a_{j,\,j+m}$ for $j \in \mathbb{Z}$. That is, they are the elements $a_{j,k}$ with $j - k$ constant.

We begin with notation for matrix elements of an operator $a \in L(l^2(S))$.

Definition 5.54. Let S be a set. For $s \in S$ let $\delta_s \in l^2(S)$ be the standard basis vector associated with s. For $a \in L(l^2(S))$ and $s, t \in S$, we define the *matrix coefficient* $a_{s,t}$ of a by $a_{s,t} = \langle a\delta_t, \delta_s \rangle$.

Remark 5.55. The indexing in Definition 5.54 is consistent with the usual conventions for entries of finite matrices. For example, let $s_0, t_0 \in S$, and let $v \in L(l^2(S))$ be the partial isometry determined by $v\delta_{t_0} = \delta_{s_0}$ and $v\delta_t = 0$ for $t \in S \setminus \{t_0\}$. Then $v_{s_0,t_0} = 1$ and $v_{s,t} = 0$ for all other pairs $(s,t) \in S \times S$. Moreover, for general $a \in L(l^2(S))$ and $t \in S$, the element $\xi = a\delta_t \in l^2(S)$ is determined by the relations $\xi(s) = \langle \xi, \delta_s \rangle = a_{s,t}$ for all $s \in S$. In particular, $a\delta_t = \sum_{s \in S} a_{s,t}\delta_s$ with convergence in norm in $l^2(S)$. Finally, we note that $(a^*)_{s,t} = \overline{a_{t,s}}$ for all $s, t \in S$.

Definition 5.56. Let G be a discrete group. Let $a \in L(l^2(G))$, and write $a = (a_{g,h})_{g,h \in G}$. We say that a is *constant on diagonals* if $a_{g,h} = a_{s,t}$ whenever $g, h, s, t \in G$ satisfy $gh^{-1} = st^{-1}$.

Theorem 5.57. Let G be a discrete group. Then

$$W_{\mathrm{r}}^*(G) = \big\{ a \in L(l^2(G)) \colon a \text{ is constant on diagonals} \big\}.$$

Proof. Let $M \subset L(l^2(G))$ be the set of all $a \in L(l^2(G))$ which are constant on diagonals. Let $N \subset L(l^2(G))$ be the set of all $b \in L(l^2(G))$ such that $b_{g,h} = b_{s,t}$ whenever $g, h, s, t \in G$ satisfy $g^{-1}h = s^{-1}t$. (Note the different placement of the inverses.) Let $v \colon G \to U(l^2(G))$ be the left regular representation (Definition 5.3), and let $w \colon G \to U(l^2(G))$ be the right regular representation, given by $(w(g)\xi)(h) = \xi(hg)$ for $g, h \in G$ and $\xi \in l^2(G)$.

We first claim that if $a \in M$ and $b \in N$, then $ab = ba$. Fix $s, t \in G$; we prove that $\langle ab\delta_s, \delta_t \rangle = \langle ba\delta_s, \delta_t \rangle$. Using several parts of Remark 5.55 at the second step, we get

$$\langle ab\delta_s, \delta_t \rangle = \langle b\delta_s, a^*\delta_t \rangle = \left\langle \sum\nolimits_{g \in G} b_{g,s}\delta_g, \, \sum\nolimits_{g \in G} \overline{a_{t,g}}\delta_g \right\rangle = \sum\nolimits_{g \in G} a_{t,g} b_{g,s}.$$

Similarly, we get the first step of the following calculation. The second step follows from the definitions of $a \in M$ and $b \in N$, and the third step is a change of variables:

$$\langle ba\delta_s, \delta_t \rangle = \sum_{g \in G} a_{g,s} b_{t,g} = \sum_{g \in G} a_{t, sg^{-1}t} b_{sg^{-1}t, s} = \sum_{g \in G} a_{t,g} b_{g,s} = \langle ab\delta_s, \delta_t \rangle.$$

This proves the claim.

We next claim that $w(G)' = M$. Let $a \in L(l^2(G))$. We have to show that $aw(g) = w(g)a$ for all $g \in G$ if and only if a is constant on diagonals. For $g, h, k \in G$, we compute

$$(aw(g))_{h,k} = \langle aw(g)\delta_k, \, \delta_h \rangle = \langle a\delta_{kg^{-1}}, \, \delta_h \rangle = a_{h,kg^{-1}}$$

and similarly

$$(w(g)a)_{h,k} = \langle w(g)a\delta_k, \, \delta_h \rangle = \langle a\delta_k, \, w(g)^*\delta_h \rangle = a_{hg,k}.$$

It is easy to check that $a_{h,kg^{-1}} = a_{hg,k}$ for all $g, h, k \in G$ if and only if a is constant on diagonals. The claim follows.

Similarly, one proves that $v(G)' = N$.

Since $w(G)$ and $v(G)$ are both closed under adjoints, it follows that M and N are von Neumann algebras. It is immediate from the definitions that $v(g)w(h) = w(h)v(g)$ for all $g, h \in G$. Therefore, using the first claim at the second step,

$$v(G)' = N \subset M'.$$

Take commutants throughout to get

$$v(G)'' = N' \supset M'' = M.$$

Since also $v(G) \subset w(G)' = M$, we use the definition at the first step to get $W_{\mathrm{r}}^*(G) = v(G)'' = M$, as was to be proved. $\square$

The following proposition gives "coefficients" of elements of $C_{\mathrm{r}}^*(G)$. It actually works not just for $C_{\mathrm{r}}^*(G)$ but for $W_{\mathrm{r}}^*(G)$, once one has extended the tracial state on $C_{\mathrm{r}}^*(G)$ to $W_{\mathrm{r}}^*(G)$.

Proposition 5.58. Let G be a discrete group, let $b \in C_{\mathrm{r}}^*(G) \subset L(l^2(G))$, and let $g \in G$. For $g \in G$, let $\delta_g \in l^2(G)$ be the standard basis vector corresponding to g. Let $\tau \colon C_{\mathrm{r}}^*(G) \to \mathbb{C}$ be the tracial state of Theorem 5.28. Then the following three numbers are equal:

(1) $\tau(b u_g^*)$.
(2) $\langle b\delta_1, \delta_g \rangle$.
(3) The constant value λ_g that the matrix of $b \in L(l^2(G))$ has on the diagonal consisting of those elements $b_{s,t}$ for $s, t \in G$ such that $st^{-1} = g$.

Proof. The equation $\lambda_g = \langle b\delta_1, \delta_g \rangle$ comes from the formula for the coefficients $b_{s,t}$, namely $b_{s,t} = \langle b\delta_t, \delta_s \rangle$ for $s, t \in G$.

We prove that $\tau(b u_g^*) = \langle b\delta_1, \delta_g \rangle$. By linearity and continuity, we may assume that $b \in \mathbb{C}[G]$. Thus, we may assume that $b = \sum_{h \in G} b_h u_h$ with $b_g = 0$ for all but finitely many $h \in G$. Then $\tau(b u_g^*) = b_g$. Also, letting $v \colon G \to U(l^2(G))$ be the left regular representation (Definition 5.3), the operator $\rho_v(b) \in L(l^2(G))$ acts as $\sum_{h \in G} b_h v(h)$, so

$$\langle b\delta_1, \delta_g \rangle = \left\langle \sum\nolimits_{h \in G} b_h v(h)\delta_1,\, \delta_g \right\rangle = \sum_{h \in G} b_h \langle \delta_h, \delta_g \rangle = b_g.$$

This completes the proof. $\square$

The last part of the proof above is simpler if we remember the proof of Theorem 5.28. We defined τ by the formula $\tau(b) = \langle b\delta_1, \delta_1 \rangle$. So, using the trace property at the first step, we have

$$\tau(b u_g^*) = \tau(u_g^* b) = \langle v(g)^* b\delta_1, \delta_1 \rangle = \langle b\delta_1, v(g)\delta_1 \rangle = \langle b\delta_1, \delta_g \rangle.$$

We can now think of an element $b \in C_{\mathrm{r}}^*(G)$ as a formal sum "$b = \sum_{g \in G} b_g u_g$". We emphasize that, in general, this sum is **only formal**. It does have one good feature.

Proposition 5.59. Let G be a discrete group, let $\tau \colon C_{\mathrm{r}}^*(G) \to \mathbb{C}$ be the tracial state of Theorem 5.28, and let $b \in C_{\mathrm{r}}^*(G)$. Suppose $\tau(b u_g) = 0$ for all $g \in G$. Then $b = 0$.

Proof. Recall from Theorem 5.28 that if $a \in C_{\mathrm{r}}^*(G)$ and $\tau(a^* a) = 0$, then $a = 0$. It therefore suffices to show that for all $a \in C_{\mathrm{r}}^*(G)$ and all $g \in G$, we have $\tau(a^* a) \geq |\tau(a u_g)|^2$. By continuity of τ and density of $\mathbb{C}[G]$ in $C_{\mathrm{r}}^*(G)$, it suffices to prove this inequality for $a \in \mathbb{C}[G]$. So assume that $a = \sum_{h \in G} a_h u_h$ with $a_h \in \mathbb{C}$ for all $h \in G$ and $a_h = 0$ for all but finitely many $h \in G$. Then, using $\tau(u_h^* u_k) \neq 0$ only if $h = k$ at the second step,

$$\tau(a^* a) = \tau\left(\sum\nolimits_{h,k \in G} \overline{a_h} a_k u_h^* u_k \right) = \sum_{k \in G} |a_k|^2 \geq |a_{g^{-1}}|^2 = |\tau(a u_g)|^2.$$

This completes the proof. □

Proposition 5.59 is useful, but it is quite weak. There are, in fact, many difficulties in understanding group C*-algebras.

Remark 5.60. Consider the special case $G = \mathbb{Z}$. Then $C^*(G)$ is isomorphic to $C(S^1)$, and the map $\lambda\colon l^1(\mathbb{Z}) \to C(S^1)$ of Exercise 5.49 is the Fourier series map: for $a = (a_n)_{n \in \mathbb{Z}_{>0}} \in l^1(\mathbb{Z})$, its image $\lambda(a)$ is the function

$$\lambda(a)(\zeta) = \sum_{n \in \mathbb{Z}} a_n \zeta^n$$

for $\zeta \in S^1$. This looks more familiar when we identify $C(S^1)$ with the set of 2π-periodic continuous functions on $\mathbb{R}$: it is

$$\lambda(a)(t) = \sum_{n \in \mathbb{Z}} a_n e^{int}$$

for $t \in \mathbb{R}$.

Every $f \in C(S^1)$ has a Fourier series. Letting μ be normalized arc length measure on S^1, its coefficients are given by

$$a_n = \int_{S^1} f(\zeta)\zeta^{-n}\, d\mu(\zeta).$$

It is well known that $\lim_{n \to \infty} a_n = \lim_{n \to -\infty} a_n = 0$, whence $a = (a_n)_{n \in \mathbb{Z}_{>0}} \in C_0(\mathbb{Z})$. However:

(1) We know of no good description of which sequences $a \in C_0(\mathbb{Z})$ are the Fourier coefficients of some $f \in C(S^1)$. Since the Fourier series map is a bijection from $l^2(\mathbb{Z})$ to $L^2(S^1)$, we do know that any such a must be in $l^2(\mathbb{Z})$. But in fact the Fourier coefficients of every element of $L^\infty(S^1)$, which is the group von Neumann algebra of $\mathbb{Z}$, are also in $l^2(\mathbb{Z})$, for the same reason. We get essentially no useful information out of a criterion for membership in a group C*-algebra which is satisfied by all elements in the group von Neumann algebra.

(2) For $a \in l^1(\mathbb{Z})$, or even in $\mathbb{C}[\mathbb{Z}]$, we know of no general way to compute the norm $\|\lambda(a)\|$ in terms of a, except by directly carrying out the computation of

$$\sup_{\zeta \in S^1} \left| \textstyle\sum_{n \in \mathbb{Z}} a_n \zeta^n \right|.$$

There are of course a few specific cases in which computations can be done. For example, let $\delta_n \in \mathbb{C}[\mathbb{Z}] \subset l^1(\mathbb{Z})$ be the element which takes the value 1 at n and is zero elsewhere. Then δ_n is unitary in $l^1(\mathbb{Z})$ and therefore also in $C^*(\mathbb{Z})$. So $\|\lambda(\delta_n)\| = 1$.

Computations of norms of some special elements of reduced group C*-algebras can be found in [1].

(3) Let δ_n be as in (2), and set $z_n = \lambda(\delta_n)$, which is the function $z_n(\zeta) = \zeta^n$ for $\zeta \in S^1$. For $f \in C(S^1)$, its sequence a of Fourier coefficients gives a formal series $\sum_{n \in \mathbb{Z}} a_n z_n$ for f. However, this series need not converge to f (or, indeed, to anything) in $C(S^1)$. In more familiar terms, this is the statement that the Fourier series of a continuous function need not converge uniformly.

(4) In fact, the series in (3) need not even converge in the weak operator topology on the von Neumann algebra, which here is isomorphic to $L^\infty(S^1)$. See Proposition 1 and the following remark in [171]. We warn the reader that erroneous claims for the convergence of this series have been made in some well known textbooks, such as in 7.11.2 of [198] and before Proposition V.7.6 of [276], as well as in some papers. (See [171] for details.) There is a topology, described in [171], in which one does have convergence. (We are grateful to Stuart White for pointing out this issue and providing the reference to [171].) Note, though, that the Cesàro means of the Fourier series of a continuous function f do converge uniformly to f. See 2.5 and Theorem 2.11 in Chapter 1 of [136]. This idea can be generalized substantially, to countable amenable groups and somewhat beyond, and to reduced crossed products rather than just reduced group C*-algebras. In [17], see Sections 5, and for example Theorem 5.6, which considers reduced crossed products by general countable amenable groups.

When G is abelian, the description of $C^*(G)$ as $C_0\big(\widehat{G}\big)$ is a concrete description of a different sort which is extremely useful. There are other groups, particularly various semisimple Lie groups, for which there are descriptions of $C^*(G)$ or $C^*_{\mathrm{r}}(G)$ which might be considered similar in spirit (although they are much more complicated). However, for many groups, including many countable amenable groups, no concrete description of $C^*(G)$ or $C^*_{\mathrm{r}}(G)$ is known.

Remark 5.61. The situation for $C^*(G)$ when G is not amenable is even worse than is suggested by Remark 5.60. For $a \in C^*(G)$, we can still use the homomorphism $\kappa\colon C^*(G) \to C^*_{\mathrm{r}}(G)$ to define "coefficients" a_g for $g \in G$, by $a_g = \tau(\kappa(a)u_g^*)$. However, since there are nonzero elements $a \in C^*(G)$ such that $\kappa(a) = 0$, these coefficients no longer even determine a uniquely.

When we get to them, we will see that the situation can be worse for crossed products. See Remark 9.19.

It seems appropriate to point out that, despite the issues presented in Remark 5.60 and Remark 5.61, in some ways $C^*(G)$ (in which we don't know the elements as functions on G, and where the natural convergence can fail) is better behaved that $l^1(G)$. For example, again take $G = \mathbb{Z}$. We can certainly write down an explicit description of all the elements of $l^1(\mathbb{Z})$. However, the (closed) ideal structure of $l^1(\mathbb{Z})$ is very complicated, and not completely known, while the ideal structure of $C^*(\mathbb{Z})$ is very simple: the closed ideals are in bijective order reversing correspondence with the closed subsets of S^1. According to Theorem 42.21 of [108] (see Definition 39.9 of [108] for the terminology), and the additional statements in 42.26 of [108], the phenomenon of intractable ideal structure occurs in $L^1(G)$ for every locally compact but noncompact abelian group G. Another example is the computation of the K-theory for crossed products. It turns out that the computation of the K-theory of crossed products by $\mathbb{Z}$, and even by nonabelian free groups, is easier than the computation of the K-theory of crossed products by $\mathbb{Z}/2\mathbb{Z}$.

6. Simplicity of the Reduced C*-Algebra of a Free Group

In this short section, we prove that $C^*_{\mathrm{r}}(F_n)$ is simple and has a unique tracial state for $n \in \{2, 3, \ldots, \infty\}$. We follow the original proof of Powers [227], with a slight simplification. A differently organized proof can be found in Section VII.7 of [52].

This result is not in the main direction of these notes, which are mainly concerned with the structure of crossed products by much smaller (in particular, amenable) groups in situations in which the action is free in some sense. It is included to provide a contrast to Theorem 15.10, a simplicity theorem which requires that the action be essentially free, and the observation that if G has more than one element, then $C^*(G)$ is never simple (since the one dimensional trivial representation gives a nontrivial homomorphism $C^*(G) \to \mathbb{C}$).

This result is the original one of its type. Simplicity of $C^*_{\mathrm{r}}(G)$ is now known for many (nonamenable) countable groups G. For recent definitive results, see [32].

Notation 6.1. Let $n \in \{2, 3, \ldots, \infty\}$. We let F_n denote the free group on n generators, and we call the generators $\gamma_1, \gamma_2, \ldots, \gamma_n$ (or $\gamma_1, \gamma_2, \ldots$ when $n = \infty$). We let $\tau \colon C^*_{\mathrm{r}}(F_n) \to \mathbb{C}$ be the tracial state of Theorem 5.28. For $g \in F_n$, we let $\delta_g \in l^2(F_n)$ be the corresponding standard basis vector. We take a *reduced word* in the generators to be an expression of the form

$$\gamma_{j(1)}^{l(1)} \cdot \gamma_{j(2)}^{l(2)} \cdots \gamma_{j(m)}^{l(m)} \tag{6.1}$$

with

$$m \in \mathbb{Z}_{\geq 0}, \qquad j(1) \neq j(2), \qquad j(2) \neq j(3), \qquad \ldots, \qquad j(m-1) \neq j(m), \tag{6.2}$$

and

$$l(1),\, l(2),\, \ldots,\, l(m) \in \mathbb{Z} \setminus \{0\}. \tag{6.3}$$

When $m = 0$, we get the empty word, representing $1 \in F_n$. We recall that every element of F_n is represented by a unique reduced word. For $m \neq 0$, we say that the reduced word (6.1) begins with $\gamma_{j(1)}^{l(1)}$ and ends with $\gamma_{j(m)}^{l(m)}$.

Lemma 6.2 (Lemma 4 of [227])**.** Let $n \in \{2, 3, \ldots, \infty\}$. Let $s \in \mathbb{Z}_{>0}$ and let $g_1, g_2, \ldots, g_s \in F_n \setminus \{1\}$. Then there exists $k \in \mathbb{Z}$ such that, for $r = 1, 2, \ldots, s$, the reduced word representing $\gamma_1^k g_r \gamma_1^{-k}$ begins and ends with nonzero powers of γ_1.

Proof. We renumber the elements $g_1, g_2, \ldots, g_s$ so that there is $s_0 \leq s$ such that $g_1, g_2, \ldots, g_{s_0}$ are not powers of γ_1 and $g_{s_0+1}, g_{s_0+2}, \ldots, g_s$ are powers of γ_1. For $r = 1, 2, \ldots, s_0$, the element g_r is then given by a reduced word of the form

$$g_r = \gamma_1^{\mu_r} \gamma_{j_r(1)}^{l_r(1)} \cdot \gamma_{j_r(2)}^{l_r(2)} \cdots \gamma_{j_r(m_r)}^{l(m_r)} \gamma_1^{\nu_r}$$

with $\gamma_{j_r(1)}^{l_r(1)} \cdot \gamma_{j_r(2)}^{l_r(2)} \cdots \gamma_{j_r(m_r)}^{l(m_r)}$ as in (6.1), (6.2), and (6.3), with $m_r \geq 1$, with $j_r(1) \neq 1$ and $j_{m_r}(1) \neq 1$, and with $\mu_r, \nu_r \in \mathbb{Z}$. If $\mu_r = 0$ or $\nu_r = 0$, the corresponding term in g_r is absent. For $r = s_0 + 1,\, s_0 + 2,\, \ldots,\, s$, there is $\nu_r \in \mathbb{Z}$ such that $g_r = \gamma_1^{\nu_r}$, and $\nu_r \neq 0$ since $g_r \neq 1$.

It is immediate that any $k \in \mathbb{Z}$ such that

$$k \notin \big\{ -\mu_1,\, -\mu_2,\, \ldots,\, -\mu_{s_0},\, \nu_1,\, \nu_2,\, \ldots,\, \nu_{s_0} \big\}$$

will satisfy the conclusion of the lemma. ∎

The following lemma generalizes Lemma 3 of [227].

Lemma 6.3. Let H be a Hilbert space, let $E \subset H$ be a closed subspace, let $p \in L(H)$ be the orthogonal projection onto E, and let $a \in L(H)$ satisfy $a(E^{\perp}) \subset E$. Then for all $\xi, \eta \in H$ we have

$$|\langle \xi, a\eta \rangle| \leq \|a\| \big(\|p\xi\| \cdot \|p\eta\| + \|p\xi\| \cdot \|(1-p)\eta\| + \|(1-p)\xi\| \cdot \|p\eta\| \big).$$

Proof. We expand

$$|\langle \xi, a\eta \rangle| \leq |\langle p\xi, ap\eta \rangle| + |\langle p\xi, a(1-p)\eta \rangle| + |\langle (1-p)\xi, ap\eta \rangle| + |\langle (1-p)\xi, a(1-p)\eta \rangle|.$$

By hypothesis, the last term is zero. Estimate

$$|\langle p\xi, ap\eta \rangle| \leq \|p\xi\| \cdot \|p\eta\| \cdot \|a\|, \qquad |\langle p\xi, a(1-p)\eta \rangle| \leq \|p\xi\| \cdot \|(1-p)\eta\| \cdot \|a\|,$$

and

$$|\langle (1-p)\xi, ap\eta \rangle| \leq \|(1-p)\xi\| \cdot \|p\eta\| \cdot \|a\|$$

to complete the proof. ∎

Lemma 6.4. Let $M \in \mathbb{Z}_{>0}$ and let

$$\lambda_1, \lambda_2, \ldots, \lambda_M, \mu_1, \mu_2, \ldots, \mu_M \in \mathbb{R}$$

be positive numbers such that $\sum_{m=1}^{M} \lambda_m^2 \leq 1$ and $\sum_{m=1}^{M} \mu_m^2 \leq 1$. Then

$$\sum_{m=1}^{M} \lambda_m \leq \sqrt{M}, \qquad \sum_{m=1}^{M} \mu_m \leq \sqrt{M}, \qquad \text{and} \qquad \sum_{m=1}^{M} \lambda_m \mu_m \leq 1.$$

Proof. Define $\lambda, \mu, \xi \in \mathbb{C}^M$ by

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_M), \quad \mu = (\mu_1, \mu_2, \ldots, \mu_M), \qquad \text{and} \qquad \xi = (1, 1, \ldots, 1).$$

Using the Cauchy-Schwarz inequality at the second step, we have

$$\sum_{m=1}^{M} \lambda_m = \langle \lambda, \xi \rangle \leq \|\lambda\|_2 \cdot \|\xi\|_2 = \left(\sum_{m=1}^{M} \lambda_m^2 \right)^{1/2} \cdot \sqrt{M} \leq \sqrt{M}.$$

This proves the first inequality. The proof of the second is the same, and the third follows by applying the Cauchy-Schwarz inequality to $\langle \lambda, \mu \rangle$. ∎

The following result is our substitute for Lemma 5 of [227], and the proof is essentially the same. However, we need not restrict to selfadjoint elements. Our statement includes that of Theorem 1 of [227], without using the iteration step in Lemma 6 of [227].

The proof obviously implies that the result holds simultaneously for all elements of any finite set in $C^*_{\mathrm{r}}(F_n)$.

Lemma 6.5. Let $n \in \{2, 3, \ldots, \infty\}$. Let $a \in C^*_{\mathrm{r}}(F_n)$ and let $\varepsilon > 0$. Then there exist $M \in \mathbb{Z}_{>0}$ and $h_1, h_2, \ldots, h_m \in F_n$ such that the linear map $T \colon C^*_{\mathrm{r}}(F_n) \to C^*_{\mathrm{r}}(F_n)$, defined by

$$T(b) = \frac{1}{M} \sum_{m=1}^{M} u_{h_m} b u_{h_m}^*,$$

satisfies $\|T(a) - \tau(a) \cdot 1\| < \varepsilon$.

Proof. We first suppose that $a \in \operatorname{span}\big(\{u_g \colon g \in F_n \setminus \{1\}\}\big)$. That is, there are $s \in \mathbb{Z}_{>0}$, $g_1, g_2, \ldots, g_s \in F_n \setminus \{1\}$, and $\lambda_1, \lambda_2, \ldots, \lambda_s \in \mathbb{C}$ such that $a = \sum_{r=1}^{s} \lambda_j u_{g_r}$. Then $\tau(a) = 0$, and we must find T of the form described in the conclusion such that $\|T(a)\| < \varepsilon$.

We may clearly assume $a \neq 0$. Choose $M \in \mathbb{Z}_{>0}$ such that

$$M > \frac{\varepsilon^2}{9\|a\|^2}.$$

Choose $k \in \mathbb{Z}$ as in Lemma 6.2, with $g_1, g_2, \ldots, g_s$ as given. For $m = 1, 2, \ldots, M$, define $h_m = \gamma_2^m \gamma_1^k$. Then, for $r = 1, 2, \ldots, s$, the reduced word representing $h_m g_r h_m^{-1}$ begins with γ_2^m and ends with γ_2^{-m}. Let $S_m \subset F_n$ be the set of all $g \in F_n$ for which the reduced word representing g begins with γ_2^m. Let $E_m \subset l^2(F_n)$ be $E_m = \overline{\operatorname{span}}(\{\delta_g \colon g \in S_m\})$. For any $g \in F_n \setminus S_m$, in the product $h_m g_r h_m^{-1} g$ the factor γ_2^{-m} at the end of $h_m g_r h_m^{-1}$ does not completely cancel, so the immediately preceding nonzero power of γ_1 is still present in the reduced word representing $h_m g_r h_m^{-1} g$. One can check that this word must then still begin with γ_2^m. We have shown that $h_m g_r h_m^{-1}(F_n \setminus S_m) \subset S_m$. It follows that $u_{h_m} u_{g_r} u_{h_m}^*(E_m^{\perp}) \subset E_m$. Since this is true for $r = 1, 2, \ldots, s$, it follows that $u_{h_m} a u_{h_m}^*(E_m^{\perp}) \subset E_m$.

Let $T \colon C^*_{\mathrm{r}}(F_n) \to C^*_{\mathrm{r}}(F_n)$ be defined as in the statement of the lemma, with this choice of M and $h_1, h_2, \ldots, h_m$. Let $\xi, \eta \in l^2(F_n)$ satisfy $\|\xi\|, \|\eta\| \leq 1$. Let $p_m \in L(H)$ be the orthogonal projection onto E_m. The spaces $E_1, E_2, \ldots, E_M$ are orthogonal, so

$$\sum_{m=1}^{M} \|p_m \xi\|^2 \leq \|\xi\|^2 = 1 \qquad \text{and} \qquad \sum_{m=1}^{M} \|p_m \eta\|^2 \leq \|\eta\|^2 = 1. \tag{6.4}$$

Using Lemma 6.3 at the second step, and (6.4) and Lemma 6.4 at the fifth step, we then have

$$\begin{aligned}
|\langle \xi, T(a)\eta\rangle| &= \frac{1}{M}\left|\sum_{m=1}^{M} \langle \xi, u_m a u_m^* \eta\rangle\right| \\
&\leq \frac{1}{M}\sum_{m=1}^{M} \|a\|\big(\|p_m\xi\|\cdot\|p_m\eta\| + \|p_m\xi\|\cdot\|(1-p_m)\eta\| + \|(1-p_m)\xi\|\cdot\|p_m\eta\|\big) \\
&\leq \frac{1}{M}\sum_{m=1}^{M} \|a\|\big(\|p_m\xi\|\cdot\|p_m\eta\| + \|p_m\xi\| + \|p_m\eta\|\big) \\
&= \frac{\|a\|}{M}\left(\sum_{m=1}^{M}\|p_m\xi\|\cdot\|p_m\eta\| + \sum_{m=1}^{M}\|p_m\xi\| + \sum_{m=1}^{M}\|p_m\eta\|\right) \\
&\leq \frac{\|a\|}{M}\big(1 + \sqrt{M} + \sqrt{M}\big) \leq \frac{3\|a\|}{\sqrt{M}}.
\end{aligned}$$

Since $\xi, \eta \in l^2(F_n)$ are arbitrary elements of norm 1, it follows that

$$\|T(a)\| \leq \frac{3\|a\|}{\sqrt{M}} < \varepsilon.$$

The special case $a \in \operatorname{span}\big(\{u_g \colon g \in F_n \setminus \{1\}\}\big)$ has been proved.

Next, suppose that $a \in \operatorname{span}\big(\{u_g \colon g \in F_n\}\big)$. Then $b = a - \tau(a) \cdot 1$ is in $\operatorname{span}\big(\{u_g \colon g \in F_n \setminus \{1\}\big)$, so there is T of the form in the conclusion such that

$\|T(b)\| < \varepsilon$. One easily checks that $T(1) = 1$. Therefore

$$\|T(a) - \tau(a) \cdot 1\| = \|T(a - \tau(a) \cdot 1)\| < \varepsilon.$$

Finally, we consider an arbitrary element $a \in C^*_{\mathrm{r}}(F_n)$. Choose $b \in \mathrm{span}\big(\{u_g \colon g \in F_n\big)$ such that $\|b - a\| < \frac{\varepsilon}{3}$. The previous paragraph provides T of the form in the conclusion such that $\|T(b) - \tau(b) \cdot 1\| < \frac{\varepsilon}{3}$. It is easy to check that $\|T\| \leq 1$. Therefore

$$\|T(a-\tau(a)\cdot 1)\| \leq \|T(a-b)\| + \big\|T((\tau(a)-\tau(b))\cdot 1)\big\| + \|T(b)-\tau(b)\cdot 1\| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.$$

This completes the proof of the lemma. $\square$

Theorem 6.6 (Theorem 2 of [227])**.** Let $n \in \{2, 3, \ldots, \infty\}$. Then $C^*_{\mathrm{r}}(F_n)$ is simple.

Proof. Let $I \subset C^*_{\mathrm{r}}(F_n)$ be a nonzero ideal. Choose $a \in I$ such that $a \neq 0$. Then $\tau(a^*a) \neq 0$ by Theorem 5.28. Lemma 6.5 provides $M \in \mathbb{Z}_{>0}$ and $h_1, h_2, \ldots, h_m \in F_n$ such that the element

$$c = \frac{1}{M} \sum_{m=1}^{M} u_{h_m} a^* a u_{h_m}^*$$

satisfies $\|c - \tau(a^*a) \cdot 1\| < \frac{1}{2}\tau(a^*a)$. Clearly $c \in I$. Then $b = \tau(a^*a)^{-1}c$ is also in I, and $\|b - 1\| < \frac{1}{2}$, so b is invertible. Therefore $I = C^*_{\mathrm{r}}(F_n)$. $\square$

The following result (for $n = 2$) is proved at the end of [227].

Theorem 6.7 ([227])**.** Let $n \in \{2, 3, \ldots, \infty\}$. Then $C^*_{\mathrm{r}}(F_n)$ has a unique tracial state.

Proof. Let σ be any tracial state on $C^*_{\mathrm{r}}(F_n)$. We prove that $\sigma = \tau$. Let $a \in C^*_{\mathrm{r}}(F_n)$ and let $\varepsilon > 0$. Use Lemma 6.5 to find $M \in \mathbb{Z}_{>0}$ and $h_1, h_2, \ldots, h_m \in F_n$ such that the element

$$c = \frac{1}{M} \sum_{m=1}^{M} u_{h_m} a u_{h_m}^*$$

satisfies $\|c - \tau(a) \cdot 1\| < \varepsilon$. We clearly have $\sigma(c) = \sigma(a)$ and $\sigma(\tau(a) \cdot 1) = \tau(a)$. So

$$|\sigma(a) - \tau(a)| = |\sigma(c - \tau(a) \cdot 1)| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\sigma(a) = \tau(a)$. $\square$

7. C*-Algebras of Locally Compact Groups

In this section, we consider the C*-algebras of general locally compact groups. Since our later focus will be mostly on discrete groups, we omit a number of proofs.

In the discrete case, in Section 5, we constructed the group C*-algebra as the closed linear span of the group elements in a suitable norm, and we constructed the group von Neumann algebra as the closed linear span of the group elements in a suitable (much weaker) topology. For the von Neumann algebra, this definition turns out to still work, but it does not give a reasonable outcome for the C*-algebra. For example, if the group G is second countable, one wants the group C*-algebra to be separable. However, if $g, h \in G$ with $g \neq h$, then $\|u_g - u_h\| = 2$. See Exercise 7.3.

Throughout this section, we let μ be a fixed left Haar measure on G.

The following definition is the generalization of Definition 5.3.

Definition 7.1. The *left regular representation of* G is the representation $v\colon G \to U(L^2(G,\mu))$ given by $(v(g)\xi)(h) = \xi(g^{-1}h)$ for $g, h \in G$ and $\xi \in L^2(G,\mu)$.

Exercise 7.2. Let G be a locally compact group. Prove that v as in Definition 7.1 is a unitary representation of G on $L^2(G)$.

The main point beyond Exercise 5.4 (the case of a discrete group) is to prove continuity. Left invariance of the measure will be needed to show that $v(g)$ is unitary.

As for discrete groups, there is also a right regular representation. There is one new feature: one must use right Haar measure, or else correct the formula by including suitable Radon-Nikodym derivatives (here, a suitable power of the modular function of Theorem 7.5 below).

Exercise 7.3. Let G be a locally compact group, let $v\colon G \to U(L^2(G))$ be the left regular representation, and let $g, h \in G$ with $g \neq h$. Prove that $\|v(g) - v(h)\| = 2$. Use this fact to prove that if G is not discrete, then $\overline{\mathrm{span}}\big(\{v(g)\colon g \in G\}\big)$ is not separable.

When we have constructed $C^*(G)$ and $C^*_{\mathrm{r}}(G)$, it will turn out that the group elements u_g are in the multiplier algebras $M(C^*(G))$ and $M(C^*_{\mathrm{r}}(G))$. (We will not prove this.) In particular, the naive analog of Exercise 5.19 certainly does not hold, and we know of no general method of describing either $C^*(G)$ or $C^*_{\mathrm{r}}(G)$ in terms of generators and relations.

Instead of $\mathbb{C}[G]$, we will use the space $C_{\mathrm{c}}(G)$ of compactly supported continuous functions on G, with the convolution defined by the analog of (5.2) in Remark 5.9.

Notation 7.4. Let X be a locally compact Hausdorff space. We denote by $C_{\mathrm{c}}(X)$ the complex vector space of all continuous functions from X to $\mathbb{C}$ which have compact support, with pointwise addition and scalar multiplication. Unless otherwise specified, we make this space a complex *-algebra using pointwise complex conjugation and pointwise multiplication, but we will frequently use other operations; in particular, if G is a group, the operations will usually be as in Definition 7.6 below. If E is any Banach space, we further denote by $C_{\mathrm{c}}(X, E)$ the vector space of all continuous functions from X to E which have compact support.

Another complication which appears for general locally compact groups is the possible failure of unimodularity. We recall for reference the basic properties of the modular function.

Theorem 7.5. Let G be a locally compact group. Make $(0, \infty)$ into a locally compact abelian group by taking the group operation to be multiplication. Then there is a unique continuous homomorphism $\Delta\colon G \to (0,\infty)$, called the *modular function* of G, such that, for every choice of Haar measure μ on G, for every $g \in G$, and every measurable set $E \subset G$, we have $\mu(Eg) = \Delta(g)\mu(E)$. Moreover, for $g \in G$ and every $a \in C_{\mathrm{c}}(G)$, we have

$$\int_G a(gh)\, d\mu(g) = \Delta(h)^{-1} \int_G a(g)\, d\mu(g), \tag{7.1}$$

and for every $a \in C_{\mathrm{c}}(G)$ we have

$$\int_G \Delta(g)^{-1} a(g^{-1})\, d\mu(g) = \int_G a(g)\, d\mu(g). \tag{7.2}$$

For the proof, see Lemma 1.61 and Lemma 1.67 of [292].

Definition 7.6. Let G be a locally compact group. Using left Haar measure μ on G, for $a, b \in C_{\mathrm{c}}(G)$ we define

$$(ab)(g) = \int_G a(h)b(h^{-1}g)\ d\mu(h) \qquad \text{and} \qquad a^*(g) = \Delta(g)^{-1}\overline{a(g^{-1})}. \tag{7.3}$$

We will need Fubini's Theorem several times, and the following lemma will be used to verify its hypotheses.

Lemma 7.7. Let G be a locally compact group, and let $a, b \in C_{\mathrm{c}}(G)$. Define $f_{a,b}\colon G \times G \to \mathbb{C}$ by $f_{a,b}(g,h) = a(h)b(h^{-1}g)$ for $g, h \in G$. Then $f_{a,b} \in C_{\mathrm{c}}(G \times G)$, and $\mathrm{supp}(f_{a,b})$ is contained in the compact set $(\mathrm{supp}(a) \cdot \mathrm{supp}(b)) \times \mathrm{supp}(a)$.

Proof. It is immediate that $f_{a,b}$ is continuous. To see that $f_{a,b}$ has compact support, define $K \subset G$ by

$$K = \mathrm{supp}(a) \cdot \mathrm{supp}(b) = \big\{ gh\colon g \in \mathrm{supp}(a) \text{ and } h \in \mathrm{supp}(b) \big\}.$$

Then K is compact because K is the image of the compact set $\mathrm{supp}(a) \times \mathrm{supp}(b) \subset G \times G$ under the multiplication map. We show that $\mathrm{supp}(f_{a,b}) \subset K \times \mathrm{supp}(a)$. So suppose $f_{a,b}(g,h) \neq 0$. Obviously $h \in \mathrm{supp}(a)$ and $h^{-1}g \in \mathrm{supp}(b)$. Therefore $g = h \cdot h^{-1}g \in K$. □

Proposition 7.8. Let G be a locally compact group. Equipped with the operations in Definition 7.6, the space $C_{\mathrm{c}}(G)$ is a complex *-algebra.

Proof. Let μ be left Haar measure on G. For $a, b \in C_{\mathrm{c}}(G)$, let $f_{a,b} \in C_{\mathrm{c}}(G \times G)$ be as in Lemma 7.7. We then have

$$(ab)(g) = \int_G f_{a,b}(g,h)\, d\mu(h).$$

Therefore $(ab)(g)$ can be nonzero only for $g \in \mathrm{supp}(a) \cdot \mathrm{supp}(b)$.

We next prove that ab is continuous. This is a standard argument, which we give for completeness. We need only consider the case $a \neq 0$. Set $M = \mu(\mathrm{supp}(a)) > 0$. Let $\varepsilon > 0$ and let $g_0 \in G$. For $h \in G$ choose open sets $U(h), V(h) \subset G$ such that $g_0 \in U(h)$, $h \in V(h)$, and for all $g \in U(h)$ and $k \in V(h)$, we have

$$\big| f_{a,b}(g,k) - f_{a,b}(g_0,h) \big| < \frac{\varepsilon}{4M}.$$

Choose $n \in \mathbb{Z}_{>0}$ and $h_1, h_2, \ldots, h_n \in G$ such that the sets $V(h_1)$, $V(h_2)$, $\ldots$, $V(h_n)$ cover $\mathrm{supp}(a)$. Set $U = \bigcap_{j=1}^n U(h_j)$, which is an open set containing g_0.

Let $g \in U$. For $h \in \mathrm{supp}(a)$, we claim that

$$\big| f_{a,b}(g,h) - f_{a,b}(g_0,h) \big| < \frac{\varepsilon}{2M}.$$

Choose $j \in \{1, 2, \ldots, n\}$ such that $h \in V(h_j)$. Then $g \in U(h_j)$, so

$$\begin{aligned} \big| f_{a,b}(g,h) - f_{a,b}(g_0,h) \big| &\leq \big| f_{a,b}(g,h) - f_{a,b}(g_0,h_j) \big| + \big| f_{a,b}(g_0,h_j) - f_{a,b}(g_0,h) \big| \\ &< \frac{\varepsilon}{4M} + \frac{\varepsilon}{4M} = \frac{\varepsilon}{2M}, \end{aligned}$$

as desired.

It now follows that

$$\begin{aligned}\big|(ab)(g) - (ab)(g_0)\big| &\leq \int_{\mathrm{supp}(a)} \big|f_{a,b}(g,h) - f_{a,b}(g_0,h)\big| \, d\mu(h) \\ &\leq \left(\frac{\varepsilon}{2M}\right) \mu(\mathrm{supp}(a)) = \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

This completes the proof that ab is continuous.

We have now shown that $(a, b) \mapsto ab$ is a well defined map $C_{\mathrm{c}}(G) \times C_{\mathrm{c}}(G) \to C_{\mathrm{c}}(G)$. It is obviously bilinear. It remains only to prove associativity and the properties of the adjoint.

Let $a, b, c \in C_{\mathrm{c}}(G)$. We compute as follows, with the second step being an application of Fubini's Theorem which is justified afterwards. The third step is a change of variables in the inner integral, replacing h with kh. For $g \in G$, we have

$$\begin{aligned}[(ab)c](g) &= \int_G \left(\int_G a(k) b(k^{-1}h) \, d\mu(k) \right) c(h^{-1}g) \, d\mu(h) \\ &= \int_G a(k) \left(\int_G b(k^{-1}h) c(h^{-1}g) \, d\mu(h) \right) d\mu(k) \\ &= \int_G a(k) \left(\int_G b(h) c(h^{-1}k^{-1}g) \, d\mu(h) \right) d\mu(k) \\ &= \int_G a(k) (bc)(k^{-1}g) \, d\mu(k) = [a(bc)](g).\end{aligned}$$

To justify the application of Fubini's Theorem at the second step, we observe that the integrand as a function of both variables is $(h, k) \mapsto f_{a,b}(h, k) c(h^{-1}g)$, which is a continuous function on $G \times G$ with support in the compact set $(\mathrm{supp}(a) \cdot \mathrm{supp}(b)) \times \mathrm{supp}(a)$. Therefore it is integrable with respect to $\mu \times \mu$.

It is obvious that $a \mapsto a^*$ is conjugate linear, and easy to check that $a^{**} = a$ for all $a \in C_{\mathrm{c}}(G)$. It remains only to check that $(ab)^* = b^* a^*$ for $a, b \in C_{\mathrm{c}}(G)$. For $g \in G$, using the change of variables from h to gh at the third step, we have

$$\begin{aligned}(b^* a^*)(g) &= \int_G \Delta(h)^{-1} \overline{b(h^{-1})} \Delta(h^{-1}g)^{-1} \overline{a((h^{-1}g)^{-1})} \, d\mu(h) \\ &= \int_G \Delta(g)^{-1} \overline{a(g^{-1}h)} \cdot \overline{b(h^{-1})} \, d\mu(h) \\ &= \Delta(g)^{-1} \int_G \overline{a(h)} \cdot \overline{b(h^{-1}g^{-1})} \, d\mu(h) = (ab)^*(g).\end{aligned}$$

This completes the proof. $\square$

Exercise 7.9. Let G be a discrete group. Prove that there is a complex *-algebra isomorphism of $\mathbb{C}[G]$ as in Definition 5.9 with $C_{\mathrm{c}}(G)$ as in Definition 7.6 and Proposition 7.8.

The main point is to make sure that the definitions of the product and adjoint match.

We will need a topology on $C_{\mathrm{c}}(G)$. To follow what we did for discrete G as closely as possible, we would use the direct limit topology. Continuity of linear functionals in this topology is determined by testing on nets $(b_i)_{i \in I}$ in $C_{\mathrm{c}}(G)$ and elements $b \in C_{\mathrm{c}}(G)$ such that $b_i \to b$ uniformly and there is some common compact set $K \subset G$ with $\mathrm{supp}(b_i) \subset K$ for all $i \in I$. See Remark 1.86 of [292] for more on this topology. (It can have other convergent nets.)

Here, it seems simpler to just use the L^1 norm, and to complete $C_c(G)$ in this norm, getting the convolution algebra $L^1(G)$. When G is discrete, this definition specializes to the algebra $l^1(G)$ of Definition 5.42.

Definition 7.10. Let G be a locally compact group. Using Haar measure in the integral, we define a norm on $C_c(G)$ by $\|b\|_1 = \int_G |b(g)| \, d\mu(g)$. We define $L^1(G)$ to be the completion of $C_c(G)$ in this norm. Justified by Proposition 7.11 below, we make $L^1(G)$ into a Banach *-algebra by extending the operations of Definition 7.6 by continuity.

There is never any problem with the integral, because we need only integrate continuous functions on compact sets. When G is second countable, so that Haar measure is σ-finite and all Borel sets are Baire sets, the resulting space is just the usual space $L^1(G)$ of integrable Borel functions on G. In our presentation, we avoid technicalities of measure theory (including but not limited to dealing with measures which are not σ-finite) by defining $L^1(G)$ to be the completion of $C_c(G)$.

Proposition 7.11. Let G be a locally compact group. Then for $a, b \in C_c(G)$, we have $\|ab\|_1 \leq \|a\|_1 \|b\|_1$ and $\|a^*\|_1 = \|a\|_1$.

Proof. For the first part, let $a, b \in C_c(G)$. Let $f_{a,b} \in C_c(G \times G)$ be as in Lemma 7.7, that is, $f_{a,b}(g,h) = a(h) b(h^{-1}g)$. Since $f_{a,b}$ is integrable with respect to $\mu \times \mu$, we can apply Fubini's Theorem at the third step in the following calculation:

$$\begin{aligned}
\|ab\|_1 &= \int_G \left| \int_G f_{a,b}(g,h) \, d\mu(h) \right| \, d\mu(g) \leq \int_G \left(\int_G |f_{a,b}(g,h)| \, d\mu(h) \right) \, d\mu(g) \\
&= \int_G \left(\int_G |f_{a,b}(g,h)| \, d\mu(g) \right) \, d\mu(h) \\
&= \int_G |a(h)| \left(\int_G |b(h^{-1}g)| \, d\mu(g) \right) \, d\mu(h) = \int_G |a(h)| \cdot \|b\|_1 \, d\mu(h) = \|a\|_1 \|b\|_1.
\end{aligned}$$

For the second part, we apply (7.2) in Theorem 7.5 at the second step to get

$$\|a^*\|_1 = \int_G \Delta(g)^{-1} \left| \overline{a(g^{-1})} \right| \, d\mu(g) = \int_G \left| \overline{a(g)} \right| \, d\mu(g) = \|a\|_1.$$

This completes the proof. $\square$

Exercise 7.12. Let G be a discrete group, and take Haar measure on G to be counting measure. Prove that there is a Banach *-algebra isomorphism of $L^1(G)$ as in Definition 7.10 and $l^1(G)$ as in Definition 5.42.

Given Exercise 7.9, this exercise is essentially trivial.

We now give the analog of the construction of Definition 5.13. At this point, we want to integrate continuous functions with compact support which have values in a Banach space. In principle, the "right" approach to Banach space valued integration is to define measurable Banach space valued functions and their integrals. This has been done; one reference is Appendix B of [292]. (Note the systematic misprint there: "separately-valued" should be "separably-valued".) Things simplify considerably if G is second countable and E is separable, but neither of these conditions is necessary for the constructions we carry out, either here or in Section 8. For continuous functions with compact support, it is easy to avoid this theory, and this is the route we take. An integration theory sufficient for this purpose is developed in Section 1.5 of [292]. We summarize the properties.

We could avoid such integrals here by always working in terms of scalar products below. This seems pointless since we won't be able to do something similar when defining multiplication in crossed products in Definition 8.2.

In the following theorem, the relation in (2) is (1.23) in [292], existence is in Lemma 1.91 of [292], and uniqueness is in the discussion at the beginning of Section 1.5 of [292].

Theorem 7.13. Let G be a locally compact group with left Haar measure μ, and let E be a Banach space. Then there is a unique linear map $I_E \colon C_{\mathrm{c}}(G, E) \to E$ with the following properties:

(1) $\|I_E(\xi)\| \leq \int_G \|\xi(g)\| \, d\mu(g)$ for all $\xi \in C_{\mathrm{c}}(G, E)$.

(2) For $\eta \in E$ and $f \in C_{\mathrm{c}}(G)$, the function $\xi(g) = f(g)\eta$ for $g \in G$ satisfies $I_E(\xi) = \left(\int_G f(g) \, d\mu(g) \right) \eta$.

Definition 7.14. Let G be a locally compact group with left Haar measure μ, and let E be a Banach space. With I_E as in Theorem 7.13, we define $\int_G \xi(g) \, d\mu(g) = I_E(\xi)$ for $\xi \in C_{\mathrm{c}}(G, E)$.

The next lemma is part of Lemma 1.91 of [292], but we give a direct proof directly from the properties of the integral given in Theorem 7.13.

Lemma 7.15. Let G be a locally compact group with left Haar measure μ, let E and F be Banach spaces, and let $a \in L(E, F)$. Then for all $\xi \in C_{\mathrm{c}}(G, E)$, we have

$$a \left(\int_G \xi(g) \, d\mu(g) \right) = \int_G a(\xi(g)) \, d\mu(g).$$

Proof. Let $I_E \colon C_{\mathrm{c}}(G, E) \to E$ and $I_F \colon C_{\mathrm{c}}(G, F) \to F$ be as in Theorem 7.13. Define $T \colon C_{\mathrm{c}}(G, E) \to C_{\mathrm{c}}(G, F)$ by $T(\xi)(g) = a(\xi(g))$ for $\xi \in C_{\mathrm{c}}(G, E)$ and $g \in G$. We must prove that $a \circ I_E = I_F \circ T$. Using Theorem 7.13(2) twice, it is easy to check that if $\xi_0 \in E$, $f \in C_{\mathrm{c}}(G)$, and we define $\xi \in C_{\mathrm{c}}(G, E)$ by $\xi(g) = f(g)\xi_0$ for $g \in G$, then

$$(a \circ I_E)(\xi) = a \left(\left[\int_G f \mu \right] \xi_0 \right) = \left(\int_G f \, d\mu \right) a\xi_0 = (I_F \circ T)(\xi).$$

Now let $\xi \in C_{\mathrm{c}}(G, E)$ be arbitrary. Let $\varepsilon > 0$. We use a partition of unity argument to prove that $\big\| (a \circ I_E)(\xi) - (I_F \circ T)(\xi) \big\| < \varepsilon$. Choose an open set $U \subset G$ such that $\mathrm{supp}(\xi) \subset U$ and the set $L = \overline{U}$ is compact. Set

$$\delta = \frac{\varepsilon}{3(\|a\| + 1)(\mu(L) + 1)}.$$

Use compactness of L and continuity of ξ to find $n \in \mathbb{Z}_{>0}$, open sets $V_1, V_2, \ldots, V_n \subset G$ which cover L, and $g_j \in V_j$ for $j = 1, 2, \ldots, n$ such that $\|\xi(g) - \xi(g_j)\| < \delta$ for $j = 1, 2, \ldots, n$ and $g \in V_j$. Choose continuous functions $f_j \colon L \to [0, 1]$ which form a partition of unity on L and such that $\mathrm{supp}(f_j) \subset V_j \cap L$ for $j = 1, 2, \ldots, n$. We may extend the functions $f_1, f_2, \ldots, f_n$ so that they are continuous functions defined on all of G, take values on $[0, 1]$, satisfy $\mathrm{supp}(f_j) \subset V_j$ for $j = 1, 2, \ldots, n$, and satisfy $\sum_{j=1}^n f_j(g) \leq 1$ for all $g \in G$. Further choose a continuous function $f \colon G \to [0, 1]$ such that $f(g) = 1$ for all $g \in \mathrm{supp}(\xi)$ and $\mathrm{supp}(f) \subset U$.

Define $\eta \in C_{\mathrm{c}}(G, E)$ by

$$\eta(g) = f(g) \sum_{j=1}^n f_j(g)\xi(g_j)$$

for $g \in G$. Then

$$\|\eta(g) - \xi(g)\| \leq f(g) \sum_{j=1}^{n} f_j(g) \|\xi(g) - \xi(g_j)\|.$$

We have $\|\xi(g) - \xi(g_j)\| < \delta$ whenever $f_j(g) \neq 0$, and $0 \leq f(g) \sum_{j=1}^{n} f_j(g) \leq 1$, so $\|\eta(g) - \xi(g)\| < \delta$. Moreover $\eta(g) = \xi(g)$ for all $g \in G \setminus L$. Theorem 7.13(1) therefore implies that $\|I_E(\xi) - I_E(\eta)\| \leq \mu(L)\delta$. So

$$\big\|(a \circ I_E)(\xi) - (a \circ I_E)(\eta)\big\| \leq \|a\| \mu(L)\delta.$$

Also

$$\|T(\xi)(g) - T(\eta)(g)\| = \|a(\eta(g)) - a(\xi(g))\| < \|a\|\delta$$

for all $g \in G$, so Theorem 7.13(1) implies

$$\big\|(I_F \circ T)(\xi) - (I_F \circ T)(\eta)\big\| \leq \|a\| \mu(L)\delta.$$

The first paragraph of the proof implies that $(a \circ I_E)(\eta) = (I_F \circ T)(\eta)$, so

$$\big\|(a \circ I_E)(\xi) - (I_F \circ T)(\xi)\big\| \leq \|a\| \mu(L)\delta + \|a\| \mu(L)\delta < \varepsilon,$$

as desired. □

The formula in the following definition should be compared with (5.5) in Definition 5.45.

Definition 7.16. Let G be a locally compact group with left Haar measure μ, let H be a Hilbert space, and let $v \colon G \to U(H)$ be a unitary representation. Then the *integrated form* of v is the representation $\rho_v \colon C_{\mathrm{c}}(G) \to L(H)$ given by

$$\rho_v(b)\xi = \int_G b(g) v(g) \xi \, d\mu(g)$$

for $b \in C_{\mathrm{c}}(G)$. Justified by Proposition 7.17 below, we extend this representation by continuity to a representation $L^1(G) \to L(H)$, which we also denote by ρ_v and call the *integrated form* of v.

We want to think of $\rho_v(b)$ as $\int_G b(g) v(g) \, d\mu(g)$. Defining ρ_v directly by this formula causes technical problems, because $g \mapsto b(g) v(g)$ is only a strong operator continuous function to $L(H)$, not a norm continuous function. The definition given is the easiest solution to these difficulties.

Proposition 7.17 (Part of Theorem 3.9 of [87]; part of Proposition 13.3.4 of [60])**.** Let G be a locally compact group, let H be a Hilbert space, and let $w \colon G \to U(H)$ be a unitary representation. Then $\rho_w \colon C_{\mathrm{c}}(G) \to L(H)$ is a *-homomorphism and $\|\rho_w(b)\| \leq \|b\|_1$ for all $b \in C_{\mathrm{c}}(G)$.

We want to make one point explicitly. Even though the function $g \mapsto w(g)$ is not required to be norm continuous, the representation ρ_w *is* norm continuous. In particular, if $a \in C_{\mathrm{c}}(G)$ and we define $a_g \in C_{\mathrm{c}}(G)$ by $a_g(h) = a(g^{-1}h)$ for $g, h \in G$, then $g \mapsto a_g$ is a continuous function from G to $L^1(G)$, and $g \mapsto \rho_w(a_g)$ is a norm continuous function from G to $L(H)$.

Proof of Proposition 7.17. The expression for $\rho_w(b)\xi$ is defined, by Definition 7.14 and Theorem 7.13. Moreover, $\rho_w(b)\xi$ is obviously linear in both $b \in C_{\mathrm{c}}(G)$ and $\xi \in H$.

Now let $b \in C_{\mathrm{c}}(G)$ and $\xi \in H$. Using Theorem 7.13(1) at the first step, and $\|w(g)\| = 1$ at the second step, we get

$$\|\rho_w(b)\xi\| \leq \int_G |b(g)| \cdot \|w(g)\xi\| \, d\mu(g) \leq \|b\|_1 \|\xi\|.$$

Thus $\rho_w(b) \in L(H)$ for all $b \in C_{\mathrm{c}}(G)$.

It remains to prove that ρ_w preserves products and adjoints. Let $a, b \in C_{\mathrm{c}}(G)$, and let $\xi, \eta \in H$. We prove that

$$\langle \rho_w(ab)\xi, \, \eta \rangle = \langle \rho_w(a)\rho_w(b)\xi, \, \eta \rangle \qquad \text{and} \qquad \langle \rho_w(b^*)\xi, \, \eta \rangle = \langle \xi, \, \rho_w(b)\eta \rangle.$$

For the first, we use Lemma 7.15 at the first, third, and fifth steps, Fubini's Theorem (justified by Lemma 7.7 and continuity of $(g, h) \mapsto f_{a,b}(g, h)\langle w(g)\xi, \eta \rangle$) at the second step, Lemma 7.15 and left translation invariance of μ at the fourth step, getting

$$\begin{aligned}
\langle \rho_w(ab)\xi, \, \eta \rangle &= \int_G \left(\int_G a(h)b(h^{-1}g) \, d\mu(h) \right) \langle w(g)\xi, \eta \rangle \, d\mu(g) \\
&= \int_G \left(\int_G a(h)b(h^{-1}g)\langle w(g)\xi, \eta \rangle \, d\mu(g) \right) d\mu(h) \\
&= \int_G \left\langle a(h)w(h) \left(\int_G b(h^{-1}g)w(h^{-1}g)\xi \, d\mu(g) \right), \, \eta \right\rangle d\mu(h) \\
&= \int_G \big\langle a(h)w(h)\rho_w(b)\xi, \, \eta \big\rangle \, d\mu(h) = \langle \rho_w(a)\rho_w(b)\xi, \, \eta \rangle.
\end{aligned}$$

For the second, we use Lemma 7.15 at the first step, $w(g)^* = w(g^{-1})$ at the second step, (7.2) (in Theorem 7.5) at the third step, and Lemma 7.15 at the fourth step, getting

$$\begin{aligned}
\langle \rho_w(b^*)\xi, \, \eta \rangle &= \int_G \Delta(g)^{-1} \big\langle \overline{b(g^{-1})} w(g)\xi, \, \eta \big\rangle \, d\mu(g) \\
&= \int_G \Delta(g)^{-1} \big\langle \xi, \, b(g^{-1})w(g^{-1})\eta \big\rangle \, d\mu(g) \\
&= \int_G \langle \xi, \, b(g)w(g)\eta \rangle \, d\mu(g) = \langle \xi, \, \rho_w(b)\eta \rangle.
\end{aligned}$$

This completes the proof. □

The following theorem is the analog for locally compact groups of Theorem 5.22 (for discrete groups).

Theorem 7.18 (Theorems 3.9 and 3.11 of [87]; Proposition 7.1.4 of [198]; Proposition 13.3.4 of [60])**.** *Let G be a locally compact group, and let H be a Hilbert space. Then the integrated form construction defines a bijection from the set of unitary representations of G on H to the set of nondegenerate continuous *-representations of $L^1(G)$ on H.*

Since our main subject is discrete groups, we will not give a proof here. We do mention one key technical point. The proof can't be done the same way as the proof of Proposition 5.14, because there is no analog in $C_{\mathrm{c}}(G)$, or even in $L^1(G)$, of the images u_g of the group elements in $\mathbb{C}[G]$. The analogs of the elements u_g can only be found in the multiplier algebra of $L^1(G)$.

Since integrated form representations of $L^1(G)$ are necessarily contractive, *all* continuous representations of $L^1(G)$ are necessarily contractive.

We now give the analog of Definition 5.16.

Definition 7.19. Let G be a locally compact group. Choose a fixed Hilbert space H_0 with dimension $\mathrm{card}(G)$, and define a unitary representation w of G to be the direct sum of all possible unitary representations of G on subspaces of H_0. We call w the *universal representation* of G.

Definition 7.20. Let G be a locally compact group, and let $w\colon G \to U(H)$ be its universal unitary representation, as in Definition 7.19. Using the notation of Definition 7.16, we define $C^*(G)$ to be the norm closure in $L(H)$ of $\rho_w(C_{\mathrm{c}}(G))$.

Equivalently, one can take $C^*(G) = \overline{\rho_w(L^1(G))}$.

Theorem 7.21 (13.9.3 of [60])**.** Let G be a locally compact group, and let H be a Hilbert space. Then the integrated form construction defines a bijection from the set of unitary representations of G on H to the set of nondegenerate representations of $C^*(G)$ on H.

Given Theorem 7.18, the proof is similar to the first part of the proof of Theorem 5.22.

If we were able to take the universal representation of G to be the direct sum of all possible representations of G, the proof would be clear. Given any representation of G, it would be the restriction of the universal representation of G to some invariant subspace, and we would simply restrict the corresponding representation of $C^*(G)$ to the same subspace.

Definition 7.22. Let G be a locally compact group, and let $v\colon G \to U(L^2(G))$ be its left regular representation (Definition 7.1). Using the notation of Definition 7.16, we define the *reduced group C^*-algebra* $C^*_{\mathrm{r}}(G)$ to be the closure $\overline{\rho_v(C_{\mathrm{c}}(G))}$ in the norm topology on $L(L^2(G))$.

Proposition 7.23. Let G be a locally compact group. Then there is a surjective homomorphism $\kappa\colon C^*(G) \to C^*_{\mathrm{r}}(G)$ obtained from Theorem 7.21 by taking the nondegenerate representation used there to be the left regular representation of G. It is uniquely determined by the property that if $f \in C_{\mathrm{c}}(G)$ and $a \in C^*(G)$ and $b \in C^*_{\mathrm{r}}(G)$ are the images of f in those two algebras, then $\kappa(a) = b$.

Proof. The result is immediate from Theorem 7.21 as soon as one knows that the left regular representation of G is continuous. This fact is Exercise 7.2. $\square$

Recall (Theorem 5.50 and Theorem 5.51; both stated for the general case) that $\kappa\colon C^*(G) \to C^*_{\mathrm{r}}(G)$ is an isomorphism if and only if G is amenable. The proof given after Theorem 5.50 covers only the discrete case. However, the proof of the general case is contained in the proof of the corresponding result for crossed products, Theorem 9.7 below. We give that proof in full below.

Evaluation at $1 \in G$ gives a tracial linear functional from $C_{\mathrm{c}}(G)$ to $\mathbb{C}$. However, this functional is not continuous with respect to $\|\cdot\|_1$. Thus, unlike in Theorem 5.28, we do not get a tracial state on $C^*_{\mathrm{r}}(G)$.

Functoriality as in Exercise 5.26 and Exercise 5.27 does not generalize very well. In particular, the the full group C*-algebra is not a functor from locally compact group and group homomorphisms to C*-algebras and homomorphisms. If G_2 is discrete and $\varphi\colon G_1 \to G_2$ is the inclusion of a subgroup, then the map in Exercise 5.26

is given at the level of $C_{\mathrm{c}}(G_1) \to C_{\mathrm{c}}(G_2)$ by extending a function on G_1 to all of G_2 by having it take the value zero on $G_2 \setminus G_1$. However, suppose $\varphi \colon G_1 \to G_2$ is the inclusion of the subgroup $G_1 = \{1\}$ in G_2, and assume that G_2 is not discrete. There is no related homomorphism $C_{\mathrm{c}}(G_1) \to C_{\mathrm{c}}(G_2)$. If we try an analogous definition of a map $L^1(G_1) \to L^1(G_2)$, since $\{1\}$ has measure zero in G_2, we get the zero map. The same kind of thing goes wrong for the inclusion of, for example, the subgroup $\mathbb{R} \times \{0\}$ in $\mathbb{R}^2$.

Things still work if the range of φ is open in G_2. We omit the proof. There are other things that can be done instead, but we do not discuss them here.

There is an approach to the theory of locally compact abelian groups which starts out by defining $\widehat{G}$ to be the maximal ideal space $\mathrm{Max}(C^*(G))$. Chapter 4 of [87] comes close to following this approach.

Remark 5.60 describes some of the difficulties with understanding and working with $C^*(\mathbb{Z})$. When the group is not discrete, everything that can go wrong before can still go wrong, although, since the canonical unitaries associated to the group elements are no longer in the group C*-algebra (only in its multiplier algebra), the situation is harder to describe. The obvious analogous case to consider is $G = \mathbb{R}$. We make explicit just one issue. The analog of Remark 5.60(1) is to ask exactly which functions on $\mathbb{R}$ have Fourier transforms (in the distributional sense) which are in $C_0(\mathbb{R}) = C^*(\mathbb{R})$. This is certainly at least as hard as, and probably harder than, asking which functions on $\mathbb{Z}$ are the sequence of Fourier coefficients of functions in $C(S^1) = C^*(\mathbb{Z})$.

If G is not amenable, the situation for $C^*(G)$ is of course also at least as bad as described in Remark 5.61, and it is harder to even formulate the problem.

There is also a group von Neumann algebra. The following definition is the analog for locally compact groups of Definition 5.53 for discrete groups.

Definition 7.24. Let G be a locally compact group. Regard $C^*_{\mathrm{r}}(G)$ as a subalgebra of $L(L^2(G))$, as in Definition 7.22. We define the *group von Neumann algebra* $W^*_{\mathrm{r}}(G)$ to be the closure of $C^*_{\mathrm{r}}(G)$ in the weak operator topology on $L(L^2(G))$.

See Section VII.3 of [277], and Definition V.7.4 of [276] for the discrete case. The notation used in [276] and [277] ($\mathcal{R}(G)$ and $\mathcal{R}_{\mathrm{r}}(G)$) is not common. The most frequently used notation seems to be $L(G)$, $\mathcal{L}(G)$, and $W^*(G)$.

Although we will not prove this here, the unitaries u_g corresponding to the group elements $g \in G$ are in $W^*_{\mathrm{r}}(G)$. In fact, taking $v \colon G \to U(L^2(G))$ to be the left regular representation (as in Definition 7.22), one has

$$W^*_{\mathrm{r}}(G) = \big\{ v(g) \colon g \in G \big\}''.$$

8. Crossed Products

In this section, we define (full) crossed products, and prove a few results closely related to the construction. We omit some of the details, especially in the case that the group is not discrete. See Sections 7.4 and 7.6 of [198], and, for considerably more detail, Sections 2.4 and 2.5 of [292].

Definition 8.1. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. A *covariant representation* of (G, A, α) on a Hilbert space H is a pair (v, π) consisting of a unitary representation $v \colon G \to U(H)$ (the unitary group of H) and a representation $\pi \colon A \to L(H)$ (the algebra of all bounded operators on

H), satisfying the *covariance condition*

$$v(g)\pi(a)v(g)^* = \pi(\alpha_g(a))$$

for all $g \in G$ and $a \in A$. It is called *nondegenerate* if π is nondegenerate.

Recall that, by convention, unitary representations are strong operator continuous. By convention, representations of C*-algebras, and of other *-algebras (such as the algebras $L^1(G, A, \alpha)$ and $C_{\mathrm{c}}(G, A, \alpha)$ introduced below) will be *-representations (and, similarly, homomorphisms are *-homomorphisms).

The crossed product C*-algebra $C^*(G, A, \alpha)$ is the universal C*-algebra for covariant representations of (G, A, α), in essentially the same way that the (full) group C*-algebra $C^*(G)$ is the universal C*-algebra for unitary representations of G, as in Theorem 7.18 (Theorem 5.22 when G is discrete). We construct it in a similar way to the group C*-algebra. We start with the analogs of $C_{\mathrm{c}}(G)$ (Definition 7.6) and of $L^1(G)$ (Definition 7.10).

To define the crossed product by a general locally compact group, one needs an integration theory for Banach space valued functions. This theory was not needed to define the convolution multiplication in $C_{\mathrm{c}}(G)$, but it was needed for later work involving $C^*(G)$, such as the integrated form of a representation (Definition 7.16). Here, we already need it for the definition of the product in Definition 8.2. A sufficient theory for our purposes is discussed before Theorem 7.13, and the main facts we need are in Theorem 7.13, Definition 7.14, and Lemma 7.15.

As in Section 7, we let μ be a fixed left Haar measure on G.

Definition 8.2. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We let $C_{\mathrm{c}}(G, A, \alpha)$ be the *-algebra of compactly supported continuous functions $a\colon G \to A$, with pointwise addition and scalar multiplication. Using Haar measure in the integral, we define multiplication by the following "twisted convolution":

$$(ab)(g) = \int_G a(h)\alpha_h(b(h^{-1}g))\, d\mu(h).$$

Let Δ be the modular function of G. We define the adjoint by

$$a^*(g) = \Delta(g)^{-1}\alpha_g(a(g^{-1})^*).$$

This does in fact make $C_{\mathrm{c}}(G, A, \alpha)$ a *-algebra; see Exercise 8.3 below. We define a norm $\|\cdot\|_1$ on $C_{\mathrm{c}}(G, A, \alpha)$ by $\|a\|_1 = \int_G \|a(g)\|\, d\mu(g)$. One checks (Exercise 8.3) that $\|ab\|_1 \leq \|a\|_1\|b\|_1$ and $\|a^*\|_1 = \|a\|_1$. Then $L^1(G, A, \alpha)$ is the Banach *-algebra obtained by completing $C_{\mathrm{c}}(G, A, \alpha)$ in $\|\cdot\|_1$.

The next exercise is the analog of Proposition 7.8. It needs Fubini's Theorem for Banach space valued integrals of continuous functions with compact support. See Proposition 1.105 of [292]. Since such functions are automatically integrable, the required result can be gotten from the usual scalar valued Fubini's Theorem by applying continuous linear functionals and using the Hahn-Banach Theorem.

Exercise 8.3. In the situation of Definition 8.2, and assuming a suitable version of Fubini's Theorem for Banach space valued integrals, prove that that multiplication in $C_{\mathrm{c}}(G, A, \alpha)$ is associative. Further prove for $a, b \in C_{\mathrm{c}}(G, A, \alpha)$ that $\|ab\|_1 \leq \|a\|_1\|b\|_1$, that $(ab)^* = b^*a^*$, and that $\|a^*\|_1 = \|a\|_1$. Finally, prove that $L^1(G, A, \alpha)$ is a Banach *-algebra.

Remark 8.4. Suppose $A = C_0(X)$, and α comes from an action of G on X. Since we complete in a suitable norm later on, it suffices to use only the dense subalgebra $C_{\mathrm{c}}(X)$ in place of $C_0(X)$. There is an obvious identification of $C_{\mathrm{c}}(G, C_{\mathrm{c}}(X))$ with $C_{\mathrm{c}}(G \times X)$. On $C_{\mathrm{c}}(G \times X)$, the formulas for multiplication and adjoint become

$$(f_1 f_2)(g, x) = \int_G f_1(h, x) f_2(h^{-1}g, \, h^{-1}x) \, d\mu(h)$$

and

$$f^*(g, x) = \Delta(g)^{-1} \overline{f(g^{-1}, \, g^{-1}x)}.$$

Exercise 8.5. Prove the formulas in Remark 8.4.

Remark 8.6. If G is discrete, we choose Haar measure to be counting measure. In this case, $C_{\mathrm{c}}(G, A, \alpha)$ is, as a vector space, the group ring $A[G]$, consisting of all finite formal linear combinations of elements in G with coefficients in A. The multiplication and adjoint are given by

$$(a \cdot g)(b \cdot h) = (a[gbg^{-1}]) \cdot (gh) = (a \alpha_g(b)) \cdot (gh) \qquad \text{and} \qquad (a \cdot g)^* = \alpha_g^{-1}(a^*) \cdot g^{-1}$$

for $a, b \in A$ and $g, h \in G$, extended linearly. This definition makes sense in the purely algebraic situation, where it is called the *skew group ring*.

When G is discrete, we also often write $l^1(G, A, \alpha)$ instead of $L^1(G, A, \alpha)$.

Notation 8.7. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. In these notes, we will adopt the following fairly commonly used notation. First, suppose A is unital. For $g \in G$, we let u_g be the element of $C_{\mathrm{c}}(G, A, \alpha)$ which takes the value 1_A at g and 0 at the other elements of G. We use the same notation for its image in $l^1(G, A, \alpha)$ (Definition 8.2 above) and in $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$ (Definitions 8.15 and 9.4 below). It is unitary, and we call it the canonical unitary associated with g.

If A is not unital, extend the action to an action $\alpha^+ \colon G \to \mathrm{Aut}(A^+)$ on the unitization A^+ of A by $\alpha_g^+(a + \lambda \cdot 1) = \alpha_g(a) + \lambda \cdot 1$. Then write u_g as above. Products $a u_g$, with $a \in A$, are still in $C_{\mathrm{c}}(G, A, \alpha)$, $l^1(G, A, \alpha)$, $C^*(G, A, \alpha)$, or $C^*_{\mathrm{r}}(G, A, \alpha)$, as appropriate.

Remark 8.8. In particular, $l^1(G, A, \alpha)$ is the set of all sums $\sum_{g \in G} a_g u_g$ with $a_g \in A$ and $\sum_{g \in G} \|a_g\| < \infty$. These sums converge in $l^1(G, A, \alpha)$, and hence also in $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$. A general element of $C^*_{\mathrm{r}}(G, A, \alpha)$ has such an expansion, but unfortunately the series one writes down generally does not converge. See Remark 9.19; as in Remark 5.60, there is usually no convergence even when $A = \mathbb{C}$ and G is amenable.

Definition 8.9. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A, and let (v, π) be a covariant representation of (G, A, α) on a Hilbert space H. (We do not assume that π is nondegenerate.) Then the *integrated form* of (v, π) is the representation $\sigma \colon C_{\mathrm{c}}(G, A, \alpha) \to L(H)$ given by

$$\sigma(a)\xi = \int_G \pi(a(g)) v(g) \xi \, d\mu(g).$$

(This representation is sometimes called $v \times \pi$ or $\pi \times v$. We will sometimes use the notation $v \ltimes \pi$.)

One needs to be more careful with the integral here, just as in Definition 7.16 and the remark afterwards, because v is generally only strong operator continuous, not norm continuous. Nevertheless, one gets $\|\sigma(a)\| \leq \|a\|_1$, so σ extends to a representation of $L^1(G, A, \alpha)$. We use the same notation σ for this extension.

Of course, one also needs to check that σ is a representation. When G is discrete, and using Notation 8.7, the formula for σ comes down to $\sigma(au_g) = \pi(a)v(g)$ for $a \in A$ and $g \in G$. Then

$$\begin{aligned}\sigma(au_g)\sigma(bu_h) &= \pi(a)v(g)\pi(b)v(g)^*v(g)v(h) = \pi(a)\pi(\alpha_g(b))v(g)v(h) \\ &= \pi(a\alpha_g(b))v(gh) = \sigma\big([a\alpha_g(b)]u_{gh}\big) = \sigma\big((au_g)(bu_h)\big).\end{aligned}$$

Exercise 8.10. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A, and let (v, π) be a nondegenerate covariant representation of (G, A, α) on a Hilbert space H. Starting from the computation above, fill in the details of the proof that the integrated form representation σ of Definition 8.9 really is a nondegenerate representation of $C_{\mathrm{c}}(G, A, \alpha)$.

Theorem 8.11 (Proposition 7.6.4 of [198])**.** Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Then the integrated form construction defines a bijection from the set of nondegenerate covariant representations of (G, A, α) on a Hilbert space H to the set of nondegenerate continuous representations of $L^1(G, A, \alpha)$ on the same Hilbert space.

(There is a misprint in the statement of Proposition 7.6.4 of [198]: it omits the nondegeneracy condition on the covariant representation, but includes nondegeneracy for the integrated form.)

Also see Propositions 2.39 and 2.40 of [292]. These are stated in terms of $C^*(G, A, \alpha)$, but, by Definition 8.15 below, that is the same thing. (The C*-algebra result is stated as Theorem 8.17 below.)

Remark 8.12. Since integrated form representations of $L^1(G, A, \alpha)$ are necessarily contractive, *all* continuous representations of $L^1(G, A, \alpha)$ are necessarily contractive.

If G is discrete and A is unital, then there are homomorphic images of both G and A inside $C_{\mathrm{c}}(G, A, \alpha)$, given (following Notation 8.7) by $g \mapsto u_g$ and $a \mapsto au_1$, so it is clear how to get a covariant representation of (G, A, α) from a nondegenerate representation of $C_{\mathrm{c}}(G, A, \alpha)$. In general, one must use the multiplier algebra of $L^1(G, A, \alpha)$, which contains copies of $M(A)$ and $M(L^1(G))$. The point is that $M(L^1(G))$ is the measure algebra of G, and therefore contains the group elements as point masses.

Exercise 8.13. Prove Theorem 8.11 when G is discrete and A is unital.

For a small taste of the general case, use approximate identities in A to do the following exercise.

Exercise 8.14. Prove Theorem 8.11 when G is discrete but A is not necessarily unital.

In the following definition, we ignore the set theoretic problem, that the collection of all nondegenerate representations of $L^1(G, A, \alpha)$ is not a set. Exercise 8.16 afterwards asks for a set theoretically correct definition, and a proof from this definition that one still has the correct universal property. The case of $C^*(G)$ for a discrete group G was done carefully in Definition 5.18 and the first part of the proof of Theorem 5.22. For locally compact G, see Definition 7.20 and Theorem 7.21 (for which we did not give a proof). It suffices to use a fixed Hilbert space whose dimension is at least $\operatorname{card}(G)\operatorname{card}(A)$.

Definition 8.15. Let $\alpha\colon G \to \operatorname{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We define the *universal representation* σ of $L^1(G, A, \alpha)$ to be the direct sum of all nondegenerate representations of $L^1(G, A, \alpha)$ on Hilbert spaces. Then we define the *crossed product* $C^*(G, A, \alpha)$ to be the norm closure of $\sigma(L^1(G, A, \alpha))$.

One could of course equally well use the norm closure of $\sigma(C_{\mathrm{c}}(G, A, \alpha))$.

Exercise 8.16. Give a set theoretically correct definition of the crossed product. The important point is to preserve the universal property in Theorem 8.17; prove that your definition does this.

It follows that every nondegenerate covariant representation of (G, A, α) gives a representation of $C^*(G, A, \alpha)$. (Take the integrated form, and restrict elements of $C^*(G, A, \alpha)$ to the appropriate summand in the direct sum in Definition 8.15.) The crossed product is, essentially by construction, the universal C*-algebra for covariant representations of (G, A, α), in the same sense that if G is a locally compact group, then $C^*(G)$ is the universal C*-algebra for unitary representations of G. Theorem 8.11 then becomes the following result, which is the analog for crossed products of Theorem 7.21 (for group C*-algebras).

Theorem 8.17 (Propositions 2.39 and 2.40 of [292]; Theorem 7.6.6 of [198])**.** Let $\alpha\colon G \to \operatorname{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A, and let H be a Hilbert space. Then the integrated form construction defines a bijection from the set of nondegenerate covariant representations of (G, A, α) on H to the set of nondegenerate representations of $C^*(G, A, \alpha)$ on H.

Exercise 8.18. Prove Theorem 8.17 when G is discrete and A is unital.

Remark 8.19. There are many notations in use for crossed products, and for related objects called reduced crossed products (to be constructed in Section 9 below). Here are most of the most common ones, listed in pairs (notation for the full crossed product first):

- $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$.
- $C^*(A, G, \alpha)$ and $C^*_{\mathrm{r}}(A, G, \alpha)$.
- $A \rtimes_\alpha G$ and $A \rtimes_{\alpha,\mathrm{r}} G$ (used in the book [292]).
- $A \times_\alpha G$ and $A \times_{\alpha,\mathrm{r}} G$ (used in the book [52]).
- $G \times_\alpha A$ and $G \times_{\alpha,\mathrm{r}} A$ (used in the book [198]).

In all of them, we may omit α if it is understood. In the notation for the reduced crossed products (especially the first two versions), the letter "r" ("reduced") is sometimes replaced by "λ" (the conventional name for the left regular representation of a group). The symbol in the third comes from the relation $C^*(N \rtimes H) \cong C^*(H, C^*(N))$, and is meant to suggest a generalized semidirect product. The first two make it easy to distinguish C*-algebra crossed products from other sorts, such as von Neumann algebra crossed products, smooth crossed products, L^1 crossed products, L^p operator crossed products, and purely algebraic crossed products (all of which will receive short shrift in these notes, but are important in their own right, sometimes in the same paper). I use the order $C^*(G, A, \alpha)$ because it matches the natural order in $C_{\mathrm{c}}(G, A, \alpha)$ and $L^1(G, A, \alpha)$.

Definition 8.20. Let G be a locally compact group, let X be a locally compact Hausdorff space, and let $(g, x) \mapsto gx$ be an action of G on X. The *transformation group C*-algebra* of (G, X), written $C^*(G, X)$, is the crossed product C*-algebra $C^*(G, C_0(X))$.

Theorem 8.21. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a unital C*-algebra A. Then $C^*(G, A, \alpha)$ is the universal C*-algebra generated by a unital copy of A (that is, the identity of A is supposed to be the identity of the generated C*-algebra) and unitaries u_g, for $g \in G$, subject to the relations $u_g u_h = u_{gh}$ for $g, h \in G$ and $u_g a u_g^* = \alpha_g(a)$ for $a \in A$ and $g \in G$.

Exercise 8.22. Based on the discussion above, write down a careful proof of Theorem 8.21.

Corollary 8.23. Let A be a unital C*-algebra, and let $\alpha \in \mathrm{Aut}(A)$. Then the crossed product $C^*(\mathbb{Z}, A, \alpha)$ is the universal C*-algebra generated by a copy of A and a unitary u, subject to the relations $uau^* = \alpha(a)$ for $a \in A$.

We now discuss functoriality of crossed products. The locally compact group G will be treated as fixed. Since we have not included full proofs earlier in this section when G is not discrete, we are not giving self contained proofs of the functoriality results. However, given the results stated earlier, the functoriality proofs are the same even when G is not discrete.

Definition 8.24. Let G be a locally compact group. A C*-algebra A equipped with an action $G \to \mathrm{Aut}(A)$ will be called a *G-algebra*, or a *G-C*-algebra*. We sometimes refer to (G, A, α) as a G-algebra or G-C*-algebra.

Recall from Definition 1.3 that if (G, A, α) and (G, B, β) are G-algebras, then a homomorphism $\varphi \colon A \to B$ is said to be equivariant if for every $g \in G$, we have $\varphi \circ \alpha_g = \beta_g \circ \varphi$. We say that φ is *G-equivariant* if the group must be specified.

Proposition 8.25. For a fixed locally compact group G, the G-algebras and equivariant homomorphisms form a category.

Proof. This is obvious. □

We will need to use degenerate covariant representations when considering functoriality for homomorphisms whose ranges are "too small" (such as being contained in proper ideals). We recall the following standard lemma on degenerate representations of C*-algebras. We omit the easy proof.

Lemma 8.26. Let A be a C*-algebra, let H_0 be a Hilbert space, and let $\pi_0 \colon A \to L(H_0)$ be a representation. Let H be the closed linear span of $\pi_0(A)H_0$. Then:

(1) The subspace H is invariant for π.
(2) The representation $\pi = \pi_0(-)|_H$ is nondegenerate.
(3) We have
$$H^\perp = \big\{ \xi \in H_0 \colon \pi(a)\xi = 0 \text{ for all } a \in A \big\}.$$
(4) The representation π_0 is the direct sum of π and the zero representation on $H^\perp$.

(5) For all $a \in A$, we have $\|\pi(a)\| = \|\pi_0(a)\|$.
(6) We have $\mathrm{Ker}(\pi) = \mathrm{Ker}(\pi_0)$.
(7) If π_0 is given as the direct sum of a nondegenerate representation on a Hilbert space H_1 and the zero representation on a Hilbert space H_2, then $H = H_1$ and $H^\perp = H_2$.

Lemma 8.27. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let I be a set, and for $i \in I$ let (v_i, π_i) be a covariant representation of (G, A, α) on a Hilbert space H_i, with integrated form σ_i. Then $\left(\bigoplus_{i \in I} v_i, \bigoplus_{i \in I} \pi_i\right)$ is a covariant representation of (G, A, α) on $\bigoplus_{i \in I} H_i$, and its integrated form is $\bigoplus_{i \in I} \sigma_i$.

Proof. The proof is routine. □

Lemma 8.28. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let (v, π) be a covariant representation of (G, A, α) on a Hilbert space H. If π is the zero representation, then the integrated form σ of (v, π) is the zero representation.

Proof. It is immediate that $\sigma(a)\xi = 0$ for all $a \in C_{\mathrm{c}}(G, A, \alpha)$ and $\xi \in H$. □

Lemma 8.29. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let (v_0, π_0) be a covariant representation of (G, A, α) on a Hilbert space H_0. (We do not assume that π_0 is nondegenerate.) Let $\sigma_0\colon C_{\mathrm{c}}(G, A, \alpha) \to L(H_0)$ be the integrated form of (v_0, π_0), as in Definition 8.9. Then $\pi_0(A)H_0$ and $\sigma_0(C^*(G, A, \alpha))H_0$ have the same closed linear spans.

Proof. Let H be the closed linear span of $\pi_0(A)H_0$. Then H is invariant under π_0 by Lemma 8.26(1).

We claim that H is invariant under v_0. It is enough to prove invariance of $\pi_0(A)H_0$. Let $g \in G$, let $a \in A$, and let $\xi \in H_0$. Then

$$v_0(g)\pi_0(a)\xi = \pi_0(\alpha_g(a))v_0(g)\xi \in \pi_0(A)H_0.$$

The claim is proved.

Set $\pi = \pi_0(-)|_H$. Then $\pi_0 = \pi \oplus 0$ by Lemma 8.26(4), and the claim implies the existence of a representation w of G on $H^\perp$ such that $v_0 = v \oplus w$. Let σ be the integrated form of (v, π). Use Lemma 8.27 and then Lemma 8.28 to get $\sigma_0 = \sigma \oplus 0$, the zero representation being on $H^\perp$. It follows from Theorem 8.11 and Definition 8.15 that σ is nondegenerate. Therefore the conclusion follows from Lemma 8.26(7). □

Corollary 8.30. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $a \in C^*(G, A, \alpha)$. Then

$$\|a\| = \sup\big(\big\{\|\sigma(a)\|\colon \sigma \text{ is the integrated form of a possibly degenerate covariant representation of } (G, A, \alpha)\big\}\big).$$

Proof. It follows from Lemma 8.29, Lemma 8.26(2), and Lemma 8.26(5) that the supremum on the right is unchanged if we restrict to nondegenerate covariant representations of (G, A, α). □

Theorem 8.31. Let G be a locally compact group. If (G, A, α) and (G, B, β) are G-algebras and $\varphi\colon A \to B$ is an equivariant homomorphism, then there is a homomorphism $\psi\colon C_{\mathrm{c}}(G, A, \alpha) \to C_{\mathrm{c}}(G, B, \beta)$ given by the formula $\psi(b)(g) = \varphi(b(g))$

for $b \in C_{\mathrm{c}}(G, A, \alpha)$ and $g \in G$, and this homomorphism extends by continuity to a homomorphism $L^1(G, A, \alpha) \to L^1(G, B, \beta)$, and then to a homomorphism $C^*(G, A, \alpha) \to C^*(G, B, \beta)$. This construction makes the crossed product construction a functor from the category of G-algebras to the category of C*-algebras.

Proof. One checks directly that ψ preserves multiplication and adjoint, and that $\|\psi(a)\|_1 \leq \|a\|_1$ for all $a \in C_{\mathrm{c}}(G, A, \alpha)$. The extension to the L^1-algebras is now immediate.

To prove that ψ extends by continuity to a homomorphism $C^*(G, A, \alpha) \to C^*(G, B, \beta)$, we let $\|\cdot\|$ denote restrictions to $C_{\mathrm{c}}(G, A, \alpha)$ and $C_{\mathrm{c}}(G, B, \beta)$ of the norms on $C^*(G, A, \alpha)$ and $C^*(G, B, \beta)$. We have to prove that $\|\psi(b)\| \leq \|b\|$ for all $b \in C_{\mathrm{c}}(G, A, \alpha)$. So let (w, ρ) be a nondegenerate covariant representation of (G, B, β) on a Hilbert space H, and let $\nu \colon C_{\mathrm{c}}(G, B, \beta) \to L(H)$ be the integrated form of (w, ρ), as in Definition 8.9. We have to prove that $\|\nu(\psi(b))\| \leq \|b\|$. Clearly $(w, \rho \circ \varphi)$ is a covariant representation of (G, A, α) on H, with integrated form $\sigma = \nu \circ \psi$. There is no reason to suppose that $(w, \rho \circ \varphi)$ is nondegenerate, but, even without nondegeneracy, Corollary 8.30 gives $\|\sigma(b)\| \leq \|b\|$. Thus $\|\nu(\psi(b))\| = \|\sigma(b)\| \leq \|b\|$, as desired. ☐

Theorem 8.32 (Lemma 2.8.2 of [200]; Theorem 2.6 of [255]; Proposition 3.9 of [292])**.** Let G be a locally compact group. Let

$$0 \longrightarrow J \xrightarrow{\iota_0} A \xrightarrow{\kappa_0} B \longrightarrow 0$$

be an exact sequence of G-algebras, with actions γ on J, α on A, and β on B. Then the sequence

$$0 \longrightarrow C^*(G, J, \gamma) \xrightarrow{\iota} C^*(G, A, \alpha) \xrightarrow{\kappa} C^*(G, B, \beta) \longrightarrow 0$$

of crossed products and induced maps is exact.

Theorem 8.32 implies in particular that if (G, J, γ) and (G, A, α) are G-algebras, and $\varphi \colon J \to A$ is an injective equivariant homomorphism whose image is an ideal, then the corresponding homomorphism $C^*(G, J, \gamma) \to C^*(G, A, \alpha)$ is injective. If the image is merely a subalgebra, the proof fails. The difficulty occurs when we extend a covariant representation of (G, J, γ) to a covariant representation of (G, A, α). If J is not an ideal, to extend a representation of J to one of A one usually needs a bigger Hilbert space, and one has trouble with how to extend the representation of G to a representation on the larger space.

The (full) crossed product should be thought of as somehow analogous to the maximal tensor product of C*-algebras. Similarly, the reduced crossed product (discussed in Section 9 below) should be thought of as somehow analogous to the minimal tensor product of C*-algebras. Compare with Example 10.1, where it is observed that if the action of G on A is trivial, then

$$C^*(G, A) \cong C^*(G) \otimes_{\max} A \qquad \text{and} \qquad C^*_{\mathrm{r}}(G, A) \cong C^*_{\mathrm{r}}(G) \otimes_{\min} A.$$

Theorem 8.32 should then be compared with Proposition 3.7.1 of [37], according to which $A \otimes_{\max} -$ is an exact functor.

The proof of Theorem 8.32 requires at least the first part of the following exercise. For this part, one can use a partition of unity argument similar to that in the proof of Lemma 7.15. For part (2), one can then apply part (1) to the error $\overline{\kappa}(a) - b$, with a smaller error, repeat, and sum the results. One can also reduce to the case in which X is compact, where one can apply C*-algebraic tensor products and the isomorphism $C(X, A) \cong C(X) \otimes A$.

Exercise 8.33. Let X be a locally compact Hausdorff space, let A and B be C*-algebras, and let $\kappa \colon A \to B$ be a surjective homomorphism. Let $\overline{\kappa} \colon C_{\mathrm{c}}(X, A) \to C_{\mathrm{c}}(X, B)$ be the linear map given by $\overline{\kappa}(a)(x) = \kappa(a(x))$ for $a \in C_{\mathrm{c}}(X, A)$ and $x \in X$. Let $b \in C_{\mathrm{c}}(X, B)$.

(1) Prove that there is a compact set $K \subset X$ such that for every $\varepsilon > 0$ there is $a \in C_{\mathrm{c}}(X, A)$ satisfying
$$\|a\|_{\infty} \leq \|b\|_{\infty} + \varepsilon, \qquad \operatorname{supp}(a) \subset K, \qquad \text{and} \qquad \|\overline{\kappa}(a) - b\|_{\infty} < \varepsilon.$$
(2) Prove that there is $a \in C_{\mathrm{c}}(X, A)$ such that $\overline{\kappa}(a) = b$ and $\operatorname{supp}(a) = \operatorname{supp}(b)$.

Proof of Theorem 8.32. We prove that κ is surjective. Since κ is a homomorphism, it suffices to prove that κ has dense range. It follows from Exercise 8.33(2) that the range of κ contains the image of $C_{\mathrm{c}}(G, B, \beta)$, and we know that the image of $C_{\mathrm{c}}(G, B, \beta)$ is dense. (Actually, Exercise 8.33(1) is good enough here, since it implies that the closure of the range of κ contains the image of $C_{\mathrm{c}}(G, B, \beta)$.) This proves surjectivity of κ.

It is immediate that $\kappa \circ \iota = 0$.

We prove that ι is injective. For this, it is convenient to identify J with the ideal $\iota_0(J) \subset A$. Let $y \in C^*(G, J, \gamma)$ be nonzero. Choose a nondegenerate covariant representation (v, π_0) of (G, J, γ) on a Hilbert space H such that the integrated form $\pi \colon C^*(G, J, \gamma) \to L(H)$ satisfies $\pi(y) \neq 0$. Since π_0 is nondegenerate and $J \subset A$ is an ideal, a standard result in the representation theory of C*-algebras shows that there is a unique representation $\rho_0 \colon A \to L(H)$ such that $\rho_0|_J = \pi_0$.

We claim that (v, ρ_0) is covariant. Let $g \in G$. Since (v, π_0) is covariant, $a \mapsto v(g)\rho(\alpha_g^{-1}(a))v(g)^*$ is a representation whose restriction to J is π_0. By uniqueness of ρ_0, we have $v(g)\rho_0(\alpha_g^{-1}(a))v(g)^* = \rho_0(a)$ for all $a \in A$, which is covariance.

Let $\rho \colon C^*(G, A, \alpha) \to L(H)$ be the integrated form of (v, ρ_0). Then $\rho \circ \iota = \pi$, so $\rho(\iota(y)) = \pi(y) \neq 0$. Therefore $\iota(y) \neq 0$.

It remains to prove that if $y \in C^*(G, A, \alpha)$ and $\kappa(y) = 0$, then y is in the range of ι. We again identify J with the ideal $\iota_0(J) \subset A$. Since ι is injective, we may use ι to identify $C^*(G, J, \gamma)$ with a subalgebra of $C^*(G, A, \alpha)$. Since $C_{\mathrm{c}}(G, J, \gamma)$ is an ideal in $C_{\mathrm{c}}(G, A, \alpha)$ and since $C_{\mathrm{c}}(G, J, \gamma)$ and $C_{\mathrm{c}}(G, A, \alpha)$ are dense in $C^*(G, J, \gamma)$ and $C^*(G, A, \alpha)$, it follows that $C^*(G, J, \gamma)$ is an ideal in $C^*(G, A, \alpha)$.

Let $y \in C^*(G, A, \alpha)$ and suppose that $y \notin C^*(G, J, \gamma)$. We show that $\kappa(y) \neq 0$. Use a nondegenerate representation of $C^*(G, A, \alpha)/C^*(G, J, \gamma)$ which does not vanish on y to find a Hilbert space H and a nondegenerate representation $\sigma \colon C^*(G, A, \alpha) \to L(H)$ such that $\sigma(y) \neq 0$ but $\sigma|_{C^*(G,J,\gamma)} = 0$. Then σ is the integrated form of a nondegenerate covariant representation (w, σ_0) of (G, A, α). Since $\sigma|_{C^*(G,J,\gamma)} = 0$, Lemma 8.29 implies that $\sigma_0|_J = 0$. So σ_0 induces a representation $\pi_0 \colon B \to L(H)$. Clearly (w, π_0) is a nondegenerate covariant representation of (G, B, β) whose integrated form π satisfies $\pi \circ \kappa = \sigma$. So $\pi(\kappa(y)) \neq 0$. Thus $\kappa(y) \neq 0$. $\square$

Theorem 8.34. Let G be a locally compact group. Let $\big((G, A_i, \alpha^{(i)})_{i \in I}, \, (\varphi_{j,i})_{i \leq j} \big)$ be a direct system of G-algebras. Let $A = \varinjlim A_i$, with action $\alpha \colon G \to \operatorname{Aut}(A)$ given by $\alpha_g = \varinjlim \alpha_g^{(i)}$ for all $g \in G$. (See Proposition 3.24.) Let
$$\psi_{j,i} \colon C^*\big(G, A_i, \alpha^{(i)}\big) \to C^*\big(G, A_j, \alpha^{(j)}\big)$$

be the map obtained from $\varphi_{j,i}$. Using these maps in the direct system of crossed products, there is a natural isomorphism $C^*(G, A, \alpha) \cong \varinjlim C^*\big(G, A_i, \alpha^{(i)}\big)$.

Proof. We show that $C^*(G, A, \alpha)$ satisfies the universal property which defines $\varinjlim C^*\big(G, A_i, \alpha^{(i)}\big)$. First, for $i \in I$ let $\varphi_i \colon A_i \to A$ be the canonical map for the direct limit of the system $\big((A_i)_{i\in I}, (\varphi_{j,i})_{i\leq j}\big)$. We have maps

$$\psi_i \colon C^*\big(G, A_i, \alpha^{(i)}\big) \to C^*(G, A, \alpha)$$

obtained from the maps φ_i by forming crossed products. Clearly $\psi_j \circ \psi_{j,i} = \psi_i$ whenever $i, j \in I$ satisfy $i \leq j$.

Now suppose we have a C*-algebra B and homomorphisms $\nu_i \colon C^*\big(G, A_i, \alpha^{(i)}\big) \to B$ such that $\nu_j \circ \psi_{j,i} = \nu_i$ whenever $i, j \in I$ satisfy $i \leq j$. We need to prove that there is a unique homomorphism $\nu \colon C^*(G, A, \alpha) \to B$ such that $\nu \circ \psi_i = \nu_i$ for all $i \in I$. Without loss of generality, B is a nondegenerate subalgebra of $L(H)$ for some Hilbert space H.

For each $i \in I$, set

$$H_i = \overline{\nu_i\big(C^*\big(G, A_i, \alpha^{(i)}\big)\big)H}.$$

Keeping Lemma 8.26 in mind for the next several paragraphs, observe that there is a nondegenerate covariant representation (v_i, π_i) of $\big(G, A_i, \alpha^{(i)}\big)$ on H_i whose integrated form is $\nu_i(-)|_{H_i}$. Extend π_i to a representation on H by forming the direct sum with the zero representation on $H_i^{\perp}$. Let $i, j \in I$ satisfy $i \leq j$. Then $H_i \subset H_j$. Moreover, H_i is an invariant subspace for v_j and, by uniqueness of the nondegenerate covariant representation determined by a nondegenerate representation of the crossed product (Theorem 8.17), we have

$$v_j(-)|_{H_i} = v_i(-) \qquad \text{and} \qquad (\pi_j \circ \varphi_{j,i})(-)|_{H_i} = \pi_i(-).$$

Moreover, both $(\pi_j \circ \varphi_{j,i})(-)$ and $\pi_i(-)$ are zero on $H_j \cap H_i^{\perp}$ and on $H_j^{\perp}$, so $\pi_j \circ \varphi_{j,i} = \pi_i$.

Since B is nondegenerate, we have $\overline{\bigcup_{i\in I} H_i} = H$. It is then easy to see that there is a unique unitary representation v of G on H such that $v(-)|_{H_i} = v_i$ for all $i \in I$.

By the universal property of $\varinjlim A_i$, there is a unique representation $\pi \colon A \to L(H)$ such that $\pi \circ \varphi_i = \pi_i$ for all $i \in I$, and moreover (using uniqueness) (v, π) is a covariant representation. Let $\nu \colon C^*(G, A, \alpha) \to L(H)$ be the integrated form of (v, π). Then one gets $\nu \circ \psi_i = \nu_i$ for all $i \in I$. Since A is generated by the images of the algebras A_i, it follows that $\nu(C^*(G, A, \alpha)) \subset B$. Uniqueness of ν follows from uniqueness of the integrated form of a covariant representation. $\square$

9. Reduced Crossed Products

So far, it is not clear that a G-algebra (G, A, α) has any covariant representations at all. In this section, we exhibit a large easily constructed class of them, called regular covariant representations. We then study the reduced crossed product, which is defined by using the universal regular representation in place of the universal representation. We will concentrate on the case of discrete groups.

As in Sections 7 and 8, we let μ be a fixed left Haar measure on G.

We will need Hilbert spaces of the form $L^2(G, H_0)$. The easy way to construct $L^2(G, H_0)$ is to take it to be the completion of $C_{\mathrm{c}}(G, H_0)$ in the norm coming from the scalar product

$$\langle \xi, \eta \rangle = \int_G \langle \xi(g), \eta(g) \rangle \, d\mu(g).$$

Definition 9.1 (7.7.1 of [198]). Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\pi_0 \colon A \to L(H_0)$ be a representation. We define the *regular covariant representation* (v, π) of (G, A, α) on the Hilbert space $H = L^2(G, H_0)$ of L^2 functions from G to H_0 as follows. For $g, h \in G$, set

$$(v(g)\xi)(h) = \xi(g^{-1}h).$$

For $a \in A$ and $g \in G$, set

$$(\pi(a)\xi)(h) = \pi_0(\alpha_{h^{-1}}(a))(\xi(h)).$$

(Exercise 9.2 asks you to prove that (v, π) really is covariant.) The integrated form of σ, as in Definition 8.9, will be called a regular representation of any of $C_{\mathrm{c}}(G, A, \alpha)$, $L^1(G, A, \alpha)$, $C^*(G, A, \alpha)$, and (when we have defined it; see Definition 9.4) $C^*_{\mathrm{r}}(G, A, \alpha)$. Justified by Lemma 9.3 below, we will refer to (v, π) as a nondegenerate covariant representation when π_0 is nondegenerate.

Exercise 9.2. In Definition 9.1, prove that (v, π) really is a covariant representation.

If $A = \mathbb{C}$, $H_0 = \mathbb{C}$, and π_0 is the obvious representation of A on H_0, then the representation of Definition 9.1 is the usual left regular representation of G (Definition 7.1; Definition 5.3 in the discrete case).

Lemma 9.3. In Definition 9.1, the representation π is nondegenerate if and only if π_0 is nondegenerate.

Proof. Suppose π_0 is degenerate. Choose a nonzero element $\xi_0 \in (\pi_0(A)H_0)^{\perp}$. Lemma 8.26(3) implies that $\pi_0(a)\xi = 0$ for all $a \in A$. Choose a nonzero function $f \in C_c(G)$. Define $\xi(g) = f(g)\xi_0$ for $g \in G$. Then ξ is a nonzero element of $L^2(G, H_0)$, and $\pi(a)\xi = 0$ for all $a \in A$. So π is degenerate.

Now assume that π_0 is nondegenerate. It suffices to show that $\overline{\pi(A)L^2(G, H_0)}$ contains all elements $\xi \in C_{\mathrm{c}}(G, H_0)$ which are elementary tensors, that is, for which there exist $f \in C_{\mathrm{c}}(G)$ and $\xi_0 \in H_0$ such that $\xi(h) = f(h)\xi_0$ for all $h \in G$.

Let ξ, f, and ξ_0 be as above, and let $\varepsilon > 0$. Recall that μ is a left Haar measure on G. Set

$$M = \big(\mu(\mathrm{supp}(f)) + 1\big)^{1/2}(\|f\| + 1).$$

Since π_0 is nondegenerate, there are $a \in A$ and $\eta_0 \in H_0$ such that

$$\|\pi_0(a)\eta_0 - \xi_0\| < \frac{\varepsilon}{2M}.$$

Since $\big\{\alpha_h(a) \colon h \in \mathrm{supp}(f)\big\}$ is compact, there is $b \in A$ such that

$$\|b\alpha_h(a) - \alpha_h(a)\| < \frac{\varepsilon}{2M(\|\eta_0\| + 1)}$$

for all $h \in \mathrm{supp}(f)$. Then

$$\|\alpha_h^{-1}(b)a - a\| < \frac{\varepsilon}{2M(\|\eta_0\| + 1)}$$

for all $h \in \operatorname{supp}(f)$. Define $\eta \in C_{\mathrm{c}}(G)$ by $\eta(h) = f(h)\pi_0(a)\eta_0$ for $h \in G$. For all $h \in \operatorname{supp}(f)$, we then have

$$\begin{aligned} \|(\pi(b)\eta)(h) - \xi(h)\| &= |f(h)| \cdot \left\|\pi_0(\alpha_h^{-1}(b))(\pi_0(a)\eta_0) - f(h)\xi_0\right\| \\ &\leq \|f\| \cdot \|\alpha_h^{-1}(b)a - a\| \cdot \|\eta_0\| + \|f\| \cdot \|\pi_0(a)\eta_0 - \xi_0\| \\ &< \frac{\varepsilon\|f\| \cdot \|\eta_0\|}{2M(\|\eta_0\| + 1)} + \frac{\varepsilon\|f\|}{2M} \\ &< \frac{\varepsilon}{2\mu(\operatorname{supp}(f)) + 1)^{1/2}} + \frac{\varepsilon}{2\mu(\operatorname{supp}(f)) + 1)^{1/2}} \\ &= \frac{\varepsilon}{\mu(\operatorname{supp}(f)) + 1)^{1/2}}. \end{aligned}$$

Therefore

$$\|\pi(b)\eta - \xi\|^2 \leq \mu(\operatorname{supp}(f)) \left(\frac{\varepsilon}{\mu(\operatorname{supp}(f)) + 1)^{1/2}}\right)^2 < \varepsilon^2,$$

so $\|\pi(b)\eta - \xi\| < \varepsilon$. □

In the following definition, we ignore a set theoretic problem analogous to those encountered previously, for example in Definition 8.15.

Definition 9.4. Let $\alpha\colon G \to \operatorname{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\lambda\colon L^1(G, A, \alpha) \to L(H)$ be the direct sum of all regular representations of $L^1(G, A, \alpha)$ coming from nondegenerate representations of A. We define the *reduced crossed product* $C^*_{\mathrm{r}}(G, A, \alpha)$ to be the norm closure of $\lambda(L^1(G, A, \alpha))$.

Exercise 9.5. Give a set theoretically correct definition of the reduced crossed product.

We use notation analogous to that of Definition 8.20 in the case of an action on a locally compact space.

Definition 9.6. Let G be a locally compact group, let X be a locally compact Hausdorff space, and let $(g, x) \mapsto gx$ be an action of G on X. The *reduced transformation group C*-algebra* of (G, X), written $C^*_{\mathrm{r}}(G, X)$, is the reduced crossed product C*-algebra $C^*_{\mathrm{r}}(G,\, C_0(X))$.

Implicit in the definition of $C^*_{\mathrm{r}}(G, A, \alpha)$ is a representation of $L^1(G, A, \alpha)$, hence of $C^*(G, A, \alpha)$. Thus, there is a homomorphism $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$. By construction, it has dense range, and is therefore surjective. Moreover, by construction, any regular representation of $L^1(G, A, \alpha)$ extends to a representation of $C^*_{\mathrm{r}}(G, A, \alpha)$.

In the context of the next theorem, see the comments before Theorem 5.50 for a discussion of amenability.

Theorem 9.7 (Theorem 7.13 of [292]; Theorem 7.7.7 of [198])**.** Let $\alpha\colon G \to \operatorname{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. If G is amenable, then $C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is an isomorphism.

The converse is true for $A = \mathbb{C}$: if $C^*(G) \to C^*_{\mathrm{r}}(G)$ is an isomorphism, then G is amenable. See Theorem 7.3.9 of [198]. But it is not true in general. For example, if G acts on itself by translation, then $C^*(G,\, C_0(G)) \to C^*_{\mathrm{r}}(G,\, C_0(G))$ is an isomorphism for every G. See Example 10.8 for the case of a discrete group.

The proof of Theorem 9.7 is similar to that of Theorem 5.50, with the algebra A just carried along.

Proof of Theorem 9.7. Let $\iota\colon C_{\mathrm{c}}(G, A, \alpha) \to C^*(G, A, \alpha)$ and $\kappa\colon C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ be the standard maps. We have to prove that $\|\kappa(\iota(b))\| \geq \|\iota(b)\|$ for all $b \in C_{\mathrm{c}}(G, A, \alpha)$. It suffices to prove the following. Let $b \in C_{\mathrm{c}}(G, A, \alpha)$, let H be a Hilbert space, let (w, σ) be a nondegenerate covariant representation of (G, A, α) on H, and let $\varepsilon > 0$. Then there is a Hilbert space E and a nondegenerate representation $\pi_0\colon A \to L(E)$ such that, if we let (y, π) be the associated regular covariant representation of Definition 9.1, then

$$\|(w \ltimes \sigma)(b)\| - \varepsilon < \|(y \ltimes \pi)(b)\|.$$

We will in fact take $\pi_0 = \sigma$.

As usual, let μ be a left Haar measure on G. Let v be the left regular representation of G on $L^2(G)$. Let (y, π) be the regular covariant representation associated to σ, which acts on $L^2(G, H) = L^2(G, \mu) \otimes H$. Thus $y_g = v_g \otimes 1$ for all $g \in G$. It is easy to check that there is a unique unitary $z \in L(L^2(G, H))$ such that $(z\xi)(g) = w_g^{-1}(\xi(g))$ for $\xi \in L^2(G, H)$ and $g \in G$.

We claim that $z(v_h \otimes w_h)z^{-1} = v_h \otimes 1$ for all $h \in G$ and that $z(1 \otimes \sigma(a))z^{-1} = \pi(a)$ for all $a \in A$. To check these, let $\xi \in L^2(G, H)$ and let $g \in G$. Then

$$\begin{aligned}\big(z(v_h \otimes w_h)\xi\big)(g) &= w_g^{-1}\big([(v_h \otimes w_h)\xi](g)\big) = w_g^{-1}\big(w_h(\xi(h^{-1}g))\big)\\ &= w_{g^{-1}h}(\xi(h^{-1}g)) = (z\xi)(h^{-1}g) = \big((v_h \otimes 1)z\xi\big)(g)\end{aligned}$$

and, using covariance of (w, σ) at the third step and the definition of π at the fifth step,

$$\begin{aligned}\big(z[1 \otimes \sigma(a)]\xi\big)(g) &= w_g^{-1}\big([(1 \otimes \sigma(a))\xi](g)\big) = w_g^{-1}\sigma(a)(\xi(g))\\ &= \sigma\big(\alpha_g^{-1}(a)\big)w_g^{-1}(\xi(g)) = \sigma\big(\alpha_g^{-1}(a)\big)\big((z\xi)(g)\big) = \big(\pi(a)z\xi\big)(g).\end{aligned}$$

This proves the claim.

Writing $1 \otimes \sigma$ for the representation $a \mapsto 1 \otimes \sigma(a)$ on $L^2(G) \otimes H = L^2(G, H)$, and recalling the notation in Definition 8.9 for integrated forms of covariant representations, the claim implies that $(v \otimes w,\, 1 \otimes \sigma)$ is a covariant representation and

$$\big\|\big((v \otimes w) \ltimes (1 \otimes \sigma)\big)(b)\big\| = \|(y \ltimes \pi)(b)\|.$$

We finish the proof by showing that

$$\big\|\big((v \otimes w) \ltimes (1 \otimes \sigma)\big)(b)\big\| > \|(w \ltimes \sigma)(b)\| - \varepsilon.$$

We may assume that $(w \ltimes \sigma)(b) \neq 0$ and $\varepsilon < \|(w \ltimes \sigma)(b)\|$. Choose $\xi_0 \in H$ such that

$$\|\xi_0\| = 1 \qquad \text{and} \qquad \|(w \ltimes \sigma)(b)\xi_0\| > \|(w \ltimes \sigma)(b)\| - \frac{\varepsilon}{2}.$$

Set

$$\delta = \left(\frac{\|(w \ltimes \sigma)(b)\| - \frac{\varepsilon}{2}}{\|(w \ltimes \sigma)(b)\| - \varepsilon}\right)^2 - 1.$$

Then $\delta > 0$. Set $S = \operatorname{supp}(b) \cup \{1\}$. Then S and S^{-1} are compact subsets of G. Since G is amenable, the main result of [76] (also see Theorem 3.1.1 there) provides a compact subset $K \subset G$ such that

$$0 < \mu(K) < \infty \qquad \text{and} \qquad \mu\big(S^{-1}K \mathbin{\triangle} K\big) < \delta\mu(K).$$

Since $1 \in S^{-1}$, the second condition implies that $\mu\big(S^{-1}K \setminus K\big) < \delta\mu(K)$. In particular, $\mu(S^{-1}K) < (1+\delta)\mu(K)$. Define $\xi \in L^2(G, H)$ by

$$\xi(g) = \begin{cases} \xi_0 & g \in S^{-1}K \\ 0 & g \notin S^{-1}K. \end{cases}$$

Then

$$\|\xi\| = \mu(S^{-1}K)^{1/2}\|\xi_0\| < (1+\delta)^{1/2}\mu(K)^{1/2}. \tag{9.1}$$

We estimate $\big\|\big((v \otimes w) \ltimes (1 \otimes \sigma)\big)(b)\xi\big\|$. For $g \in K$ we have, at the fourth step using $\xi(h^{-1}g) = \xi_0$ whenever $b(h) \neq 0$,

$$\begin{aligned} \big(\big[(v \otimes w) \ltimes (1 \otimes \sigma)\big](b)\xi\big)(g) &= \int_G \big(\big[(1 \otimes \sigma)(b(h))\big](v_h \otimes w_h)\xi\big)(g)\, d\mu(h) \\ &= \int_G \sigma(b(h))w_h(\xi(h^{-1}g))\, d\mu(h) \\ &= \int_G \sigma(b(h))w_h\xi_0\, d\mu(h) = (w \ltimes \sigma)(b)\xi_0. \end{aligned}$$

Therefore

$$\big\|\big((v \otimes w) \ltimes (1 \otimes \sigma)\big)(b)\xi\big\| \geq \mu(K)^{1/2}\|(w \ltimes \sigma)(b)\xi_0\| > \mu(K)^{1/2}\left(\|(w \ltimes \sigma)(b)\| - \frac{\varepsilon}{2}\right),$$

from which it follows using (9.1) that

$$\begin{aligned} \big\|\big((v \otimes w) \ltimes (1 \otimes \sigma)\big)(b)\big\| &> \frac{\mu(K)^{1/2}\left(\|(w \ltimes \sigma)(b)\| - \frac{\varepsilon}{2}\right)}{(1+\delta)^{1/2}\mu(K)^{1/2}} \\ &= (1+\delta)^{-1/2}\left(\|(w \ltimes \sigma)(b)\| - \frac{\varepsilon}{2}\right) = \|(w \ltimes \sigma)(b)\| - \varepsilon, \end{aligned}$$

as desired. □

Theorem 9.8. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Then $C_{\mathrm{c}}(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is injective.

We will prove this below in the case of a discrete group. The proof of the general case can be found in Lemma 2.26 of [292]. It is, I believe, true that $L^1(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is injective, and this can probably be proved by working a little harder in the proof of Lemma 2.26 of [292], but I have not carried out the details and I do not know a reference.

Theorem 9.9 (Theorem 7.7.5 of [198])**.** Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\pi_0\colon A \to L(H_0)$ be any nondegenerate injective representation. Then the integrated form of the regular representation associated to π_0 is injective on $C^*_{\mathrm{r}}(G, A, \alpha)$.

We will not prove this in general, but we will obtain the result when G is discrete, as a special case of Proposition 9.16(2) below.

We now further analyze the reduced crossed product $C^*_{\mathrm{r}}(G, A, \alpha)$ when G is discrete. One of the consequences will be the discrete group case of Theorem 9.9, but some of what we do does not have a good analog for groups which are not discrete. The main tool is the structure of regular representations of $C^*_{\mathrm{r}}(G, A, \alpha)$. When G is discrete, we can write $L^2(G, H_0)$ as a Hilbert space direct sum $\bigoplus_{g \in G} H_0$, and elements of it can be thought of as families $(\xi_g)_{g \in G}$.

The main result is Proposition 9.16, which in particular contains the faithfulness of the conditional expectation from the reduced crossed product to the original algebra. Faithfulness is proved in Theorem 4.12 of [295]; also see some of the preceding results there. The development there differs somewhat from ours. We have not found a reference for the following development, although we presume that there is one. The closest we have come is Section 1.2 of [177], especially Lemma 1.2.3 and Lemma 1.2.5 there, where it is specifically assumed that $G = \mathbb{Z}$. The proofs in [177] are more complicated than what we give here. Since [177] treats the full rather than the reduced crossed product, the proofs there must also implicitly prove that the map $C^*(\mathbb{Z}, A, \alpha) \to C^*_{\mathrm{r}}(\mathbb{Z}, A, \alpha)$ is an isomorphism. (We are grateful to Sriwulan Adji for calling our attention to this reference.)

Lemma 9.10. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let $\sigma \colon C^*_{\mathrm{r}}(G, A, \alpha) \to H = L^2(G, H_0)$ be the integrated form of the associated regular representation. Let

$$a = \sum_{g \in G} a_g u_g \in C^*_{\mathrm{r}}(G, A, \alpha),$$

with $a_g = 0$ for all but finitely many g. For $\xi \in H$, we then have

$$(\sigma(a)\xi)(h) = \sum_{g \in G} \pi_0(\alpha_h^{-1}(a_g))\big(\xi(g^{-1}h)\big)$$

for all $h \in G$.

Proof. This is a calculation. □

In particular, picking off coordinates in $L^2(G, H_0)$ gives the following result.

Corollary 9.11. Let the hypotheses be as in Lemma 9.10, and let

$$a = \sum_{g \in G} a_g u_g \in C^*_{\mathrm{r}}(G, A, \alpha)$$

as there. For $g \in G$, let $s_g \in L(H_0, H)$ be the isometry which sends $\eta \in H_0$ to the function $\xi \in L^2(G, H_0)$ given by

$$\xi(h) = \begin{cases} \eta & h = g \\ 0 & h \neq g. \end{cases}$$

Then

$$s_h^* \sigma(a) s_k = \pi_0\big(\alpha_h^{-1}(a_{hk^{-1}})\big)$$

for all $h, k \in G$.

Proof. This is an easy calculation from Lemma 9.10. □

Lemma 9.12. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $\|\cdot\|$ be the C*-algebra norm on $C^*(G, A, \alpha)$ restricted to $C_{\mathrm{c}}(G, A, \alpha)$, let $\|\cdot\|_{\mathrm{r}}$ be the C*-algebra norm on $C^*_{\mathrm{r}}(G, A, \alpha)$ restricted to $C_{\mathrm{c}}(G, A, \alpha)$, and let $\|\cdot\|_\infty$ be the supremum norm. Then for every $a \in C_{\mathrm{c}}(G, A, \alpha)$, we have $\|a\|_\infty \leq \|a\|_{\mathrm{r}} \leq \|a\| \leq \|a\|_1$.

Proof. The middle of this inequality follows from the definitions.

The last part follows from the observation in Remark 8.12 that all continuous representations of $L^1(G, A, \alpha)$ are norm reducing. Here is a direct proof: for $a = \sum_{g \in G} a_g u_g \in C_c(G, A, \alpha)$, with all but finitely many of the a_g equal to zero, we have

$$\left\| \sum\nolimits_{g \in G} a_g u_g \right\| \leq \sum\nolimits_{g \in G} \|a_g\| \cdot \|u_g\| = \sum\nolimits_{g \in G} \|a_g\| = \left\| \sum\nolimits_{g \in G} a_g u_g \right\|_1 .$$

We prove the first part of the inequality. Let $a = \sum_{g \in G} a_g u_g$, with all but finitely many of the a_g equal to zero, and let $g \in G$. Let $\pi_0 \colon A \to L(H_0)$ be an injective nondegenerate representation. With the notation of Corollary 9.11, we have

$$\|a_g\| = \|\pi_0(a_g)\| = \|s_1^* \sigma(a) s_{g^{-1}}\| \leq \|\sigma(a)\| \leq \|a\|_{\mathrm{r}}.$$

This completes the proof. $\square$

Remark 9.13. Lemma 9.12 implies that the map $a \mapsto a u_1$, from A to $C^*_{\mathrm{r}}(G, A, \alpha)$, is injective. We routinely identify A with its image in $C^*_{\mathrm{r}}(G, A, \alpha)$ under this map, thus treating it as a subalgebra of $C^*_{\mathrm{r}}(G, A, \alpha)$.

Of course, we can do the same with the full crossed product $C^*(G, A, \alpha)$.

Corollary 9.14. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. Then the maps $C_c(G, A, \alpha) \to C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ are bijective.

Proof. When G is finite, $\| \cdot \|_1$ is equivalent to $\| \cdot \|_\infty$ as defined in Lemma 9.12, and $C_c(G, A, \alpha)$ is complete in both. Lemma 9.12 implies that both C* norms are equivalent to these norms, so $C_c(G, A, \alpha)$ is complete in both C* norms. $\square$

When G is discrete but not finite, things are much more complicated. We can get started.

Proposition 9.15. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Then for each $g \in G$, there is a linear map $E_g \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ with $\|E_g\| \leq 1$ such that if

$$a = \sum_{g \in G} a_g u_g \in C_c(G, A, \alpha),$$

then $E_g(a) = a_g$. Moreover, for every representation π_0 of A, and with s_g as in Corollary 9.11, we have

$$s_h^* \sigma(a) s_k = \pi_0\big(\alpha_h^{-1}(E_{hk^{-1}}(a))\big)$$

for all $h, k \in G$.

Proof. The first part is immediate from the first inequality in Lemma 9.12. The last statement follows from Corollary 9.11 by continuity. $\square$

Thus, for any $a \in C^*_{\mathrm{r}}(G, A, \alpha)$, and therefore also for $a \in C^*(G, A, \alpha)$, it makes sense to talk about its coefficients a_g. As we have already seen in Remark 5.60, even when $A = \mathbb{C}$ the obvious series made with these coefficients need not converge to a (or to anything). See Remark 9.19 for more information. If $C^*(G, A, \alpha) \neq C^*_{\mathrm{r}}(G, A, \alpha)$ (which can happen if G is not amenable, but not if G is amenable; see Theorem 9.7), the coefficients $(a_g)_{g \in G}$ do not even uniquely determine the element a. (See further discussion of the case $A = \mathbb{C}$ in Remark 5.61.) This is why we only consider reduced crossed products here.

Proposition 9.16. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let the maps $E_g\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as in Proposition 9.15. Then:

(1) If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $E_g(a) = 0$ for all $g \in G$, then $a = 0$.
(2) If $\pi_0\colon A \to L(H_0)$ is a nondegenerate representation such that $\bigoplus_{g \in G} \pi_0 \circ \alpha_g$ is injective, then the regular representation σ of $C^*_{\mathrm{r}}(G, A, \alpha)$ associated to π_0 is injective.
(3) If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $g \in G$, then $\|E_g(a)\|^2 \leq \|E_1(a^*a)\|$.
(4) If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $E_1(a^*a) = 0$, then $a = 0$.

Proposition 9.16(2) implies the discrete group case of Theorem 9.9.

Proof of Proposition 9.16. We prove (1). Let $\pi_0\colon A \to L(H_0)$ be a representation, and let the notation be as in Corollary 9.11. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ satisfies $E_g(a) = 0$ for all $g \in G$, then $s_h^* \sigma(a) s_k = 0$ for all $h, k \in G$, whence $\sigma(a) = 0$. Since π_0 is arbitrary, it follows that $a = 0$. This proves (1).

For (2), suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1} g^{-1}$ in Proposition 9.15, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$. So $E_l(a) = 0$. This is true for all $l \in G$, so $a = 0$.

We now prove (3). As before, let

$$a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha).$$

Then

$$a^*a = \sum_{g, h \in G} u_g^* a_g^* a_h u_h = \sum_{g, h \in G} \alpha_g^{-1}(a_g a_h^*) u_{g^{-1}h},$$

so

$$E_1(a^*a) = \sum_{g \in G} \alpha_g^{-1}\big(E_g(a)^* E_g(a)\big).$$

In particular, for each fixed g, we have $E_1(a^*a) \geq \alpha_g^{-1}\big(E_g(a)^* E_g(a)\big)$. By continuity, this inequality holds for all $a \in C^*_{\mathrm{r}}(G, A, \alpha)$. So

$$\|E_1(a^*a)\| \geq \big\|\alpha_g^{-1}\big(E_g(a)^* E_g(a)\big)\big\| = \big\|E_g(a)^* E_g(a)\big\| = \|E_g(a)\|^2,$$

as desired.

Part (4) now follows easily. If $E_1(a^*a) = 0$, then by (3) we have $E_g(a)^* E_g(a) = 0$ for all g. Therefore $a = 0$ by Part (1). □

The map E_1 used in Proposition 9.16(4) is an example of what is called a *conditional expectation* (from $C^*_{\mathrm{r}}(G, A, \alpha)$ to A) that is, it has the properties given in the following exercise. (Some of them are redundant.) Proposition 9.16(4) asserts that this conditional expectation is faithful.

Exercise 9.17. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $E = E_1\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as in Proposition 9.15. Prove that E has the following properties:

(1) $E(a) = a$ for all $a \in A$.
(2) $E(E(b)) = E(b)$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
(3) If $b \geq 0$ then $E(b) \geq 0$.
(4) $\|E(b)\| \leq \|b\|$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
(5) If $a \in A$ and $b \in C^*_{\mathrm{r}}(G, A, \alpha)$, then $E(ab) = aE(b)$ and $E(ba) = E(b)a$.

Definition 9.18. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. The map $E = E_1\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ of Proposition 9.15, determined by

$$E\left(\sum\nolimits_{g\in G} a_g u_g\right) = a_1$$

when $\sum_{g\in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$, is called the *standard conditional expectation from* $C^*_{\mathrm{r}}(G, A, \alpha)$ *to* A. It is usually written E. When $A = \mathbb{C}$ (and the action is trivial), we obtain a tracial state on $C^*_{\mathrm{r}}(G)$, which we call the *standard tracial state*.

The standard tracial state already appeared in Theorem 5.28.

Remark 9.19. Unfortunately, in general the series $\sum_{g\in G} a_g u_g$ does not converge in $C^*_{\mathrm{r}}(G, A, \alpha)$. Indeed, we saw in Remark 5.60(3) that this already fails for the trivial action of $\mathbb{Z}$ on $\mathbb{C}$.

As suggested by Remark 5.60(1), it can be very difficult to determine exactly which families $(a_g)_{g\in G}$ correspond to elements of $C^*_{\mathrm{r}}(G, A, \alpha)$. If G is discrete abelian, then there is a good alternate description of $C^*(G)$. Since $C^*(G)$ is commutative and unital, it must be isomorphic to $C(X)$ for some compact Hausdorff space X, and the right choice is the Pontryagin dual $\widehat{G}$. This was already proved in Theorem 5.38. In general, the computation of $C^*(G)$ and $C^*_{\mathrm{r}}(G)$ is a difficult problem, as is suggested by Remark 5.60. Answers are known for some groups, particularly semisimple Lie groups (which of course are not discrete).

Even if one understands completely what all the elements of $C^*_{\mathrm{r}}(G)$ are, and even if the action $\alpha\colon G \to \mathrm{Aut}(A)$ is trivial, understanding the elements of the reduced crossed product requires that one understand all the elements of the completed tensor product $C^*_{\mathrm{r}}(G) \otimes_{\min} A$. If G is abelian, one gets $C\big(\widehat{G}, A\big)$. However, as far as I know, this problem is also in general intractable.

When the group is not amenable, for full crossed products instead of reduced crossed products, one of course has the generalization of the difficulty described in Remark 5.61 with the full group C*-algebra.

There is just one bright spot, although we will not prove it here. The Cesàro means of the Fourier series of a continuous function always converge uniformly to the function, and, as already mentioned in Remark 5.60(4), this fact has generalizations to crossed products by discrete amenable groups and even some cases beyond that. See Section 5 of [17]. The case $G = \mathbb{Z}$ is Theorem VIII.2.2 of [52].

Remark 5.60 is meant to point out the difficulties in dealing with crossed products by infinite groups. Despite all this, for some problems, finite groups are harder. As suggested after Remark 5.61, we have excellent information about the K-theory of crossed products by $\mathbb{Z}$ [221] and by $\mathbb{R}$ [42], and even for both reduced crossed products by free groups F_n [222] and the corresponding full crossed products (in [49] see Theorem 2.1(c), Definition 2.2, and Theorem 2.4(c)). See [219] for a generalization of the result on reduced crossed products by F_n. The result for full crossed products by F_n holds despite the fact that the the conditional expectation of Definition 9.18 is usually not faithful on full crossed products, so that an element is not even uniquely determined by its “coefficients”. All these formulas imply, in particular, that if the K-theory of the original algebra is zero, then so is the K-theory of the crossed product. There is no such formula for the K-theory of crossed products by the two element group $\mathbb{Z}/2\mathbb{Z}$, in which not even any completion is needed. There even exists a C*-algebra A which is contractible (a much stronger condition than

$K_*(A) = 0$) and an action $\alpha\colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(A)$ such that $K_*\big(C^*(\mathbb{Z}/2\mathbb{Z},\, A,\, \alpha)\big) \neq 0$. (Examples can be constructed using some of the examples in Section 3 of [201]. We omit the details.)

We now discuss functoriality of reduced crossed products. Again, given the results above, the the proofs are no harder for general locally compact groups G than in the discrete case, but we can't claim that the presentation of the general case is self contained.

Lemma 9.20. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let I be a set, and for $i \in I$ let ρ_i be a representation of A on a Hilbert space H_i. Let (v_i, π_i) be the associated regular representation, and let σ_i be its integrated form. Then the regular representation associated with $\bigoplus_{i \in I} \rho_i$ is $\big(\bigoplus_{i \in I} v_i,\, \bigoplus_{i \in I} \pi_i\big)$. Its integrated form is $\bigoplus_{i \in I} \sigma_i$.

Proof. The proof of the first statement is routine. The second statement follows from Lemma 8.27. ∎

Lemma 8.29 has the following analog for regular covariant representations.

Lemma 9.21. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\rho_0\colon A \to L(H_0)$ be a representation of A on a Hilbert space H_0. (We do not assume that ρ_0 is nondegenerate.) Let $\sigma_0\colon C^*(G, A, \alpha) \to L(L^2(G, H_0))$ be the integrated form (as in Definition 8.9) of the regular covariant representation associated to ρ_0 (as in Definition 9.1). Let H be the closed linear span of $\pi_0(A)H_0$. Then the closed linear span of $\sigma_0(C^*(G, A, \alpha))L^2(G, H_0)$ is $L^2(G, H)$.

Proof. The subspace H is invariant under ρ_0 by Lemma 8.26(1). Set $\rho = \rho_0(-)|_H$. Then $\rho_0 = \rho \oplus 0$ by Lemma 8.26(4).

Let (v_0, π_0), (v, π), and (w, ζ) be the regular covariant representations of (G, A, α) on $L(L^2(G, H_0))$, $L(L^2(G, H))$, and $L(L^2(G, H)^{\perp})$ associated to ρ_0, ρ, and the zero representation on $H^{\perp}$. Let σ and ν be the integrated forms of (v, π) and (w, ζ). Lemma 9.20 gives $v_0 = v \oplus w$, $\pi_0 = \pi \oplus \zeta$, and $\sigma_0 = \sigma \oplus \nu$. Clearly ζ is the zero representation of A, so ν is the zero representation of $C^*(G, A, \alpha)$, while π is nondegenerate by Lemma 9.3. Apply Lemma 8.26(7) to the direct sum decomposition $\sigma_0 = \sigma \oplus \nu$, and then apply Lemma 8.26(4). ∎

Corollary 9.22. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Let $\kappa\colon C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ be the quotient map. Let $a \in C^*(G, A, \alpha)$. Then

$$\begin{aligned}\|\kappa(a)\| = \sup\big(\big\{\|\sigma(a)\|\colon &\sigma \text{ is the integrated form of a possibly} \\ &\text{degenerate regular covariant representation of } (G, A, \alpha)\big\}\big).\end{aligned}$$

Proof. It follows from Lemma 9.21, Lemma 8.26(2), and Lemma 8.26(5) that the supremum on the right is unchanged if we restrict to covariant representations of (G, A, α) coming from nondegenerate representations of A. ∎

Theorem 9.23. Let G be a locally compact group. If (G, A, α) and (G, B, β) are G-algebras and $\varphi\colon A \to B$ is an equivariant homomorphism, then the homomorphism $C^*(G, A, \alpha) \to C^*(G, B, \beta)$ of Theorem 8.31 induces a homomorphism $C^*_{\mathrm{r}}(G, A, \alpha) \to C^*_{\mathrm{r}}(G, B, \beta)$. This construction makes the reduced crossed product construction a functor from the category of G-algebras to the category of C*-algebras.

Proof. One observes that if $\pi_0 \colon B \to L(H_0)$ is a representation, and if

$$\sigma \colon C_{\mathrm{c}}(G, B, \beta) \to L(L^2(G, H_0))$$

is the associated regular representation, then $\sigma \circ \psi$ is the regular representation associated with the representation $\pi_0 \circ \varphi \colon A \to L(H_0)$. In view of Corollary 9.22, it follows that $C^*(G, A, \alpha) \to C^*(G, B, \beta)$ induces a well defined homomorphism $C^*_{\mathrm{r}}(G, A, \alpha) \to C^*_{\mathrm{r}}(G, B, \beta)$. The properties of a functor are easy to check. □

The analog of Theorem 8.32 for reduced crossed products is in general false. Counterexamples are hard to find, and the history is confusing; we refer to the (brief) discussion in the introduction to [16]. Since the reduced crossed product is functorial, the maps in the sequence are defined. In fact, exactness can only fail in the middle. Indeed, we have the following result.

Theorem 9.24. Let G be a locally compact group, let (G, A, α) and (G, B, β) be G-algebras, and let $\varphi \colon A \to B$ be an equivariant homomorphism. Let

$$\psi \colon C^*_{\mathrm{r}}(G, A, \alpha) \to C^*_{\mathrm{r}}(G, B, \beta)$$

be the corresponding homomorphism of the reduced crossed products.

(1) If φ is injective then so is ψ.
(2) If $\varphi(A)$ is an ideal in B, then $\psi\big(C^*_{\mathrm{r}}(G, A, \alpha)\big)$ is an ideal in $C^*_{\mathrm{r}}(G, B, \beta)$.
(3) If φ is surjective then so is ψ.
(4) If $\varphi(A)$ is a nonzero proper ideal in B, then $\psi\big(C^*_{\mathrm{r}}(G, A, \alpha)\big)$ is a nonzero proper ideal in $C^*_{\mathrm{r}}(G, B, \beta)$.

Proof. For (1), choose a nondegenerate injective representation π_0 of B on a Hilbert space H, let (v, π) be the associated regular covariant representation of (G, B, β) (Definition 9.1), and let $\sigma \colon C^*_{\mathrm{r}}(G, B, \beta) \to L(L^2(G, H))$ be its integrated form (Definition 8.9). Then $\pi_0 \circ \varphi$ is an injective representation of A on H (not necessarily nondegenerate).

Use Lemma 8.26(4) to find a closed subspace $M \subset H$ and a nondegenerate representation $\rho \colon A \to L(M)$ such that $\pi_0 \circ \varphi$ is the direct sum of ρ and the zero representation on $M^{\perp}$. Then ρ is injective by Lemma 8.26(6). Let $\mu \colon C^*_{\mathrm{r}}(G, A, \alpha) \to L(L^2(G, M))$ be the integrated form of the regular covariant representation associated to ρ. Theorem 9.9 implies that μ is injective, and Lemma 9.21 implies that μ is a direct summand in the representation $\sigma \circ \psi$. Therefore ψ must be injective.

In the next two parts, we let

$$\iota_A \colon C_{\mathrm{c}}(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha) \qquad \text{and} \qquad \iota_B \colon C_{\mathrm{c}}(G, B, \beta) \to C^*_{\mathrm{r}}(G, B, \beta)$$

be the standard maps.

We prove (2). Since ι_A and ι_B have dense ranges, to show that $\psi\big(C^*_{\mathrm{r}}(G, A, \alpha)\big)$ is an ideal, it suffices to prove that that for $a \in C_{\mathrm{c}}(G, A, \alpha)$ and $b \in C_{\mathrm{c}}(G, B, \beta)$, we have $\psi(\iota_A(a))\iota_B(b) \in C^*_{\mathrm{r}}(G, B, \beta)$ and $\iota_B(b)\psi(\iota_A(a)) \in C^*_{\mathrm{r}}(G, B, \beta)$. In fact, both are obviously in $\iota_B(C_{\mathrm{c}}(G, B, \beta))$.

We prove (3). One checks that φ induces a surjective map $C_{\mathrm{c}}(G, A, \alpha) \to C_{\mathrm{c}}(G, B, \beta)$. Therefore the range of ψ contains $\iota_B(C_{\mathrm{c}}(G, B, \beta))$. So ψ has dense range, and is therefore surjective.

Finally, we prove (4). The subalgebra $\psi\big(C^*_{\mathrm{r}}(G, A, \alpha)\big)$ is an ideal by (2), and is nonzero by (1). Let $\pi \colon B \to B/\varphi(A)$ be the quotient map, and let

$$\sigma \colon C^*_{\mathrm{r}}(G, B, \alpha) \to C^*_{\mathrm{r}}(G,\, B/\varphi(A),\, \beta)$$

be the corresponding homomorphism of the reduced crossed products. Clearly $\sigma \circ \psi = 0$, so $\psi\big(C^*_{\mathrm{r}}(G, A, \alpha)\big) \subset \mathrm{Ker}(\sigma)$. Since σ is surjective by (3), it follows that $\psi\big(C^*_{\mathrm{r}}(G, A, \alpha)\big) \neq C^*_{\mathrm{r}}(G, B, \beta)$. $\square$

Remark 9.25. We describe dual actions without proof; see [274] for details. Let A be any C*-algebra, let G be a locally compact abelian group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action. Let $\widehat{G}$ be the Pontryagin dual of G (Definition 5.30). For $\sigma \in \widehat{G}$, there is an automorphism $\widehat{\alpha}_\sigma$ of $C^*(G, A, \alpha)$ given on $C_{\mathrm{c}}(G, A, \alpha)$ by $\widehat{\alpha}_\sigma(a)(g) = \overline{\sigma(g)}a(g)$ for $a \in C_{\mathrm{c}}(G, A, \alpha)$, $\sigma \in \widehat{G}$, and $g \in G$. Moreover, $\widehat{\alpha} \colon \widehat{G} \to \mathrm{Aut}\big(C^*(G, A, \alpha)\big)$ is a continuous action of $\widehat{G}$ on $C^*(G, A, \alpha)$, called the dual action.

One can also use $\sigma(g)$ in place of $\overline{\sigma(g)}$. The choice $\overline{\sigma(g)}$ seems to be more common. It agrees with the conventions in [274] (see the beginning of Section 3 there) and [292] (see the beginning of Section 7.1 there), but disagrees with the choice in [198] (see Proposition 7.8.3 there). To see the reason for the choice $\overline{\sigma(g)}$, consider the case $G = S^1$ and $A = \mathbb{C}$. For $f \in C_{\mathrm{c}}(G)$, we have $\widehat{\alpha}_n(f)(\zeta) = \zeta^{-n}$ for $n \in \mathbb{Z}$ and $\zeta \in S^1$. The n-th Fourier coefficient of f is then $\widehat{f}(n) = \int_G \widehat{\alpha}_n(f)(\zeta)\, d\zeta$, giving the corresponding Fourier series $f(\zeta) = \sum_{n \in \mathbb{Z}} \widehat{f}(n)\zeta^n$. If one uses $\sigma(g)$ in the definition of the dual action, then some extra signs are required in these formulas.

If G is discrete then $\widehat{G}$ is compact, and the conditional expectation $E \colon C^*(G, A, \alpha) \to A$ of Definition 9.18 is given by $E(a) = \int_{\widehat{G}} \widehat{\alpha}_\sigma(a)\, d\sigma$, using normalized Haar measure in the integral. Whether or not G is discrete, the crossed product by the dual action is $K(L^2(G)) \otimes A$. This result is Takai duality ([274]; Theorem 7.1 of [292]; Theorem 7.9.3 of [198]). It is a generalization of the abelian case of Example 10.8 below: if $A = \mathbb{C}$, then $C^*(G, A, \alpha) = C^*(G) \cong C_0\big(\widehat{G}\big)$, and the dual action is just translation on $\widehat{G}$.

Exercise 9.26. Adopt the notation of Remark 9.25. Prove that the formula $\widehat{\alpha}_\sigma(a)(g) = \overline{\sigma(g)}a(g)$, for $a \in C_{\mathrm{c}}(G, A, \alpha)$, $\sigma \in \widehat{G}$, and $g \in G$, extends to a continuous action of $\widehat{G}$ on $C^*(G, A, \alpha)$.

Exercise 9.26 is easiest when G is discrete and A is unital, in which case one can use the description in Theorem 8.21 of $C^*(G, A, \alpha)$ in terms of generators and relations.

Exercise 9.27. Adopt the notation of Remark 9.25, and assume that G is discrete. Let ν be normalized Haar measure on $\widehat{G}$. Prove that for all $a \in C^*(G, A, \alpha)$, the the conditional expectation $E \colon C^*(G, A, \alpha) \to A$ of Definition 9.18 satisfies

$$E(a) = \int_{\widehat{G}} \widehat{\alpha}_\sigma(a)\, d\nu(\sigma)$$

for all $a \in C^*(G, A, \alpha)$, as claimed in Remark 9.25. Hint: Prove this for $a \in C_{\mathrm{c}}(G, A, \alpha)$ first.

10. Computation of Some Examples of Crossed Products

We give some explicit elementary computations of crossed products, mostly involving finite groups. These examples serve several purposes. First, they give, in a comparatively elementary context, an explicit sense of what crossed products look like. In particular, our calculations motivate the statements of various general theorems, some of which we give without proof. Second, a number of interesting examples of actions and their crossed products have been constructed by taking direct limits of some of the kinds of examples we consider. The computation of some of these crossed products depends on knowing enough detail in examples of some of the types discussed here that one can calculate direct limits of them. We include in this section several examples of computations of crossed products by actions constructed using direct limits.

Some of our examples can be found in Section 2.5 of [292]. For the most part, however, we have not found calculations in the literature in the explicit form which we give here.

Throughout this section, we will use the *-algebra $C_{\mathrm{c}}(G, A, \alpha)$ of compactly supported continuous functions $a \colon G \to A$, with pointwise addition and scalar multiplication, with multiplication given by convolution as in Definition 8.2, and with the adjoint defined there. By construction (Definition 8.15, together with density of $C_{\mathrm{c}}(G, A, \alpha)$ in $L^1(G, A, \alpha)$, as in Definition 8.2), the image of this algebra in $C^*(G, A, \alpha)$ is dense, and by Theorem 9.8 the map $C_{\mathrm{c}}(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)$ is injective. It follows that the map $C_{\mathrm{c}}(G, A, \alpha) \to C^*(G, A, \alpha)$ is injective. We therefore routinely identify $C_{\mathrm{c}}(G, A, \alpha)$ with a dense subalgebra of $C^*(G, A, \alpha)$ and also, depending on context, with a dense subalgebra of $C^*_{\mathrm{r}}(G, A, \alpha)$.

Our group G will almost always be discrete. In this case and if A is unital, for $g \in G$ we let $u_g \in C_{\mathrm{c}}(G, A, \alpha)$ be the canonical unitary corresponding to g, as in Notation 8.7. Also following Notation 8.7, we use the same notation for the corresponding unitaries in $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$. When A is not unital, we follow the conventions for the nonunital case in Notation 8.7. (In particular, u_g denotes corresponding elements in the multiplier algebras of $C^*(G, A, \alpha)$ and $C^*_{\mathrm{r}}(G, A, \alpha)$.)

When G is discrete, $C_{\mathrm{c}}(G, A, \alpha)$ is the set of functions from G to A which have finite support. Following the notation of Lemma 9.10 and later results in Section 9, and as suggested by Remark 8.8, in both the unital and nonunital cases we regularly identify $C_{\mathrm{c}}(G, A, \alpha)$ with the set of sums $a = \sum_{g \in G} a_g u_g$ in which $a_g \in A$ for all $g \in G$ and $a_g = 0$ for all but finitely many $g \in G$. When G is finite, as is the case in many of our examples, $C_{\mathrm{c}}(G, A, \alpha)$ is then just the set of all sums $a = \sum_{g \in G} a_g u_g$ in which $a_g \in A$ for all $g \in G$, and the map from $C_{\mathrm{c}}(G, A, \alpha)$ to $C^*(G, A, \alpha)$ is bijective (Corollary 9.14).

Example 10.1. If G acts trivially on the C*-algebra A, then

$$C^*(G, A) \cong C^*(G) \otimes_{\max} A \qquad \text{and} \qquad C^*_{\mathrm{r}}(G, A) \cong C^*_{\mathrm{r}}(G) \otimes_{\min} A.$$

The case of the full crossed product, in fact, the generalization to the case of an inner action, is Example 2.53 of [292].

For the full crossed product, first assume G is discrete and A is unital. Then Theorem 8.21 implies that $C^*(G, A)$ is the universal unital C*-algebra generated by a unital copy of A and a commuting unitary representation of G in the algebra. Since $C^*(G)$ is the universal unital C*-algebra generated by a unitary representation of G in the algebra, this is exactly the universal property of the maximal tensor product.

The proof for the general case is essentially the same. The basic point (omitting the technicalities) is that a covariant representation consists of commuting representations of A and G, and hence of A and $C^*(G)$.

For the reduced crossed product, the point is that a regular nondegenerate covariant representation of (G, A) has the form $(\lambda \otimes 1_{H_0},\, 1_{L^2(G)} \otimes \pi_0)$ for an arbitrary nondegenerate representation $\pi_0 \colon A \to L(H_0)$ and with $\lambda \colon G \to U(L^2(G))$ being the left regular representation. By Proposition 9.16(2), it suffices to take π_0 to be a single injective representation. Now we are looking at $C^*_{\mathrm{r}}(G)$ on one Hilbert space and A on another, and taking the tensor product of the Hilbert spaces. This is exactly how one gets the minimal tensor product of two C*-algebras.

Note how full and reduced crossed products parallel maximal and minimal tensor products.

Remark 10.2. More generally, let A and B be C*-algebras, let $\alpha \colon G \to \mathrm{Aut}(A)$ be any action, and let $\beta \colon G \to \mathrm{Aut}(B)$ be the trivial action. Even if β is not trivial, one gets actions $\alpha \otimes_{\mathrm{max}} \beta$ of G on $A \otimes_{\mathrm{max}} B$ and $\alpha \otimes_{\mathrm{min}} \beta$ of G on $A \otimes_{\mathrm{min}} B$ which, interpreting tensor products of elements of A and B as being in $A \otimes_{\mathrm{max}} B$ or $A \otimes_{\mathrm{min}} B$ as appropriate, are uniquely determined by

$$(\alpha \otimes_{\mathrm{max}} \beta)_g (a \otimes b) = \alpha_g(a) \otimes \beta_g(b) \qquad \text{and} \qquad (\alpha \otimes_{\mathrm{min}} \beta)_g (a \otimes b) = \alpha_g(a) \otimes \beta_g(b)$$

for $a \in A$, $b \in B$, and $g \in G$. If β is trivial, these formulas become

$$(\alpha \otimes_{\mathrm{max}} \beta)_g (a \otimes b) = \alpha_g(a) \otimes b \qquad \text{and} \qquad (\alpha \otimes_{\mathrm{min}} \beta)_g (a \otimes b) = \alpha_g(a) \otimes b,$$

and one has

$$C^*(G,\, A \otimes_{\mathrm{max}} B,\, \alpha \otimes_{\mathrm{max}} \beta) \cong C^*(G, A, \alpha) \otimes_{\mathrm{max}} B$$

and

$$C^*_{\mathrm{r}}(G,\, A \otimes_{\mathrm{min}} B,\, \alpha \otimes_{\mathrm{min}} \beta) \cong C^*_{\mathrm{r}}(G, A, \alpha) \otimes_{\mathrm{min}} B.$$

Exercise 10.3. Prove Remark 10.2 when G is discrete and A and B are both unital.

Exercise 10.26, Exercise 10.27, and Exercise 10.28 contain a generalization.

Example 10.4. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an inner action of a discrete group G on a unital C*-algebra A. Thus, there is a homomorphism $g \mapsto z_g$ from G to $U(A)$ such that $\alpha_g(a) = z_g a z_g^*$ for all $g \in G$ and $a \in A$. (See Example 3.4.) We claim that $C^*(G, A, \alpha) \cong C^*(G) \otimes_{\mathrm{max}} A$. (This is true even if G is not discrete. See Exercise 10.5, or Example 2.53 of [292].)

It is also true that $C^*_{\mathrm{r}}(G, A, \alpha) \cong C^*_{\mathrm{r}}(G) \otimes_{\mathrm{min}} A$.

We prove the claim. Let $\iota \colon G \to \mathrm{Aut}(A)$ be the trivial action of G on A. As in Notation 8.7, for $g \in G$ let $u_g \in C_{\mathrm{c}}(G, A, \alpha)$ be the standard unitary, but let $v_g \in C_{\mathrm{c}}(G, A, \iota)$ be the standard unitary in the crossed product by the trivial action. Define $\varphi_0 \colon C_{\mathrm{c}}(G, A, \alpha) \to C_{\mathrm{c}}(G, A, \iota)$ by $\varphi_0(a u_g) = a z_g v_g$ for $a \in A$ and $g \in G$, and extend linearly. This map is obviously bijective (the inverse sends $a v_g$ to $a z_g^* u_g$) and isometric for $\| \cdot \|_1$. For multiplicativity, it suffices to check the following, for $a, b \in A$ and $g, h \in H$, using the fact that v_g commutes with all elements of A:

$$\begin{aligned} \varphi_0(a u_g) \varphi_0(b u_h) &= a z_g v_g b z_h v_h = a z_g b z_g^* z_{gh} v_g v_h \\ &= a \alpha_g(b) z_{gh} v_{gh} = \varphi_0\big(a \alpha_g(b) u_{gh}\big) = \varphi_0\big((a u_g)(b u_h)\big). \end{aligned}$$

Also,

$$\begin{aligned} \varphi_0(a u_g)^* &= (a z_g v_g)^* = v_g^* z_g^* a^* = (z_g^* a^* z_g) z_g^* v_g^* \\ &= \alpha_{g^{-1}}(a^*) z_{g^{-1}} v_{g^{-1}} = \varphi_0\big(\alpha_{g^{-1}}(a^*) u_{g^{-1}}\big) = \varphi_0\big((a u_g)^*\big). \end{aligned}$$

So φ_0 is an isometric isomorphism of *-algebras, and therefore extends to an isomorphism of the universal C*-algebras as in Theorem 8.21. Now use Example 10.1.

For use in Example 10.21, we write out explicitly what happens when $G = \mathbb{Z}/2\mathbb{Z}$.

Let $v_0 \in C^*(\mathbb{Z}/2\mathbb{Z})$ be the image of the nontrivial element of the group. Then $\lambda + \mu v_0 \mapsto (\lambda + \mu, \, \lambda - \mu)$ is an isomorphism from $C^*(\mathbb{Z}/2\mathbb{Z})$ to $\mathbb{C} \oplus \mathbb{C}$. (The algebra $C^*(\mathbb{Z}/2\mathbb{Z})$ is the universal C*-algebra generated by a unitary with square 1, and the corresponding unitary in $\mathbb{C} \oplus \mathbb{C}$ is $(1, \, -1)$. But one can check directly that the map above is an isomorphism.)

For the crossed product of a unital C*-algebra A by the trivial action ι of $\mathbb{Z}/2\mathbb{Z}$, let $v \in C^*(\mathbb{Z}/2\mathbb{Z}, \, A, \, \iota)$ be the standard unitary associated to the nontrivial element of the group. Then $a + bv \mapsto (a + b, \, a - b)$ is an isomorphism from $C^*(\mathbb{Z}/2\mathbb{Z}, \, A, \, \iota)$ to $A \oplus A$. This map is a homomorphism because the copy $\{(a, a) \colon a \in A\} \subset A \oplus A$ of A and the unitary $(1, \, -1) \in A \oplus A$ satisfy the appropriate commutation relations. One proves that this map is an isomorphism from $C_{\mathrm{c}}(\mathbb{Z}/2\mathbb{Z}, \, A, \, \iota)$ to $A \oplus A$ by explicitly writing down an inverse. Corollary 9.14 now shows it is an isomorphism from $C^*(\mathbb{Z}/2\mathbb{Z}, \, A, \, \iota)$ to $A \oplus A$. (For a faster proof, just tensor the isomorphism of the previous paragraph with id_A.)

Now suppose that $z \in A$ is a unitary of order 2. Let $g_0 \in \mathbb{Z}/2\mathbb{Z}$ be the nontrivial group element, and let $\alpha \colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(A)$ be the action such that $\alpha_{g_0} = \mathrm{Ad}(z)$. Let $u = u_{g_0} \in C^*(\mathbb{Z}/2\mathbb{Z}, \, A, \, \alpha)$. Then $a + bu \mapsto (a + bz, \, a - bz)$ is an isomorphism from $l^1(\mathbb{Z}/2\mathbb{Z}, \, A, \, \alpha)$ to $A \oplus A$. (Of course, once one has the formula, one can prove this directly.)

Exercise 10.5. Prove the following generalization of Example 10.4. Let $\alpha, \beta \colon G \to \mathrm{Aut}(A)$ be two actions of a locally compact group G on a C*-algebra A which are exterior equivalent in the sense of Remark 3.9. Prove that

$$C^*(G, A, \alpha) \cong C^*(G, A, \beta) \qquad \text{and} \qquad C^*_{\mathrm{r}}(G, A, \alpha) \cong C^*_{\mathrm{r}}(G, A, \beta).$$

The case of the full crossed product is done in the proof of Theorem 2.8.3(5) of [200]. (The compactness hypothesis in the theorem is not needed for the relevant part of the proof.)

Exercise 10.6. Let $\alpha \colon (\mathbb{Z}/2\mathbb{Z})^2 \to \mathrm{Aut}(M_2)$ be as in Example 3.5. Prove that the crossed product $C^*((\mathbb{Z}/2\mathbb{Z})^2, \, M_2, \, \alpha)$ is isomorphic to M_4.

Since the group is finite and the algebra is finite dimensional, this exercise can be done with linear algebra. It shows that the hypothesis in Example 10.4 can't be weakened from "inner" to "pointwise inner".

For the next example, we need notation for standard matrix units. (We have already used the usual version of this notation, when $S = \{1, 2, \ldots, n\}$, a number of times.)

Notation 10.7. For any index set S, let $\delta_s \in l^2(S)$ be the standard basis vector, determined by

$$\delta_s(t) = \begin{cases} 1 & t = s \\ 0 & t \neq s. \end{cases}$$

For $j, k \in S$, we let the "matrix unit" $e_{j,k}$ be the rank one operator on $l^2(S)$ given by $e_{j,k}\xi = \langle \xi, \delta_k \rangle \delta_j$. This gives the product formula $e_{j,k} e_{l,m} = \delta_{k,l} e_{j,m}$. Conventional matrix units for M_n are obtained by taking $S = \{1, 2, \ldots, n\}$, but we will sometimes want to take S to be a discrete (even finite) group. For $S = \{1, 2\}$, with the obvious choice of matrix representation, we get

$$e_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad e_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Example 10.8. We prove that if G is discrete and acts on itself by translation, then the crossed product is $K(l^2(G))$. When G is finite, this result is proved in Lemma 2.50 of [292]. (The conclusion is true for general locally compact groups. See Theorem 4.24 of [292].) More generally (compare with Remark 10.2, but we will not give a proof), if G acts on $G \times X$ by translation on the first factor and trivially on the second factor, then

$$C^*(G, \, G \times X) \cong K(l^2(G)) \otimes C_0(X) \cong C_0(X, \, K(l^2(G))).$$

In fact, the action on X need not be trivial. The map $(h, x) \mapsto (h, \, h^{-1}x)$ is an isomorphism from $G \times X$ with a general action of G on X to $G \times X$ with the trivial action of G on X. (For those familiar with the appropriate part of the representation theory of locally compact groups, this fact is related to the fact that the tensor product of the regular representation of a group and any other representation is a direct sum of copies of the regular representation.)

Let $\alpha \colon G \to \mathrm{Aut}(C_0(G))$ denote the action. For $g \in G$, we let u_g be the standard unitary as in Notation 8.7, and we let $\delta_g \in C_0(G)$ be the function $\chi_{\{g\}}$. Then $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$. Also, $\mathrm{span}\big(\{\delta_g \colon g \in G\}\big)$ is dense in $C_0(G)$. For $g, h \in G$, the element

$$v_{g,h} = \delta_g u_{gh^{-1}} \tag{10.1}$$

is in $C^*(G, \, C_0(G), \, \alpha)$. Moreover, for $g_1, h_1, g_2, h_2 \in G$, we have

$$\begin{aligned} v_{g_1,h_1} v_{g_2,h_2} &= \delta_{g_1} u_{g_1 h_1^{-1}} \delta_{g_2} u_{g_2 h_2^{-1}} \\ &= \delta_{g_1} \alpha_{g_1 h_1^{-1}}(\delta_{g_2}) u_{g_1 h_1^{-1}} u_{g_2 h_2^{-1}} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}. \end{aligned}$$

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is v_{g_1,h_2}. Similarly, $v_{g,h}^* = v_{h,g}$. That is, the elements $v_{g,h}$ satisfy the relations for a system of matrix units indexed by G. Also, $\mathrm{span}\big(\{v_{g,h} \colon g, h \in G\}\big)$ is dense in $l^1(G, \, C_0(G), \, \alpha)$, and hence in $C^*(G, \, C_0(G), \, \alpha)$.

For any finite set $F \subset G$, we thus get a homomorphism

$$\psi_F \colon L(l^2(F)) \to C_{\mathrm{c}}(G, \, C_0(G), \, \alpha)$$

sending the matrix unit $e_{g,h} \in L(l^2(F))$ (Notation 10.7) to $v_{g,h}$. Let

$$\varphi_F \colon L(l^2(F)) \to C^*(G, \, C_0(G), \, \alpha)$$

be the result of composing with the map from $C_{\mathrm{c}}(G, \, C_0(G), \, \alpha)$ to $C^*(G, \, C_0(G), \, \alpha)$. Set

$$K_0 = \bigcup_{F \subset G \text{ finite}} L(l^2(F)).$$

Putting our homomorphisms together gives a homomorphism

$$\varphi^{(0)} \colon K_0 \to C^*(G, \, C_0(G), \, \alpha).$$

Since for each F the restriction to $L(l^2(F))$ is a homomorphism of C*-algebras, it follows that $\big\| \varphi^{(0)}(x) \big\| \leq \|x\|$ for all $x \in K_0$. Therefore $\varphi^{(0)}$ extends by continuity to a homomorphism

$$\varphi\colon K(l^2(G)) \to C^*(G,\, C_0(G),\, \alpha).$$

The homomorphism φ is surjective because it has dense range, and it is injective because $K(l^2(G))$ is simple.

It follows that $C^*(G,\, C_0(G),\, \alpha)$ is simple. The natural map $C^*_{\mathrm{r}}(G,\, C_0(G),\, \alpha) \to C^*(G,\, C_0(G),\, \alpha)$ is then necessarily an isomorphism.

We point out that in Example 10.8, the full and reduced crossed products are the same even if the group is not amenable.

Example 10.9. Fix $n \in \mathbb{Z}_{>0}$, and consider the action of $G = \mathbb{Z}/n\mathbb{Z}$ on S^1 generated by rotation by $2\pi/n$, that is, the homeomorphism $h(\zeta) = e^{2\pi i/n}\zeta$ for $\zeta \in S^1$. (This action is from Example 2.16.)

We describe what to expect. Every point in S^1 has a closed invariant neighborhood which is equivariantly homeomorphic to $G \times I$ for some closed interval $I \subset \mathbb{R}$, with the translation action on G and the trivial action on I. This leads to quotients of $C^*(G, S^1, h)$ isomorphic to $M_n \otimes C(I)$. (See Theorem 8.32 and the general version of Example 10.8.) Since S^1 itself is not such a product, one does not immediately get an isomorphism $C^*(G, S^1, h) \cong M_n \otimes C(Y)$ for any Y. Instead, one gets the section algebra of a locally trivial bundle over Y with fiber M_n. However, the appropriate space Y is the orbit space $S^1/G \cong S^1$, and all locally trivial bundles over S^1 with fiber M_n are in fact trivial. Thus, one gets $C^*(G, S^1, h) \cong C(S^1, M_n)$ after all.

We carry out the details. Let $\alpha \in \mathrm{Aut}(C(S^1))$ be the order n automorphism given by $\alpha(f) = f \circ h^{-1}$ for $f \in C(S^1)$. Thus, $\alpha(f)(\zeta) = f(e^{-2\pi i/n}\zeta)$ for $\zeta \in S^1$. Let $s \in M_n$ be the shift unitary

$$s = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

The key computation, which we leave to the reader, is

$$s\,\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)\, s^* = \mathrm{diag}(\lambda_n, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \tag{10.2}$$

for $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$. Set

$$B = \bigl\{ f \in C([0,1],\, M_n)\colon f(0) = s f(1) s^* \bigr\}.$$

Define $\varphi_0\colon C(S^1) \to B$ by sending $f \in C(S^1)$ to the continuously varying diagonal matrix

$$\varphi_0(f)(t) = \mathrm{diag}\bigl(f\bigl(e^{2\pi i t/n}\bigr),\, f\bigl(e^{2\pi i (t+1)/n}\bigr),\, \ldots,\, f\bigl(e^{2\pi i (t+n-1)/n}\bigr) \bigr).$$

(For fixed t, the diagonal entries are obtained by evaluating f at the points in the orbit of $e^{2\pi i t/n}$.) The diagonal entries of $f(0)$ are gotten from those of $f(1)$ by a forwards cyclic shift, so $\varphi_0(f)$ really is in B. For the same reason, we get

$$\begin{aligned}\varphi_0(\alpha(f))(t) &= \operatorname{diag}\bigl(f\bigl(e^{2\pi i(t-1)/n}\bigr),\, f\bigl(e^{2\pi it/n}\bigr),\, \ldots,\, f\bigl(e^{2\pi i(t+n-2)/n}\bigr)\bigr) \\ &= s\varphi_0(f)(t)s^*.\end{aligned}$$

Now let $v \in C([0,1],\, M_n)$ be the constant function with value s. Then $v \in B$. The calculation just done implies that

$$\varphi_0\bigl(\alpha^k(f)\bigr) = v^k \varphi_0(f) v^{-k}$$

for $0 \leq k \leq n-1$. Also clearly $v^n = 1$. We write the group elements as $0,\, 1,\, \ldots,\, n-1$, by abuse of notation treating them as integers when convenient. The universal property of the crossed product therefore implies that there is a homomorphism $\varphi\colon C^*(G, S^1, h) \to B$ such that $\varphi|_{C(S^1)} = \varphi_0$ and (with u_k as in Notation 8.7) $\varphi(u_k) = v^k$ for $0 \leq k \leq n-1$.

We prove directly that φ is bijective. By Corollary 9.14, we can rewrite φ as the map $C(\mathbb{Z}/n\mathbb{Z} \times S^1) \to B$ given by

$$\varphi(f) = \sum_{k=0}^{n-1} \varphi_0(f(k, -))v^k.$$

Injectivity now reduces to the fact that if $a_0, a_1, \ldots, a_{n-1} \in M_n$ are diagonal matrices, and $\sum_{k=0}^{n-1} a_k s^k = 0$, then $a_0 = a_1 = \cdots = a_{n-1} = 0$. To see this explicitly, suppose that for $k = 0, 1, \ldots, n-1$, we have

$$a_k = \operatorname{diag}\bigl(\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_n^{(k)}\bigr)$$

with $\lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_n^{(k)} \in \mathbb{C}$. Then

$$\sum_{k=0}^{n-1} a_k s^k = \begin{pmatrix} \lambda_1^{(0)} & \lambda_1^{(n-1)} & \lambda_1^{(n-2)} & \cdots & \lambda_1^{(1)} \\ \lambda_2^{(1)} & \lambda_2^{(0)} & \lambda_2^{(n-1)} & \cdots & \lambda_2^{(2)} \\ \lambda_3^{(2)} & \lambda_3^{(1)} & \lambda_3^{(0)} & \cdots & \lambda_3^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_n^{(n-1)} & \lambda_n^{(n-2)} & \lambda_n^{(n-3)} & \cdots & \lambda_n^{(0)} \end{pmatrix}.$$

For surjectivity, let $a \in B$, and write

$$a(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) & \cdots & a_{1,n}(t) \\ a_{2,1}(t) & a_{2,2}(t) & \cdots & a_{2,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(t) & a_{n,2}(t) & \cdots & a_{n,n}(t) \end{pmatrix}$$

with $a_{j,k} \in C([0,1])$ for $1 \leq j, k \leq n$. The condition $a \in B$ implies that, taking the indices mod n in $\{1, 2, \ldots, n\}$, we have $a_{j,k}(1) = a_{j+1,\, k+1}(0)$ for all j and k. Therefore the formula

$$f\bigl(l,\, e^{2\pi i(t+j)/n}\bigr) = a_{j+1,\, j+1-l}(t)$$

for $t \in [0,1]$, $j = 1, 2, \ldots, n$, and $l = 0, 1, \ldots, n-1$, with $j+1-l$ taken mod n in $\{1, 2, \ldots, n\}$, gives a well defined element of $C(\mathbb{Z}/n\mathbb{Z} \times S^1)$. One checks that $\varphi(f) = a$.

It remains to prove that $B \cong C(S^1, M_n)$. Since $U(M_n)$ is connected, there is a unitary path $t \mapsto s_t$, defined for $t \in [0,1]$, such that $s_0 = 1$ and $s_1 = s$. Define $\psi\colon C(S^1, M_n) \to B$ by $\psi(f)(t) = s_t^* f(e^{2\pi i t}) s_t$. For $f \in C(S^1, M_n)$, we have

$$\psi(f)(1) = s^* f(1) s = s^* \psi(f)(0) s,$$

so $\psi(f)$ really is in B. It is easily checked that ψ is bijective.

Example 10.10. Let $X = S^n = \{x \in \mathbb{R}^{n+1} \colon \|x\|_2 = 1\}$, and let $\mathbb{Z}/2\mathbb{Z}$ act by sending the nontrivial group element to the order 2 homeomorphism $x \mapsto -x$. (This is Example 2.28.) The "local structure" of the crossed product $C^*(\mathbb{Z}/2\mathbb{Z},\, X)$ is the same as in Example 10.9. However, for $n \geq 2$ the resulting bundle is no longer trivial. The crossed product is isomorphic to the section algebra of a locally trivial but nontrivial bundle over the real projective space $\mathbb{R}P^n = S^n/(\mathbb{Z}/2\mathbb{Z})$ with fiber M_2. See Proposition 4.15 of [292].

The bundles one gets from free proper actions are, however, often stably trivial. Theorem 14 of [98] implies that the bundle always comes from a bundle of Hilbert spaces, and, if the algebra is separable, Theorem 10.7.15 of [60] implies that the Dixmier-Douady invariant is zero. If the fibers are infinite dimensional, and if the quotient space has finite covering dimension or if the map $X \to X/G$ is locally trivial, then the crossed product is $K \otimes C_0(X/G)$. See Theorems 10.8.4 and 10.8.8 of [60], and Corollary 15 of [98]. Proposition 2.52 of [292] gives a fairly explicit description of the crossed product by a free action of $\mathbb{Z}/2\mathbb{Z}$ on a compact space X, although the question of triviality of the resulting bundle is not addressed.

Example 10.11. Let $X = \mathbb{Z}/n\mathbb{Z}$, and let $\mathbb{Z}$ act on X by translation. We will give a direct proof that that $C^*(\mathbb{Z}, X) \cong M_n \otimes C(S^1)$. This is a special case of Example 2.12. In the general case (see Theorem 10.13 below), it turns out that

$$C^*(G,\, G/H) \cong K(L^2(G/H)) \otimes C^*(H).$$

There is no twisting.

Identify $\mathbb{Z}/n\mathbb{Z}$ with $\{1, 2, \ldots, n\}$. (We start at 1 instead of 0 to be consistent with common matrix unit notation.) Let $\alpha \in \mathrm{Aut}(C(\mathbb{Z}/n\mathbb{Z}))$ be $\alpha(f)(k) = f(k-1)$, with the argument taken mod n in $\{1, 2, \ldots, n\}$. (Equivalently, $\alpha(\chi_{\{k\}}) = \chi_{\{k+1\}}$, with $k+1$ taken to be 1 when $k = n$.) In $C(S^1)$ let z be the function $z(\zeta) = \zeta$ for all ζ. In $M_n(C(S^1)) \cong M_n \otimes C(S^1)$, abbreviate $e_{j,k} \otimes 1$ to $e_{j,k}$, and let v be the unitary

$$v = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & z \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

(This unitary differs from the unitary s in Example 10.9 only in that here the upper right corner entry is z instead of 1.)

Define $\varphi_0\colon C(\mathbb{Z}/n\mathbb{Z}) \to M_n \otimes C(S^1)$ by $\varphi_0(\chi_{\{k\}}) = e_{k,k}$ for $k = 1, 2, \ldots, n$, and extending linearly. Then one checks that $v\varphi_0(f)v^* = \varphi_0(\alpha(f))$ for all $f \in C(\mathbb{Z}/n\mathbb{Z})$. Letting u be the standard unitary in $C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ from the generator $1 \in \mathbb{Z}$ (called

u_1 in Notation 8.7), there is therefore a homomorphism $\varphi\colon C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to M_n \otimes C(S^1)$ such that $\varphi|_{C(\mathbb{Z}/n\mathbb{Z})} = \varphi_0$ and $\varphi(u) = v$. We claim that φ is an isomorphism.

The following description of $M_n \otimes C(S^1)$ will be useful: it is the universal unital C*-algebra generated by a system $(e_{j,k})_{1 \leq j,k \leq n}$ of matrix units such that $\sum_{j=1}^{n} e_{j,j} = 1$ and a central unitary y. The generators $e_{j,k}$ are the matrix units we have already used, and the central unitary is $1 \otimes z$. (Proof: Exercise 10.12 below.)

To prove that φ is surjective, it suffices to prove that its image contains $1 \otimes z$ and contains $e_{j,k}$ for $j, k = 1, 2, \ldots, n$. The image contains $1 \otimes z$ because $v^n = 1 \otimes z$. For $j = 1, 2, \ldots, n$, the image contains $e_{j,j} = \varphi_0(\chi_{\{j\}})$. The image therefore also contains $e_{j+1,\,j} = e_{j+1,\,j+1} v e_{j,j}$ for $j = 1, 2, \ldots, n-1$. It now easily follows that the image contains $e_{j,k}$ for all j and k.

To prove injectivity, we claim that it suffices to prove that whenever A is a unital C*-algebra, $\psi_0\colon C(\mathbb{Z}/n\mathbb{Z}) \to A$ is a unital homomorphism, and $w \in A$ is a unitary such that $w\psi_0(f)w^* = \psi_0(\alpha(f))$ for all $f \in C(\mathbb{Z}/n\mathbb{Z})$, then there is a homomorphism $\gamma\colon M_n \otimes C(S^1) \to A$ such that $\gamma \circ \varphi_0 = \psi_0$ and $\gamma(v) = w$. To prove the claim, we use the universal property of the crossed product (Theorem 8.21). Take $A = C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$, let ψ_0 be the inclusion of $C(\mathbb{Z}/n\mathbb{Z})$ in $C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$, and let $w = u$ (the standard unitary in $C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$). Let $\gamma\colon M_n \otimes C(S^1) \to C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ be the corresponding homomorphism. Then

$$\gamma \circ \varphi\colon C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$$

satisfies $(\gamma \circ \varphi)(a) = a$ for $a \in C(\mathbb{Z}/n\mathbb{Z})$ and $(\gamma \circ \varphi)(u) = u$. So $\gamma \circ \varphi = \mathrm{id}_{C^*(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})}$. Therefore φ is injective.

It remains to construct γ, and it suffices to define γ on the generators. For $j = 1, 2, \ldots, n$, we define $f_{j,j} = \psi_0(\chi_{\{j\}})$. For $1 \leq k < j \leq n$, we define $f_{j,k} = f_{j,j} w^{j-k} f_{k,k}$ and $f_{k,j} = f_{j,k}^*$. One easily checks that $(f_{j,k})_{1 \leq j,k \leq n}$ is a system of matrix units such that $\sum_{j=1}^{n} f_{j,j} = 1$, and that w^n is a unitary which commutes with $f_{j,k}$ for $j, k = 1, 2, \ldots, n$. Accordingly, we may define γ by $\gamma(1 \otimes z) = w^n$ and $\gamma(e_{j,k}) = f_{j,k}$ for $j, k = 1, 2, \ldots, n$. It is obvious that $\gamma \circ \varphi_0 = \psi_0$. To compute $(\gamma \circ \varphi)(u)$, we observe that

$$\varphi(u) = v = (1 \otimes z) e_{1,n} + \sum_{j=1}^{n-1} e_{j+1,\,j}.$$

Therefore, using the definitions of the $f_{j,k}$ for $j \neq k$ at the third step and the relations $f_{j+1,\,j+1} = w f_{j,j} w^*$ for $j = 1, 2, \ldots, n-1$ and $f_{1,1} = w^{-(n-1)} f_{n,n} w^{n-1}$ at the fourth step, we get

$$\begin{aligned}
(\gamma \circ \varphi)(u) = \gamma(v) &= w^n f_{1,n} + \sum_{j=1}^{n-1} f_{j+1,\,j} \\
&= w^n f_{1,1} w^{-(n-1)} f_{n,n} + \sum_{j=1}^{n-1} f_{j+1,\,j+1} w f_{j,j} = w f_{n,n} + \sum_{j=1}^{n-1} w f_{j,j} = w.
\end{aligned}$$

This completes the proof.

Exercise 10.12. Prove the description of $M_n \otimes C(S^1)$ in terms of generators and relations used in Example 10.11.

The relations are essentially the ones which define $M_n \otimes_{\max} C(S_1)$.

The outcome of Example 10.11 holds much more generally.

Theorem 10.13 (Corollary 2.10 of [99]; Theorem 4.30 of [292])**.** Let G be a locally compact group, let $H \subset G$ be a closed subgroup, and let G act on G/H by translation. (This is Example 2.12.) Assume that there is a measurable cross section from G/H to G. Then

$$C^*(G, G/H) \cong K(L^2(G/H)) \otimes C^*(H).$$

There is no twisting.

Several generalizations are worth mentioning.

Theorem 10.14 (Corollary 2.8 of [99])**.** Let G be a locally compact group, let $H \subset G$ be a closed subgroup, and let G act on G/H by translation. Assume that there is a measurable cross section from G/H to G. Let G also act on a C*-algebra A. Then, using the diagonal action,

$$C^*(G, C_0(G/H) \otimes A) \cong K(L^2(G/H)) \otimes C^*(H, A).$$

Theorem 10.15 (Theorem 4.1 of [99])**.** Let G be a locally compact group, let $H \subset G$ be a closed subgroup, and let G act on G/H by translation. Let X be a locally compact G-space such that there is a surjective continuous equivariant map $p\colon X \to G/H$. Assume that there is a measurable cross section from G/H to G. Let Y be the inverse image under p of the point $H \in G/H$. Then

$$C^*(G, X) \cong K(L^2(G/H)) \otimes C^*(H, Y).$$

Example 10.16. The following example (not done in detail here) combines the features of Examples 10.9 and 10.11.

Regard the action of Example 10.9 as an action of $\mathbb{Z}$ rather than of $\mathbb{Z}/n\mathbb{Z}$. (This action of $\mathbb{Z}$ also appears in Example 2.16, where it is called a rational rotation.) That is, fix $n \in \mathbb{Z}_{>0}$, and consider the action of $G = \mathbb{Z}$ on S^1 generated by rotation by $2\pi/n$, equivalently, generated by the homeomorphism $h(\zeta) = e^{2\pi i/n}\zeta$ for $\zeta \in S^1$.

The crossed product is a special case of what is known as a rational rotation algebra. (The general case uses generating rotations by $2\pi k/n$, not just $2\pi/n$.) The heuristic argument of Example 10.9 and the outcome of Example 10.11 suggest that the crossed product should be the section algebra of a locally trivial bundle over S^1 with fiber $C(S^1, M_n)$. It is not hard to show that this is in fact what happens. (Exercise: Do it.) The resulting bundle is not trivial. In fact, it can be easily seen that it is also the section algebra of a locally trivial bundle over $S^1 \times S^1$ with fiber M_n. This bundle is also nontrivial. The bundles for general rational rotation algebras are computed in [116]. (See Example 8.46 of [292].)

Remark 10.17. In Examples 10.9 and 10.11, we have seen two sources of ideals in a reduced crossed product $C^*_{\mathrm{r}}(G, A, \alpha)$: invariant ideals in A, and group elements which act trivially on A. There is a theorem due to Gootman and Rosenberg which gives a description of the primitive ideals of any crossed product $C^*(G, A)$ with G amenable, and which, very roughly, says that they all come from some combination of these two sources. (One does not even need to restrict to discrete groups.) To be a little more precise, every primitive ideal in $C^*(G, A)$ is "induced" from an ideal J in a crossed product by the stabilizer subgroup of some primitive ideal P of A, with J closely related to P. The theorem is Theorem 8.21 of [292]; see Definition 8.18 of [292] for the terminology. The proof of the Gootman-Rosenberg Theorem is quite long. (Starting from about the same assumed background as these notes, it occupies a large part of the book [292].)

Example 10.18. Take $X = S^1 = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\}$, and let $\mathbb{Z}/2\mathbb{Z}$ act by sending the nontrivial group element to the order two homeomorphism $\zeta \mapsto \overline{\zeta}$. (This is Example 2.29.) Let $\alpha \in \mathrm{Aut}(C(S^1))$ be the corresponding automorphism. We compute the crossed product, but we first describe what to expect. By considering Theorem 8.32 and Examples 10.1 and 10.9, we should expect that the points 1 and -1 contribute quotients isomorphic to $\mathbb{C} \oplus \mathbb{C}$, and that for $\zeta \neq \pm 1$, the pair of points $\big(\zeta, \overline{\zeta}\big)$ contributes a quotient isomorphic to M_2. We will in fact show that $C^*(\mathbb{Z}/2\mathbb{Z}, \, X)$ is isomorphic to the C*-algebra

$$B = \big\{ f \in C([-1, \, 1], \, M_2) \colon f(1) \text{ and } f(-1) \text{ are diagonal matrices} \big\}.$$

First, let $C_0 \subset M_2$ be the subalgebra consisting of all matrices of the form $\left(\begin{smallmatrix} \lambda & \mu \\ \mu & \lambda \end{smallmatrix}\right)$ with $\lambda, \mu \in \mathbb{C}$. (The reader should check that C_0 is actually a subalgebra.) Then define

$$C = \big\{ f \colon [-1, \, 1] \to M_2 \colon f \text{ is continuous and } f(1), \, f(-1) \in C_0 \big\}.$$

Let $v \in C$ be the constant function $v(t) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for all $t \in [-1, \, 1]$. Define $\varphi_0 \colon C(S^1) \to C$ by

$$\varphi_0(f)(t) = \begin{pmatrix} f\big(t + i\sqrt{1 - t^2}\big) & 0 \\ 0 & f\big(t - i\sqrt{1 - t^2}\big) \end{pmatrix}$$

for $f \in C(S^1)$ and $t \in [-1, \, 1]$. One checks that the conditions at ± 1 for membership in C are satisfied. Moreover, $v^2 = 1$ and $v \varphi_0(f) v^* = \varphi_0(\alpha(f))$ for $f \in C(S^1)$. Therefore there is a homomorphism $\varphi \colon C^*(\mathbb{Z}/2\mathbb{Z}, \, X) \to C$ such that $\varphi|_{C(S^1)} = \varphi_0$ and φ sends the standard unitary u in $C^*(\mathbb{Z}/2\mathbb{Z}, \, X)$ to v. It is given by the formula

$$\varphi(f_0 + f_1 u)(t) = \begin{pmatrix} f_0\big(t + i\sqrt{1 - t^2}\big) & f_1\big(t + i\sqrt{1 - t^2}\big) \\ f_1\big(t - i\sqrt{1 - t^2}\big) & f_0\big(t - i\sqrt{1 - t^2}\big) \end{pmatrix}$$

for $f_1, f_2 \in C(S^1)$ and $t \in [-1, \, 1]$.

We claim that φ is an isomorphism. Since

$$C^*(\mathbb{Z}/2\mathbb{Z}, \, X) = \big\{ f_0 + f_1 u \colon f_1, f_2 \in C(S^1) \big\}$$

by Corollary 9.14, it is easy to check injectivity. For surjectivity, let

$$a(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) \\ a_{2,1}(t) & a_{2,2}(t) \end{pmatrix}$$

define an element $a \in C$. Then

$$a_{1,1}(-1) = a_{2,2}(-1) \qquad \text{and} \qquad a_{2,1}(-1) = a_{1,2}(-1), \tag{10.3}$$

and

$$a_{1,1}(1) = a_{2,2}(1) \qquad \text{and} \qquad a_{2,1}(1) = a_{1,2}(1). \tag{10.4}$$

Now set

$$f_0(\zeta) = \begin{cases} a_{1,1}(\mathrm{Re}(\zeta)) & \mathrm{Im}(\zeta) \geq 0 \\ a_{2,2}(\mathrm{Re}(\zeta)) & \mathrm{Im}(\zeta) \leq 0 \end{cases}$$

and

$$f_1(\zeta) = \begin{cases} a_{1,2}(\mathrm{Re}(\zeta)) & \mathrm{Im}(\zeta) \geq 0 \\ a_{2,1}(\mathrm{Re}(\zeta)) & \mathrm{Im}(\zeta) \leq 0 \end{cases}$$

for $\zeta \in S^1$. The relations (10.3) and (10.4) ensure that f_0 and f_1 are well defined at ± 1, and are continuous. One easily checks that $\varphi(f_0 + f_1 u) = a$. This proves surjectivity.

The algebra C is not quite what was promised. Set

$$w = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$

which is a unitary in M_2. Then the required isomorphism $\psi \colon C^*(\mathbb{Z}/2\mathbb{Z},\, X) \to B$ is given by $\psi(a)(t) = w\varphi(a)(t)w^*$. (Check this!)

In this example, one choice of matrix units in M_2 was convenient for the free orbits, while another choice was convenient for the fixed points. It seemed better to compute everything in terms of the choice convenient for the free orbits, and convert afterwards.

Exercise 10.19. Let $\mathbb{Z}/2\mathbb{Z}$ act on $[-1, 1]$ via $x \mapsto -x$. Compute the crossed product.

Exercise 10.20. Let $\mathbb{Z}/2\mathbb{Z}$ act on

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \colon x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$

via $(x_1, x_2, \ldots, x_n, x_{n+1}) \mapsto (x_1,\, x_2,\, \ldots,\, x_n,\, -x_{n+1})$. Compute the crossed product.

In [78], there is a detailed analysis of the structure of crossed products of compact spaces by compact groups, in terms of sections of suitable bundles of C*-algebras, usually not locally trivial but locally trivial over suitable subspaces of the base space.

The crossed products and fixed point algebras of the actions of finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$ (discussed in Example 3.12) on the rational rotation algebras (take $\theta \in \mathbb{Q}$ in Example 3.12; the case $\theta = 0$ is the action on $S^1 \times S^1$ in Example 2.30) have been computed in Theorems 6.1, 1.2, and 1.3 of [31] (for $\mathbb{Z}/2\mathbb{Z}$), in the theorem at the end of Section 1 of [84] (for $\mathbb{Z}/3\mathbb{Z}$), in Theorem 6.2.1 of [83] (for $\mathbb{Z}/4\mathbb{Z}$), and in the theorem at the end of Section 1 of [85] (for $\mathbb{Z}/6\mathbb{Z}$). (For $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$, the proofs are only given for the corresponding computation of the fixed point algebras.) The rational rotation algebras are not commutative, but they are close to commutative, being section algebras of locally trivial bundles over $S^1 \times S^1$ whose fiber is a single matrix algebra.

Example 10.21. We compute the crossed product by one of the specific examples at the end of Example 3.25, namely the action of $\mathbb{Z}/2\mathbb{Z}$ on the 2^∞ UHF algebra A generated by $\bigotimes_{n=1}^{\infty} \mathrm{Ad}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We simply write α for the automorphism given by the nontrivial group element. (In Example 13.6, this action is shown to have the Rokhlin property.)

Write $A = \varinjlim M_{2^n}$, with maps $\varphi_n \colon M_{2^n} \to M_{2^{n+1}}$ given by $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ for $a \in M_{2^n}$ and $n \in \mathbb{Z}_{\geq 0}$. Define unitaries $z_n \in M_{2^n}$ inductively by $z_0 = 1$ and $z_{n+1} = \left(\begin{smallmatrix} z_n & 0 \\ 0 & -z_n \end{smallmatrix}\right)$. (In tensor product notation, and with an appropriate choice of isomorphism $M_{2^n} \otimes M_2 \to M_{2^{n+1}}$, these are $\varphi_n(a) = a \otimes 1_{M_2}$ and $z_{n+1} = z_n \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.)

Let

$$\overline{\varphi}_n \colon C^*\big(\mathbb{Z}/2\mathbb{Z},\, M_{2^n},\, \mathrm{Ad}(z_n)\big) \to C^*\big(\mathbb{Z}/2\mathbb{Z},\, M_{2^{n+1}},\, \mathrm{Ad}(z_{n+1})\big)$$

be the corresponding map on the crossed products. By Theorem 8.34, the crossed product $C^*(\mathbb{Z}/2\mathbb{Z},\, A,\, \alpha)$ is the direct limit of the resulting direct system.

In the crossed product $C^*\big(\mathbb{Z}/2\mathbb{Z},\, M_{2^n},\, \mathrm{Ad}(z_n)\big)$, let u_n be the standard unitary corresponding to the nontrivial group element. (This notation is not entirely consistent with Notation 8.7.) From the discussion at the end of Example 10.4, we get the isomorphisms

$$\sigma_n \colon C^*\big(\mathbb{Z}/2\mathbb{Z},\, M_{2^n},\, \mathrm{Ad}(z_n)\big) \to M_{2^n} \oplus M_{2^n}$$

given by $a + b u_n \mapsto (a + b z_n,\, a - b z_n)$. We now need a map

$$\psi_n \colon M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}} \oplus M_{2^{n+1}}$$

which makes the following diagram commute:

$$\begin{array}{ccc} C^*\big(\mathbb{Z}/2\mathbb{Z},\, M_{2^n},\, \mathrm{Ad}(z_n)\big) & \xrightarrow{\ \sigma_n\ } & M_{2^n} \oplus M_{2^n} \\ {\scriptstyle \overline{\varphi}_n}\big\downarrow & & \big\downarrow{\scriptstyle \psi_n} \\ C^*\big(\mathbb{Z}/2\mathbb{Z},\, M_{2^{n+1}},\, \mathrm{Ad}(z_{n+1})\big) & \xrightarrow{\ \sigma_{n+1}\ } & M_{2^{n+1}} \oplus M_{2^{n+1}}. \end{array}$$

That is, ψ_n sends

$$\sigma_n(a + b u_n) = (a + b z_n,\, a - b z_n)$$

to

$$\begin{aligned} &\sigma_{n+1}(\varphi_n(a) + \varphi_n(b) u_{n+1}) \\ &\quad = \left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} z_n & 0 \\ 0 & -z_n \end{pmatrix},\ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} - \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} z_n & 0 \\ 0 & -z_n \end{pmatrix} \right) \\ &\quad = \left(\begin{pmatrix} a + b z_n & 0 \\ 0 & a - b z_n \end{pmatrix},\ \begin{pmatrix} a - b z_n & 0 \\ 0 & a + b z_n \end{pmatrix} \right). \end{aligned}$$

So we take

$$\psi_n(b, c) = \left(\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix},\ \begin{pmatrix} c & 0 \\ 0 & b \end{pmatrix} \right)$$

for $b, c \in M_{2^n}$.

Those familiar with Bratteli diagrams will now be able to write down the Bratteli diagram for the crossed product. Here, we give a direct identification of the direct limit. Inductively define unitaries $x_n, y_n \in M_{2^n}$ by $x_0 = y_0 = 1$ and

$$x_{n+1} = \begin{pmatrix} x_n & 0 \\ 0 & y_n \end{pmatrix} \qquad \text{and} \qquad y_{n+1} = \begin{pmatrix} 0 & y_n \\ x_n & 0 \end{pmatrix}$$

for $n \in \mathbb{Z}_{\geq 0}$. Then define $\lambda_n \colon M_{2^n} \to M_{2^n} \oplus M_{2^n}$ by $\lambda_n(a) = (x_n a x_n^*,\, y_n a y_n^*)$ for $a \in M_{2^n}$, and define $\mu_n \colon M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}}$ by

$$\mu_n(b, c) = \begin{pmatrix} x_n^* b x_n & 0 \\ 0 & y_n^* c y_n \end{pmatrix}$$

for $b, c \in M_{2^n}$. Then one checks that $\mu_n \circ \lambda_n = \varphi_n$ and $\lambda_{n+1} \circ \mu_n = \psi_n$ for all n. It follows that the direct limit of the system

$$\mathbb{C} \oplus \mathbb{C} \xrightarrow{\psi_0} M_2 \oplus M_2 \xrightarrow{\psi_1} M_4 \oplus M_4 \xrightarrow{\psi_2} M_8 \oplus M_8 \xrightarrow{\psi_3} \cdots,$$

which is the crossed product $C^*(\mathbb{Z}/2\mathbb{Z},\, A,\, \alpha)$, is isomorphic to the direct limit of the system

$$\mathbb{C} \xrightarrow{\varphi_0} M_2 \xrightarrow{\varphi_1} M_4 \xrightarrow{\varphi_2} M_8 \xrightarrow{\varphi_3} \cdots,$$

which is the original algebra A.

It follows from Lemma 3.26 that the action in Example 10.21 is not inner. The result of the computation of the crossed product implies this as well. Indeed, the crossed product is simple, so comparison with Example 10.4 shows that the action is not inner.

The theorem of Gootman and Rosenberg described in Remark 10.17 gives no information here.

The fact that we got the same algebra back in Example 10.21 is somewhat special, but the general principle of the computation is much more generally applicable. We sketch a slightly different example in which we do not get the same algebra back.

Example 10.22. Let α the action of $\mathbb{Z}/2\mathbb{Z}$ on the 3^{∞} UHF algebra A generated by

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

for $n \in \mathbb{Z}_{>0}$ and $a \in M_{3^n}$. Again, we also write α for the automorphism given by the nontrivial group element. (This action has the tracial Rokhlin property but not the Rokhlin property. See Remark 14.9 and Example 13.23.)

Write $A = \varinjlim M_{3^n}$, with maps $\varphi_n \colon M_{3^n} \to M_{3^{n+1}}$ given by $a \mapsto \mathrm{diag}(a, a, a)$ for $n \in \mathbb{Z}_{\geq 0}$ and $a \in M_{3^n}$. Define unitaries $z_n \in M_{3^n}$ inductively by $z_0 = 1$ and

$$z_{n+1} = \begin{pmatrix} z_n & 0 & 0 \\ 0 & z_n & 0 \\ 0 & 0 & -z_n \end{pmatrix}.$$

Let

$$\overline{\varphi}_n \colon C^*\big(\mathbb{Z}/2\mathbb{Z},\, M_{3^n},\, \mathrm{Ad}(z_n)\big) \to C^*\big(\mathbb{Z}/2\mathbb{Z},\, M_{3^{n+1}},\, \mathrm{Ad}(z_{n+1})\big)$$

be the corresponding map on the crossed products, so that $C^*(\mathbb{Z}/2\mathbb{Z},\, A,\, \alpha)$ is the direct limit of the resulting direct system. Let $u_n \in C^*\big(\mathbb{Z}/2\mathbb{Z},\, M_{3^n},\, \mathrm{Ad}(z_n)\big)$ be the standard unitary, as in Example 10.21. The isomorphism

$$\sigma_n \colon C^*\big(\mathbb{Z}/2\mathbb{Z},\, M_{3^n},\, \mathrm{Ad}(z_n)\big) \to M_{3^n} \oplus M_{3^n}$$

is still $a + b u_n \mapsto (a + b z_n,\, a - b z_n)$. Using calculations similar to those of Example 10.21, one sees that the map

$$\psi_n \colon M_{3^n} \oplus M_{3^n} \to M_{3^{n+1}} \oplus M_{3^{n+1}}$$

should now be given by

$$\psi_n(b, c) = \big(\mathrm{diag}(b, b, c),\ \mathrm{diag}(c, c, b)\big).$$

Again, one can immediately write down the Bratteli diagram for the crossed product. Instead, we directly calculate the (unordered) K_0-group of the crossed product. It is the direct limit $\varinjlim K_0(M_{3^n} \oplus M_{3^n})$, with the maps being

$$(\psi_n)_* \colon K_0(M_{3^n} \oplus M_{3^n}) \to K_0(M_{3^{n+1}} \oplus M_{3^{n+1}}).$$

The calculation is based on the observation that the map $(\psi_n)_* \colon \mathbb{Z}^2 \to \mathbb{Z}^2$ is given by the matrix $(\psi_n)_* = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, which has eigenvector $(1, -1)$ with eigenvalue 1 and eigenvector $(1, 1)$ with eigenvalue 3. (Usually one will not be so lucky: the calculations will be messier.)

We claim that we can identify $K_0\big(C^*(\mathbb{Z}/2\mathbb{Z},\, A,\, \alpha)\big)$ with

$$H = \left\{ (k, l) \in \mathbb{Z} \oplus \mathbb{Z}\left[\tfrac{1}{3}\right] : k + l \in 2 \cdot \mathbb{Z}\left[\tfrac{1}{3}\right] \right\},$$

and with the class [1] being sent to $(0, 1)$. For $n \in \mathbb{Z}_{\geq 0}$, define $f_n \colon \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}\left[\frac{1}{3}\right]$ by

$$f_n(r, s) = \left(r - s, \, \frac{r+s}{3^n} \right)$$

for $r, s \in \mathbb{Z}$. One checks immediately that $f_n = f_{n+1} \circ (\psi_n)_*$. Therefore the group homomorphisms f_n combine to yield a homomorphism $f \colon \varinjlim \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}\left[\frac{1}{3}\right]$, whose range is easily seen to be in H. This homomorphism is injective because f_n is injective for all $n \in \mathbb{Z}_{\geq 0}$.

It remains only to show that if $(r, s) \in H$ then there exist $n \in \mathbb{Z}_{\geq 0}$ and $k, l \in \mathbb{Z}$ such that $f_n(k, l) = (r, s)$. Choose $n \in \mathbb{Z}_{\geq 0}$ such that $3^n s \in \mathbb{Z}$. Set

$$k = \frac{3^n s + r}{2} \qquad \text{and} \qquad l = \frac{3^n s - r}{2}.$$

It is easy to see that either $r \in 2\mathbb{Z}$ and $s \in 2 \cdot \mathbb{Z}\left[\frac{1}{3}\right]$ or $r \notin 2\mathbb{Z}$ and $s \notin 2 \cdot \mathbb{Z}\left[\frac{1}{3}\right]$, and that in either case $k, l \in \mathbb{Z}$. Thus $(k, l) \in \mathbb{Z}^2$, and clearly $f_n(k, l) = (r, s)$. This completes the calculation.

The following two exercises are much harder than most of the exercises in these notes. The first combines the methods of Example 10.18 (see Exercise 10.20) and the methods of Example 10.22, and the second uses Example 10.10 in place of Exercise 10.20. The computations asked for in the exercises are an important part of Propositions 4.6 and 4.2 of [209], which describe the properties of two significant examples of crossed products. Both actions are shown in [209] to have the tracial Rokhlin property, but do not have the Rokhlin property.

Exercise 10.23. Let $m \in \mathbb{Z}_{>0}$. Define $h \colon S^{2m} \to S^{2m}$ by

$$h(x_0, x_1, \ldots, x_{2m}) = (-x_0, x_1, \ldots, x_{2m})$$

for $x = (x_0, x_1, \ldots, x_{2m}) \in S^{2m}$, and let $\beta \in \mathrm{Aut}(C(S^{2m}))$ be the corresponding automorphism of order 2. For $r \in \mathbb{Z}_{>0}$ and $b \in S^{2m}$, define $\psi_{r,b} \colon C(S^{2m}) \to M_{2r+1} \otimes C(S^{2m})$ by

$$\psi_{r,b}(f)(x) = \mathrm{diag}\big(f(x), \, f(b), \, f(h(b)), \, f(b), \, f(h(b)), \, \ldots, \, f(b), \, f(h(b))\big)$$

for $x \in S^{2m}$, where $f(b)$ and $f(h(b))$ each occur r times. Choose a dense sequence $(x(n))_{n \in \mathbb{Z}_{>0}}$ in S^{2m}, such that no point x_n is a fixed point of h, and choose a sequence $(r(n))_{n \in \mathbb{Z}_{>0}}$ of strictly positive integers. Set

$$s(n) = [2r(1) + 1][2r(2) + 1] \cdots [2r(n) + 1],$$

and set $A_n = M_{s(n)} \otimes C(S^{2m})$, which, when appropriate, we think of as

$$M_{2r(1)+1} \otimes M_{2r(2)+1} \otimes \cdots \otimes M_{2r(n)+1} \otimes C(S^{2m}).$$

Define $\varphi_n \colon A_{n-1} \to A_n$ by $\varphi_n = \mathrm{id}_{M_{s(n-1)}} \otimes \psi_{r(n), \, x(n)}$. Then set $A = \varinjlim A_n$.

For $r \in \mathbb{Z}_{>0}$ define a unitary $w_r \in M_{2r+1}$ by

$$w_r = \mathrm{diag}\left(1, \, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \, \ldots, \, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

Then define an automorphism $\alpha_n \in \mathrm{Aut}(A_n)$ of order 2 by

$$\alpha_n = \mathrm{Ad}(w_{r(1)} \otimes w_{r(2)} \otimes \cdots \otimes w_{r(n)}) \otimes \beta.$$

One checks that $\varphi_n \circ \alpha_{n-1} = \alpha_n \circ \varphi_n$, so that the automorphisms α_n define an automorphism $\alpha \in \mathrm{Aut}(A)$ of order 2.

Compute the crossed product $C^*(\mathbb{Z}/2\mathbb{Z},\, A, \alpha)$, at least sufficiently well to determine its K-theory.

Exercise 10.24. Repeat Exercise 10.23, with just one change: the formula for h is now $h(x) = -x$ for all $x \in S^{2m}$. The algebras in the direct system for the crossed product are harder to describe, since they are section algebras of nontrivial bundles (see Example 10.10), but the full description is not needed in order to compute the K-theory of the resulting direct limit.

Example 10.25. Let $\theta \in \mathbb{R}$. Recall from Example 3.10 that the rotation algebra A_θ is the universal C*-algebra generated by unitaries u and v satisfying $vu = e^{2\pi i \theta} uv$.

Let $h_\theta \colon S^1 \to S^1$ be the homeomorphism $h_\theta(\zeta) = e^{2\pi i \theta}\zeta$. (Recall Example 2.16.) We claim that there is an isomorphism $\varphi \colon A_\theta \to C^*(\mathbb{Z}, S^1, h_\theta)$ which sends u to the standard unitary u_1 in the crossed product (see Notation 8.7), and sends v to the function $z \in C(S^1)$ defined by $z(\zeta) = \zeta$ for all $\zeta \in S^1$. (In Corollary 8.23, the unitary u_1 was called u, so we are essentially sending u to u.)

The proof of the claim is by comparison of universal properties. First, one checks that $z u_1 = e^{2\pi i \theta} u_1 z$, so at least there is a homomorphism φ with the properties claimed. Next, define a homomorphism $\psi_0 \colon C(S^1) \to A_\theta$ by $\psi_0(f) = f(v)$ (continuous functional calculus) for $f \in C(S^1)$. For $n \in \mathbb{Z}$, we have, using $vu = e^{2\pi i \theta} uv$ at the second step,

$$u \psi_0(z^n) u^* = (u v u^*)^n = e^{-2\pi i n \theta} v^n = \psi_0\big(e^{-2\pi i n \theta} z^n\big) = \psi_0\big(z^n \circ h_\theta^{-1}\big).$$

Since the functions z^n span a dense subspace of $C(S^1)$, it follows that $u \psi_0(f) u^* = \psi_0\big(f \circ h_\theta^{-1}\big)$ for all $f \in C(S^1)$. By Corollary 8.23, there is a homomorphism $\psi \colon C^*(\mathbb{Z}, S^1, h_\theta) \to A_\theta$ such that $\psi|_{C(S^1)} = \psi_0$ and $\psi(u_1) = u$.

We have $(\psi \circ \varphi)(u) = u$ and $(\psi \circ \varphi)(v) = v$. Since u and v generate A_θ, we conclude that $\psi \circ \varphi = \mathrm{id}_{A_\theta}$. Similarly, $(\varphi \circ \psi)(z) = z$ and $(\varphi \circ \psi)(u_1) = u_1$, the elements z and u_1 generate $C^*(\mathbb{Z}, S^1, h_\theta)$ (since z generates $C(S^1)$), and therefore $\varphi \circ \psi = \mathrm{id}_{C^*(\mathbb{Z}, S^1, h_\theta)}$.

We will see below that for $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the algebra $C^*(\mathbb{Z}, S^1, h_\theta)$ is simple. (See Theorems 15.10 and 15.12 below, and also Proposition 2.56 of [292].) On the other hand, if $\theta = p/q$ in lowest terms, with $q > 0$, then A_θ turns out to be the section algebra of a locally trivial bundle over $S^1 \times S^1$ with fiber M_q. (See Example 8.46 of [292].) The bundles have trivial Dixmier-Douady class, so are stably trivial, but they are not trivial. They are analyzed in [116].

We finish this section with several further results on crossed products by tensor products of actions, given as exercises. Remark 10.2 and Exercise 10.3 can be obtained from Exercise 10.26 and Exercise 10.27 by taking $H = G$ and restricting to the diagonal subgroup $\{(g, g) \colon g \in G\} \subset G \times G$, or (for Exercise 10.3) taking H to be trivial.

Exercise 10.26. Let G and H be topological groups, let A and B be C*-algebras, and let $\alpha \colon G \to \mathrm{Aut}(A)$ and $\beta \colon H \to \mathrm{Aut}(B)$ be actions of G and H on A and B.

(1) Prove that there is a unique action $\gamma \colon G \times H \to \mathrm{Aut}(A \otimes_{\max} B)$ such that for all $g \in G$, $h \in H$, $a \in A$, and $b \in B$, we have $\gamma_{(g,h)}(a \otimes b) = \alpha_g(a) \otimes \beta_h(b)$.
(2) Prove that there is a unique action $\rho \colon G \times H \to \mathrm{Aut}(A \otimes_{\min} B)$ such that for all $g \in G$, $h \in H$, $a \in A$, and $b \in B$, we have $\rho_{(g,h)}(a \otimes b) = \alpha_g(a) \otimes \beta_h(b)$.

When G and H are locally compact, the full crossed product in Exercise 10.26(1) and the reduced crossed product in Exercise 10.26(2) are

$$C^*(G, A, \alpha) \otimes_{\max} C^*(H, B, \beta) \qquad \text{and} \qquad C^*_{\mathrm{r}}(G, A, \alpha) \otimes_{\min} C^*_{\mathrm{r}}(H, B, \beta).$$

See Exercise 10.27 and Exercise 10.28 for the case in which G and H are discrete.

Exercise 10.27. Let G and H be discrete groups, let A and B be C*-algebras, and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon H \to \mathrm{Aut}(B)$ be actions of G and H on A and B. Let $\gamma\colon G \times H \to \mathrm{Aut}(A \otimes_{\max} B)$ be the action of Exercise 10.26(1), satisfying $\gamma_{(g,h)}(a \otimes b) = \alpha_g(a) \otimes \beta_h(b)$ for $g \in G$, $h \in H$, $a \in A$, and $b \in B$. Prove that

$$C^*(\gamma,\, G \times H,\, A \otimes_{\max} B) \cong C^*(G, A, \alpha) \otimes_{\max} C^*(H, B, \beta).$$

Exercise 10.28. Let G and H be discrete groups, let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon H \to \mathrm{Aut}(B)$ be as in Exercise 10.27, and let $\rho\colon G \times H \to \mathrm{Aut}(A \otimes_{\min} B)$ be the action of Exercise 10.26(2), satisfying $\gamma_{(g,h)}(a \otimes b) = \alpha_g(a) \otimes \beta_h(b)$ for $g \in G$, $h \in H$, $a \in A$, and $b \in B$. Prove that

$$C^*_{\mathrm{r}}(\rho,\, G \times H,\, A \otimes_{\min} B) \cong C^*_{\mathrm{r}}(G, A, \alpha) \otimes_{\min} C^*_{\mathrm{r}}(H, B, \beta).$$

Exercise 10.29. Let G, X, and the action of G on X be as in Example 2.24. (That is, X is the group $\prod_{n=1}^{\infty} \mathbb{Z}/k_n\mathbb{Z}$, and G is the subgroup $\bigoplus_{n=1}^{\infty} \mathbb{Z}/k_n\mathbb{Z}$, taken as discrete and acting by translation.) Prove that $C^*(G, X) \cong \bigotimes_{n=1}^{\infty} M_{k_n}$.

If there were only finitely many factors in the product, this computation would follow from Exercise 10.27. With infinitely many factors, one must take a direct limit.

We mention some explicit computations of crossed products that are found elsewhere: VIII.4.1 of [52] (crossed products of the Cantor set by odometer actions); Section VIII.9 of [52] (the crossed product of $S^1 = \mathbb{R}/\mathbb{Z}$ by the group $\mathbb{Z}[\frac{1}{2}] \subset \mathbb{R}$ regarded as a discrete group and acting by translation, and also the crossed product of a particular Bunce-Deddens algebra by a particular action of $\mathbb{Z}/2\mathbb{Z}$).

Part 3. Some Structure Theory for Crossed Products by Finite Groups

11. Introductory Remarks on the Structure of C*-Algebras

Our main interest is in structural results for crossed products. We want simplicity, but we really want much more than that. We particularly want theorems which show that certain crossed products are in classes of C*-algebras known to be covered by the Elliott classification program, so that the crossed product can be identified up to isomorphism by computing its K-theory and other invariants. In many cases, one settles for related weaker structural results, such as stable rank one, real rank zero, order on traces determined by projections, strict comparison of positive elements, or Z-stability. Some results with conclusions of this sort are stated in these notes, but mostly without proof.

We provide definitions of some of these conditions here: stable rank one, real rank zero, order on traces determined by projections, and property (SP). (Strict comparison of positive elements will be discussed later. See Definition 21.1.) We also define tracial rank zero. We state various results relating these conditions, and prove some of them. For use in these proofs, and some later proofs, we prove an assortment of standard lemmas on Murray-von Neumann equivalence of projections. The proofs mostly consist of repeated application of continuous functional calculus. Many of these results are in Section 2.5 of [152].

Definition 11.1. Let A be a unital C*-algebra. We say that A has *stable rank one* if the invertible elements in A are dense in A. If A is not unital, we say that A has stable rank one if its unitization A^+ does.

The (topological) stable rank $\mathrm{tsr}(A)$ of a general C*-algebra A (not necessarily unital) was introduced in [241]. (See Definition 1.4 there.) It can take arbitrary values in $\mathbb{Z}_{>0} \cup \{\infty\}$. Definition 11.1 gives the value most relevant for classification, since, apart from the purely infinite case, almost all known classification results apply only to C*-algebras with stable rank one. For further information, see Section V.3.1 of [24] (without proofs), and for the case of stable rank one, including some consequences, see Sections 3.1 and 3.2 of [152]. The topological stable rank of $C(X)$ is related to the covering dimension of X, which is discussed after Corollary 16.2.

It is clear that M_n has stable rank one.

Theorem 11.2. Let A be a C*-algebra. Then the following are equivalent:

(1) A has stable rank one.
(2) There is $n \in \mathbb{Z}_{>0}$ such that $M_n(A)$ has stable rank one.
(3) For all $n \in \mathbb{Z}_{>0}$, the algebra $M_n(A)$ has stable rank one.
(4) $K \otimes A$ has stable rank one.

Proof. See Theorem 3.3 and Theorem 3.6 of [241]. (Theorem 3.3 actually only does the unital case. To get the nonunital case, one needs Theorem 4.4 and Theorem 4.11 of [241].) □

Usually the stable rank of $M_n(A)$ is smaller than that of A. (There is an exact formula. See Theorem 6.1 of [241].) It is easily checked that $C(X)$ has stable rank one if X is the Cantor set, $[0, 1]$, or S^1. In fact, $C(X)$ has stable rank one if and only if the covering dimension of X is at most one. More generally, by Proposition 1.7 of [241], the algebra $C(X)$ has stable rank n if and only if the covering dimension of X is $2n-1$ or $2n$. (We will say more about covering dimension near the beginning of Section 16. The formal definition is Definition 16.7.)

Definition 11.3. Let A be a C*-algebra. We say that A has *real rank zero* if the selfadjoint elements with finite spectrum are dense in the selfadjoint part of A.

Again, this is the bottom case of a rank which takes arbitrary values in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$. The general version is a kind of generalization of having the invertible selfadjoint elements be dense in the selfadjoint part of A. See the beginning of Section 1 of [35]. The case real rank zero is discussed in Section V.7 of [52], with various examples, although one of the basic results (A has real rank zero if and only if $M_n(A)$ has real rank zero) is not explicitly stated. For further information, see Section V.3.2 of [24] (without proofs), and for the case of real rank zero, including some consequences, see Sections 3.1 and 3.2 of [152].

The following C*-algebras all have real rank zero: M_n, $C(X)$ when X is the Cantor set, $K(H)$, all AF algebras, all von Neumann algebras, and all purely infinite simple C*-algebras. (See Theorem V.7.4 of [52] for the purely infinite simple case.)

The real rank of $C(X)$ is the covering dimension of X. (See Proposition 1.1 of [35].) The real rank of $M_n(C(X))$ is usually smaller than that of $C(X)$ (again, there is an exact formula; see Corollary 3.2 of [18]), but the behavior is unknown when $C(X)$ is replaced by a general C*-algebra A.

Property (SP) is a condition which is considerably weaker than real rank zero, but which will play an important role later.

Definition 11.4. Let A be a C*-algebra. Then A is said to have *property (SP)* if every nonzero hereditary subalgebra in A contains a nonzero projection.

It is fairly easy to show that real rank zero implies property (SP). See Proposition 11.13 below. The converse is known to be false, even in the simple case. The examples A_2 and A_3 in [26] are counterexamples.

We state here some results about simple C*-algebras with property (SP) which will be needed later. They involve Murray-von Neumann equivalence of projections, so we start by giving our notation for Murray-von Neumann equivalence and proving some standard results. Murray-von Neumann equivalence will also often be needed later.

Notation 11.5. Let A be a C*-algebra, and let $p, q \in A$ be projections. We write $p \sim q$ to mean that p and q are Murray-von Neumann equivalent in A, that is, there exists $v \in A$ such that $v^* v = p$ and $v v^* = q$. We write $p \precsim q$ if p is Murray-von Neumann equivalent to a subprojection of q.

There are two other commonly used equivalence relations on projections, namely homotopy and unitary equivalence, so one needs to be careful with the meaning of $p \sim q$ when reading papers. There is also a relation on positive elements, used in connection with the Cuntz semigroup, which is commonly written with the same symbol. (See Definition 18.1(2) below.) This relation does not always agree with Murray-von Neumann equivalence on projections. However, the most common meaning of $\sim$ is Murray-von Neumann equivalence.

We give several standard functional calculus lemmas for working with projections. Proofs are included for the convenience of the reader.

For convenience, we recall polar decomposition in unital C*-algebras.

Lemma 11.6. Let A be a unital C*-algebra, and let $a \in A$ be invertible. Then $a (a^* a)^{-1/2}$ and $(a a^*)^{-1/2} a$ are unitary.

Proof. We only prove the first; the second is similar. Set $u = a (a^* a)^{-1/2}$. Then

$$u^* u = (a^* a)^{-1/2} a^* a (a^* a)^{-1/2} = 1$$

and

$$u u^* = a (a^* a)^{-1/2} (a^* a)^{-1/2} a^* = a (a^* a)^{-1} a^* = 1.$$

Thus u is unitary. □

The following lemma is contained in Proposition 4.6.6 of [23]. See Chapter 4 of [23] for much other related material.

Lemma 11.7. Let A be a C*-algebra, and let $p, q \in A$ be projections such that $\| p - q \| < 1$. Then $p \sim q$.

In fact, p is unitarily equivalent to q: the unitary u in the proof satisfies $u^* p u = q$.

Proof of Lemma 11.7. Define

$$a = (2p - 1)(2q - 1) + 1 \in A^+.$$

Using $\| 1 - 2p \| \leq 1$ at the third step, we get

$$\| a - 2 \| = \| 4pq - 2p - 2q \| \leq 2 \| 1 - 2p \| \| p - q \| < 2.$$

Therefore a is invertible. Then $u = a (a^* a)^{-1/2}$ is unitary by Lemma 11.6.

We have

$$pa = p(2q-1) + p = 2pq = (2p-1)q + q = aq.$$

Taking adjoints gives

$$a^*p = qa^*.$$

Combining these equations gives

$$(aa^*)q = q(aa^*).$$

Therefore

$$(aa^*)^{-1/2}q = q(aa^*)^{-1/2}.$$

So

$$uq = a(aa^*)^{-1/2}q = aq(aa^*)^{-1/2} = pa(aa^*)^{-1/2} = pu.$$

Now $v = uq$ satisfies $v^*v = q$ and $vv^* = p$. □

Lemma 11.8. Let A be a C*-algebra, and let $p, q \in A$ be projections. Suppose that $\|pq - q\| < 1$. Then $q \precsim p$.

Proof. We have

$$\|qpq - q\| \leq \|q\|\|pq - q\| < 1.$$

Therefore qpq is an invertible element of qAq. Let x be the inverse of qpq in qAq. Set $s = x^{1/2}qp$. Then $ss^* = q$. Therefore s^*s is a projection. Clearly $s^*s \in pAp$, so $s^*s \leq p$. □

At one point, we will need a quantitative version of the argument in Lemma 11.7. The estimate is not the best possible, but is chosen for convenience. (All we really need is that for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\|p - q\| < \delta$ then there is a unitary u such that $uqu^* = p$ and $\|u - 1\| < \varepsilon$.)

Lemma 11.9. Let A be a unital C*-algebra, and let $p, q \in A$ be projections such that $\|p - q\| \leq \frac{1}{6}$. Then there is a unitary $u \in A$ such that

$$\|u - 1\| \leq 10\|p - q\| \qquad \text{and} \qquad uqu^* = p.$$

Proof. We follow the proof of Lemma 11.7 with a slight change. Define

$$b = \frac{1}{2}\big[(2p-1)(2q-1) + 1\big].$$

Then the calculation in the proof of Lemma 11.7 shows that $\|b - 1\| \leq \|p - q\| \leq \frac{1}{6}$. So b is invertible, and we define $u = b(b^*b)^{-1/2}$. This element is the same unitary as in the proof of Lemma 11.7, so $uq = pu$ as there, whence $uqu^* = p$.

Since $\|b - 1\| \leq \frac{1}{6}$, we certainly have $\|b\| \leq 2$. Therefore

$$\|b^*b - 1\| \leq \|b^* - 1\|\|b\| + \|b - 1\| \leq 3\|b - 1\| \leq \frac{1}{2}.$$

One can check that if $\lambda \in \mathbb{R}$ satisfies $|\lambda - 1| \leq \frac{1}{2}$, then $\big|\lambda^{-1/2} - 1\big| \leq \sqrt{2}|\lambda - 1|$, so that

$$\begin{aligned} \|u - 1\| &\leq \|b\|\big\|(b^*b)^{-1/2} - 1\big\| + \|b - 1\| \leq 2\sqrt{2}\|b^*b - 1\| + \|b - 1\| \\ &\leq 2\sqrt{2}\cdot 3\|b - 1\| + \|b - 1\| = \big(6\sqrt{2} + 1\big)\|b - 1\| \leq \big(6\sqrt{2} + 1\big)\|p - q\|. \end{aligned}$$

Since $6\sqrt{2} + 1 < 10$, this completes the proof. □

Lemma 11.10. Let A be a C*-algebra, and let $a \in A_{\mathrm{sa}}$ satisfy $\|a^2 - a\| < \frac{1}{4}$. Then there is a projection $p \in A$ such that

$$\|p - a\| \leq \frac{2\|a^2 - a\|}{1 + \sqrt{1 - 4\|a^2 - a\|}}.$$

Proof. Set $r = \|a^2 - a\|$. Since $r < \frac{1}{4}$, the sets

$$S_0 = \left\{ \lambda \in \left(-\infty, \tfrac{1}{2} \right) : |\lambda^2 - \lambda| \leq r \right\}$$

and

$$S_1 = \left\{ \lambda \in \left(\tfrac{1}{2}, \infty \right) : |\lambda^2 - \lambda| \leq r \right\}$$

are disjoint. Moreover $\mathrm{sp}(a) \subset S_0 \cup S_1$. Therefore we can define a projection $p \in A$ by $p = \chi_{S_1}(a)$. We need to estimate $\|p - a\|$. Clearly

$$\|p - a\| \leq \max \left(\sup_{\lambda \in S_0} |\lambda|, \ \sup_{\lambda \in S_1} |\lambda - 1| \right). \tag{11.1}$$

By inspection of the shape of the graph of the function $\lambda \mapsto \lambda^2 - \lambda$ on $\mathbb{R}$, it is easy to see that both the supremums in (11.1) are equal to $\sup(S_0)$, and that moreover the number $s = \sup(S_0)$ is completely determined by the relations $s \in \left[0, \frac{1}{2}\right)$ and $s - s^2 = r$. It is easily checked directly that the number

$$s = \frac{2r}{1 + \sqrt{1 - 4r}}$$

satisfies both these conditions. ☐

Corollary 11.11. For every $\varepsilon > 0$ there is $\delta > 0$ such that whenever A is a C*-algebra and $a \in A_{\mathrm{sa}}$ satisfies $\|a^2 - a\| < \delta$, then there is a projection $p \in A$ such that $\|p - a\| < \varepsilon$.

Proof. Using

$$\lim_{r \to 0^+} \frac{2r}{1 + \sqrt{1 - 4r}} = 0,$$

this is immediate from Lemma 11.10. ☐

Lemma 11.12. For every $\varepsilon > 0$ there is $\delta > 0$ such that whenever A is a C*-algebra, $B \subset A$ is a subalgebra, and $p \in A$ is a projection such that $\mathrm{dist}(p, B) < \delta$, then there is a projection $q \in B$ such that $\|p - q\| < \varepsilon$.

Proof. Choose $\delta_0 > 0$ following Corollary 11.11 with $\frac{\varepsilon}{2}$ in place of ε. Set $\delta = \min\left(1, \frac{\varepsilon}{2}, \frac{\delta_0}{4}\right)$. Let A be a C*-algebra, let $B \subset A$ be a subalgebra, and $p \in A$ be a projection such that $\mathrm{dist}(p, B) < \delta$. Choose $c \in B$ such that $\|p - c\| < \delta$. Set $b = \frac{1}{2}(c + c^*)$. Then $b \in B_{\mathrm{sa}}$ and $\|p - b\| < \delta$. Using $p^2 = p$ at the first step and $\delta \leq 1$ at the third step, we have

$$\|b^2 - b\| \leq \|b\|\|b - p\| + \|b - p\|\|p\| + \|b - p\| = (\|b\| + 2)\|b - p\| \leq 4\|b - p\| < \delta_0.$$

The choice of δ_0 provides a projection $q \in B$ such that $\|p - b\| < \frac{\varepsilon}{2}$. Now

$$\|q - p\| \leq \|q - b\| + \|b - p\| < \delta + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof. ☐

Proposition 11.13. Let A be a C*-algebra with real rank zero. Then A has property (SP).

Much more is true: every hereditary subalgebra in A has an approximate identity consisting of projections (in the nonseparable case, not necessarily increasing). See Theorem 2.6 of [35].

Proof of Proposition 11.13. Let $B \subset A$ be a nonzero hereditary subalgebra. Choose $b \in B_{+}$ such that $\|b\| = 1$. Choose $\delta_0 > 0$ as in Lemma 11.12 for $\varepsilon = 1$. Set $\delta = \min\left(1, \frac{\delta_0}{4}\right)$. Choose $c \in A_{\mathrm{sa}}$ with finite spectrum such that $\|c - b\| < \delta$. Write $c = \sum_{j=1}^{n} \lambda_j p_j$ for nonzero orthogonal projections $p_1, p_2, \ldots, p_n \in A$ and numbers $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ such that $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Then

$$\|c\| < 1 + \delta \leq 2, \qquad |\lambda_n - 1| < \delta, \qquad \text{and} \qquad c p_n c = \lambda_n p_n.$$

Therefore

$$\|b p_n b - p_n\| \leq \|b - c\| \|p_n\| \|b\| + \|c\| \|p_n\| \|b - c\| + |\lambda_n - 1| \|p_n\| < 4\delta \leq \delta_0.$$

Therefore the choice of δ_0 provides a projection $p \in B$ such that $\|p - p_n\| < 1$. Since $p_n \neq 0$, we deduce from Lemma 11.7 that $p \neq 0$. ☐

Lemma 11.14. Let A be a C*-algebra, let $a \in A_{+}$, and let $p \in A$ be a projection. Suppose that there is $v \in A$ such that $\|v^* a v - p\| < 1$. Then there is a projection $q \in \overline{aAa}$ such that q is Murray-von Neumann equivalent to p.

Although we have not yet defined Cuntz subequivalence (see Definition 18.1(1) below), we state a consequence in these terms. If $a \in A_{+}$, $p \in A$ is a projection, and $p \precsim a$, then $\overline{aAa}$ contains a projection which is Murray-von Neumann equivalent p.

Proof of Lemma 11.14. Define $b \in A$ by $b = a^{1/2} v p$. Then $b^* b \in pAp$ and

$$\|b^* b - p\| = \|p(v^* a v - p)p\| \leq \|p\| \cdot \|v^* a v - p\| \cdot \|p\| < 1.$$

Therefore $b^* b$ is an invertible element of pAp, and, taking functional calculus in pAp, we can form $(b^* b)^{-1/2}$. Define $s \in A$ by $s = b(b^* b)^{-1/2}$. Then

$$s^* s = (b^* b)^{-1/2} b^* b (b^* b)^{-1/2} = p.$$

Therefore $s s^*$ is a projection. Since (with $(b^* b)^{-1}$ evaluated in pAp) we have

$$s s^* = a^{1/2} v p (b^* b)^{-1} p v^* a \in \overline{aAa},$$

the result follows. ☐

Lemma 11.15. Let $r \in (0, \infty)$, and let $f \colon [0, r] \to \mathbb{C}$ be a continuous function. Then for any C*-algebra C and any $c \in C$ with $\|c\| \leq r^{1/2}$, we have $c f(c^* c) = f(c c^*) c$.

Proof. We first observe that for any C*-algebra C, any $c \in C$, and any $n \in \mathbb{Z}_{\geq 0}$, we have $c (c^* c)^n = (c c^*)^n c$. Therefore $c h(c^* c) = h(c c^*) c$ whenever h is a polynomial.

Now let f be arbitrary. If C is not unital, we work in C^{+}. Let $\varepsilon > 0$; we prove that $\|c f(c^* c) - f(c c^*) c\| < \varepsilon$. Choose a polynomial h such that $|h(\lambda) - f(\lambda)| < \varepsilon / (3 r^{1/2})$ for all $\lambda \in [0, r]$. Then

$$\|h(c^* c) - f(c^* c)\| \leq \sup_{\lambda \in [0, r]} |h(\lambda) - f(\lambda)| \leq \frac{\varepsilon}{3 r^{1/2}},$$

so

$$\|c h(c^* c) - c f(c^* c)\| \leq \|c\| \cdot \left(\frac{\varepsilon}{3 r^{1/2}}\right) \leq \frac{\varepsilon}{3}.$$

Similarly,

$$\|h(cc^*)c - f(cc^*)c\| \leq \frac{\varepsilon}{3}.$$

Therefore

$$\|cf(c^*c) - f(cc^*)c\| \leq \|cf(c^*c) - ch(c^*c)\| + \|h(cc^*)c - f(cc^*)c\| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.$$

This completes the proof. □

The following lemma is essentially in Section 1 of [44]. (Also see the proof of Lemma 4.1 of [185].)

Lemma 11.16. Let A be a C*-algebra, and let $c \in A$. Then for any projection $p \in \overline{cAc^*}$, there exists a projection $q \in \overline{c^*Ac}$ such that $p \sim q$.

Much more is true. There is an isomorphism $\varphi \colon \overline{c^*Ac} \to \overline{cAc^*}$ (this is in 1.4 of [44]) such that $\varphi(p) \sim p$ for all projections $p \in \overline{c^*Ac}$ (this is easily deduced from [44]). In fact, using Cuntz equivalence (which we have not defined), for every $a \in \big(\overline{c^*Ac}\big)_+$, $\varphi(a)$ is Cuntz equivalent in A to a. This fact is made explicit in Lemma 3.8 of [195].

Proof of Lemma 11.16. For each $\varepsilon > 0$ define continuous functions $f_\varepsilon, g_\varepsilon \colon [0, \infty) \to [0, 1]$ by

$$f_\varepsilon(\lambda) = \begin{cases} 0 & \lambda \leq \frac{\varepsilon}{2} \\ \frac{2}{\varepsilon\lambda}\left(\lambda - \frac{\varepsilon}{2}\right) & \frac{\varepsilon}{2} \leq \lambda \leq \varepsilon \\ \frac{1}{\lambda} & \varepsilon \leq \lambda \end{cases} \quad \text{and} \quad g_\varepsilon(\lambda) = \begin{cases} 0 & \lambda \leq \frac{\varepsilon}{2} \\ \frac{2}{\varepsilon}\left(\lambda - \frac{\varepsilon}{2}\right) & \frac{\varepsilon}{2} \leq \lambda \leq \varepsilon \\ 1 & \varepsilon \leq \lambda. \end{cases}$$

Then $g_\varepsilon(\lambda) = \lambda f_\varepsilon(\lambda)$ for all $\lambda \in [0, \infty)$.

The net $(g_\varepsilon(cc^*))_{\varepsilon > 0}$ is an approximate identity for $\overline{cAc^*}$. In particular, there is $\varepsilon > 0$ such that

$$\big\|g_\varepsilon(cc^*)pg_\varepsilon(cc^*) - p\big\| < 1.$$

Define $a = f_\varepsilon(c^*c)c^*pcf_\varepsilon(c^*c)$, which is a positive element in $\overline{cAc^*}$. Then, using Lemma 11.15 twice at the second step,

$$\begin{aligned} \|cac^* - p\| &= \big\|cf_\varepsilon(c^*c)c^*pcf_\varepsilon(c^*c)c^* - p\big\| \\ &= \big\|f_\varepsilon(cc^*)cc^*pf_\varepsilon(cc^*)cc^* - p\big\| = \big\|g_\varepsilon(cc^*)pg_\varepsilon(cc^*) - p\big\| < 1. \end{aligned}$$

Now Lemma 11.14 provides a projection q in the hereditary subalgebra generated by a, and hence in the hereditary subalgebra generated by cc^*, such that $q \sim p$. The hereditary subalgebra generated by cc^* is $\overline{cAc^*}$. □

The following lemma is essentially Lemma 3.1 of [150], but no proof is given there.

Lemma 11.17 (Lemma 1.9 of [208])**.** Let A be a simple C*-algebra with property (SP). Let $B \subset A$ be a nonzero hereditary subalgebra, and let $p \in A$ be a nonzero projection. Then there is a nonzero projection $q \in B$ such that $q \precsim p$.

Proof. Choose a nonzero positive element $a \in B$. Since A is simple, there exists $x \in A$ such that $c = axp$ is nonzero. Since A has property (SP), there is a nonzero projection $q \in \overline{cAc^*}$. Then $q \in B$, and by Lemma 11.16 there is a projection $e \in \overline{c^*Ac}$ such that $e \sim q$. We have $\overline{c^*Ac} \subset pAp$, so $e \leq p$. □

We need to know that an infinite dimensional simple unital C*-algebra contains an arbitrarily large finite number of nonzero orthogonal positive elements. In the literature, this is usually derived from a result on page 61 of [2], according to which a C*-algebra which is not "scattered" contains a selfadjoint element whose spectrum is $[0, 1]$. In the next three lemmas, we give instead an elementary proof of the statement we need, which applies to any infinite dimensional C*-algebra.

Lemma 11.18. Let A be a C*-algebra, let $e, f \in A$ be projections, and suppose that

$$eAe = \{\lambda e \colon \lambda \in \mathbb{C}\} \qquad \text{and} \qquad fAf = \{\lambda f \colon \lambda \in \mathbb{C}\}.$$

Then $\dim(eAf) \leq 1$.

Proof. We may assume that $e, f \neq 0$ and $eAf \neq \{0\}$.

Choose a nonzero element $c \in eAf$. Then $c^* c$ is a nonzero element of fAf, so there is $\gamma \in (0, \infty)$ such that $c^* c = \gamma f$. Define $s = \gamma^{-1/2} c$. Then $s^* s = f$. We show that $eAf = \mathrm{span}(s)$.

Let $a \in eAf$. Then $as^* \in eAe$, so there is $\lambda \in \mathbb{C}$ such that $as^* = \lambda e$. Now

$$a = af = as^* s = \lambda s,$$

as desired. □

Lemma 11.19. Let A be a unital C*-algebra, and let $p \in A$ be a projection such that pAp and $(1-p)A(1-p)$ are finite dimensional. Then A is finite dimensional.

Proof. Since pAp and $(1-p)A(1-p)$ are finite direct sums of matrix algebras, we can find mutually orthogonal rank one projection $e_1, e_2, \ldots, e_m \in pAp$ and $f_1, f_2, \ldots, f_n \in (1-p)A(1-p)$ such that

$$\sum_{j=1}^{m} e_j = p \qquad \text{and} \qquad \sum_{k=1}^{n} f_k = 1 - p.$$

In particular,

$$e_j A e_j = \{\lambda e_j \colon \lambda \in \mathbb{C}\} \qquad \text{and} \qquad f_k A f_k = \{\lambda f_k \colon \lambda \in \mathbb{C}\}$$

for $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$. Now

$$pA(1-p) = \sum_{j=1}^{m} \sum_{k=1}^{n} e_j A f_k,$$

so $\dim(pA(1-p)) \leq mn$ by Lemma 11.18. Similarly $\dim((1-p)Ap) \leq mn$. This completes the proof. □

Lemma 11.20. Let A be an infinite dimensional C*-algebra. Then there exists a sequence $a_1, a_2, \ldots$ in A consisting of nonzero positive orthogonal elements.

In the proof, the case dealt with at the end, in which $\mathrm{sp}(a)$ is finite for all $a \in A_{\mathrm{sa}}$, can't actually occur.

Proof of Lemma 11.20. We first observe that it suffices to prove the result when A is unital. Indeed, if A is not unital, $a_1, a_2, \ldots$ is such a sequence in A^+, and $\pi \colon A^+ \to \mathbb{C}$ is the map associated with the unitization, then there can be at most one $n \in \mathbb{Z}_{>0}$ such that $\pi(a_n) \neq 0$.

We therefore assume that A is unital.

Suppose that there is $a \in A_{\mathrm{sa}}$ such that $\mathrm{sp}(a)$ is infinite. Choose a sequence in $\mathrm{sp}(a)$ whose terms are all distinct, choose a convergent subsequence, and (deleting at most one term) choose a subsequence $(\lambda_n)_{n \in \mathbb{Z}_{>0}}$ such that the limit is not one of the terms. Then there are disjoint open sets $U_1, U_2, \ldots \subset \mathbb{R}$ such that $\lambda_n \in U_n$ for all $n \in \mathbb{Z}_{>0}$. For $n \in \mathbb{Z}_{>0}$, choose a nonzero continuous function $f_n \colon \mathbb{R} \to [0,1]$ with compact support contained in U_n, and set $a_n = f_n(a)$. Then the sequence $a_1, a_2, \ldots$ satisfies the conclusion of the lemma.

Now suppose that $\mathrm{sp}(a)$ is finite for all $a \in A_{\mathrm{sa}}$.

We claim that if B is a unital C*-algebra with $B \not\cong \mathbb{C}$ and such that every element of B_{sa} has finite spectrum, then B has a nontrivial projection. Indeed, there must be an element $b \in B_{\mathrm{sa}}$ which is not a scalar, so $\mathrm{sp}(b)$ is a finite set with more than one element. Therefore functional calculus produces a nontrivial projection.

In particular, there is a nontrivial projection $p_1 \in A$. By Lemma 11.19, and replacing p_1 with $1 - p_1$ if necessary, we can assume that $p_1 A p_1$ is infinite dimensional. Clearly $\mathrm{sp}(a)$ is finite for all $a \in (p_1 A p_1)_{\mathrm{sa}}$. Therefore there is a nontrivial projection $p_2 \in p_1 A p_1$, and we may assume that $(p_1 - p_2) A (p_1 - p_2)$ is infinite dimensional. Proceed by induction. Then taking $p_0 = 1$ and $a_n = p_{n-1} - p_n$ for $n \in \mathbb{Z}_{>0}$ gives a sequence $a_1, a_2, \ldots$ as in the conclusion of the lemma. □

Lemma 11.21 (Lemma 1.10 of [208]; Lemma 3.2 of [150])**.** Let A be an infinite dimensional simple unital C*-algebra with property (SP). Let $B \subset A$ be a nonzero hereditary subalgebra, and let $n \in \mathbb{Z}_{>0}$. Then there exist nonzero Murray-von Neumann equivalent mutually orthogonal projections $p_1, p_2, \ldots, p_n \in B$.

Proof. Use Lemma 11.20 to choose nonzero positive orthogonal elements

$$a_1, a_2, \ldots, a_n \in A.$$

Choose a nonzero projection $e_1 \in \overline{a_1 A a_1}$. Inductively use Lemma 11.17 to find nonzero projections

$$e_2 \in \overline{a_2 A a_2},\ e_3 \in \overline{a_3 A a_3},\ \ldots,\ e_n \in \overline{a_n A a_n}$$

such that $e_j \precsim e_{j-1}$ for $j = 2, 3, \ldots, n$. Set $p_n = e_n$. Since $p_n \precsim e_{n-1}$, there is $p_{n-1} \leq e_{n-1}$ such that $p_{n-1} \sim p_n$. Then $p_{n-1} \precsim e_{n-2}$, so the same reasoning gives $p_{n-2} \leq e_{n-2}$ such that $p_{n-2} \sim p_{n-1}$. Construct $p_{n-3}, p_{n-4}, \ldots, p_1$ similarly. □

Lemma 11.22 (Lemma 1.11 of [208])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $n \in \mathbb{Z}_{>0}$. Then A has property (SP) if and only if $M_n \otimes A$ has property (SP). Moreover, in this case, for every nonzero hereditary subalgebra $B \subset M_n \otimes A$, there exists a nonzero projection $p \in A$ such that $1 \otimes p$ is Murray-von Neumann equivalent to a projection in B.

Proof. Let $(e_{j,k})_{1 \leq j,k \leq n}$ be a system of matrix units for M_n.

If $M_n \otimes A$ has property (SP), then so do all its hereditary subalgebras, including $\mathbb{C} e_{1,1} \otimes A \cong A$.

Now assume that A has property (SP), and let $B \subset M_n \otimes A$ be a nonzero hereditary subalgebra. Choose $x \in B \setminus \{0\}$. There is $j \in \{1, 2, \ldots, n\}$ such that $(e_{j,j} \otimes 1)x \neq 0$. Then $C = (e_{j,j} \otimes 1)x(M_n \otimes A)x^*(e_{j,j} \otimes 1)$ is a nonzero hereditary subalgebra in $(e_{j,j} \otimes 1)(M_n \otimes A)(e_{j,j} \otimes 1) \cong A$. Because A has property (SP), there is a projection $f \in A \setminus \{0\}$ such that $e_{j,j} \otimes f \in C$. Use Lemma 11.21 to choose mutually orthogonal nonzero Murray-von Neumann equivalent projections $f_1, f_2, \ldots, f_n \in A$ such that $f_j \leq f$ for all k. Then

$$1 \otimes f_1 = \sum_{k=1}^{n} e_{k,k} \otimes f_1 \sim \sum_{k=1}^{n} e_{j,j} \otimes f_k \leq e_{j,j} \otimes f.$$

Furthermore, Lemma 11.16 tells us that $e_{j,j} \otimes f$ is Murray-von Neumann equivalent to a projection in

$$\overline{x^*(e_{j,j} \otimes 1)(M_n \otimes A)(e_{j,j} \otimes 1)x} \subset B.$$

This completes the proof. □

Now we consider tracial states. See the beginning of Section 6.2 of [174].

Definition 11.23. Let A be a C*-algebra. A *tracial state* on A is a state $\tau \colon A \to \mathbb{C}$ with the additional property that $\tau(ba) = \tau(ab)$ for all $a, b \in A$. We define the *tracial state space* $\mathrm{T}(A)$ of A to be the set of all tracial states on A, equipped with the relative weak* topology inherited from the Banach space dual of A.

That is, a tracial state is a normalized trace. Tracial states have actually already occurred, in Theorem 5.28 and in Theorem 6.7.

Recall (Corollary 3.3.4 of [174]) that if A is a unital C*-algebra and $\omega \colon A \to \mathbb{C}$ is a linear functional such that $\omega(1) = 1$ and $\|\omega\| = 1$, then ω is automatically positive, and hence a state. In particular, if $\tau \colon A \to \mathbb{C}$ is a linear functional such that $\tau(1) = 1$, $\|\tau\| = 1$, and $\tau(ba) = \tau(ab)$ for all $a, b \in A$, then τ is a tracial state.

Example 11.24. Let $n \in \mathbb{Z}_{>0}$. Define $\tau \colon M_n \to \mathbb{C}$ by

$$\tau\left(\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}\right) = \frac{1}{n}\sum_{k=1}^{n} a_{k,k}.$$

Then τ is a tracial state.

The tracial state on M_n in Example 11.24 is just a normalization of the usual trace on the $n \times n$ matrices.

The following example generalizes Example 11.24.

Example 11.25. Let A be a C*-algebra, let τ_0 be a tracial state on A, and let $n \in \mathbb{Z}_{>0}$. Define $\tau \colon M_n(A) \to \mathbb{C}$ by

$$\tau\left(\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}\right) = \frac{1}{n}\sum_{k=1}^{n} \tau_0(a_{k,k}).$$

Then τ is a tracial state.

For consistency with K-theory, we usually want to use the unnormalized version, namely $\sum_{k=1}^{n} \tau_0(a_{k,k})$. For example, see Definition 16.14.

Example 11.26. Let X be a compact metric space, and let μ be a Borel probability measure on X. Then the formula

$$\tau(f) = \int_X f \, d\mu$$

defines a tracial state on $C(X)$.

Of course, all that is really happening in Example 11.26 is that every state on a commutative C*-algebra is automatically tracial.

Given Example 11.26, the following is a special case of Example 11.24.

Example 11.27. Let $n \in \mathbb{Z}_{>0}$, let X be a compact metric space, and let μ be a Borel probability measure on X. Let $\tau_0 \colon M_n \to \mathbb{C}$ be the tracial state of Example 11.24. Then the formula

$$\tau(a) = \int_X \tau_0(a(x))\, d\mu(x)$$

defines a tracial state on $C(X, M_n)$.

Explicitly, if

$$a(x) = \begin{pmatrix} a_{1,1}(x) & a_{1,2}(x) & \cdots & a_{1,n}(x) \\ a_{2,1}(x) & a_{2,2}(x) & \cdots & a_{2,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(x) & a_{n,2}(x) & \cdots & a_{n,n}(x) \end{pmatrix}$$

for $x \in X$, then

$$\tau(a) = \int_X \frac{1}{n}\left(\sum_{k=1}^{n} a_{k,k}(x)\right) d\mu(x).$$

Exercise 11.28. Let $n \in \mathbb{Z}_{>0}$, let X be a compact metric space, and let τ be a tracial state on $C(X, M_n)$. Prove that there exists a Borel probability measure μ on X such that τ is obtained from μ as in Example 11.27.

Example 11.29. Let G be a discrete group. Then the continuous linear functional $\tau \colon C^*_{\mathrm{r}}(G) \to \mathbb{C}$ such that $\tau(u_1) = 1$ and $\tau(u_g) = 0$ for $g \in G \setminus \{1\}$ (see Theorem 5.28) is proved there to be a tracial state.

Example 11.30. This example is a generalization of Example 11.29. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let τ_0 be a tracial state on A, which is G-invariant in the sense that $\tau_0(\alpha_g(a)) = \tau_0(a)$ for all $a \in A$ and $g \in G$. Let $E \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be the standard conditional expectation (Definition 9.18). Define $\tau \colon C^*_{\mathrm{r}}(G, A, \alpha) \to \mathbb{C}$ by $\tau = \tau_0 \circ E$. Then τ is a tracial state on $C^*_{\mathrm{r}}(G, A, \alpha)$.

Using Exercise 9.17, it is easy to check that τ is a state. It remains to prove that $E(ab) = E(ba)$ for $a, b \in C^*_{\mathrm{r}}(G, A, \alpha)$. By continuity, it suffices to prove this when

$$a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha) \qquad \text{and} \qquad b = \sum_{g \in G} b_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$$

with all but finitely many of the a_g and b_g equal to zero. We have, changing variables at the third step,

$$ab = \sum_{g,h \in G} a_g u_g b_h u_h = \sum_{g,h \in G} a_g \alpha_g(b_h) u_{gh} = \sum_{g \in G}\left(\sum_{k \in G} a_k \alpha_k(b_{k^{-1}g}) u_g\right)$$

and similarly

$$ba = \sum_{g \in G}\left(\sum_{k \in G} b_k \alpha_k(a_{k^{-1}g}) u_g\right).$$

Therefore

$$\tau(ab) = \sum_{k\in G} \tau_0\big(a_k \alpha_k(b_{k^{-1}})\big) \qquad \text{and} \qquad \tau(ba) = \sum_{k\in G} \tau_0\big(b_k \alpha_k(a_{k^{-1}})\big).$$

Starting with the second expression, we change k to k^{-1} at the first step, use the trace property of τ_0 at the second step, and then G-invariance of τ_0 at the third step, to get

$$\begin{aligned}\tau(ba) &= \sum_{k\in G} \tau_0\big(b_{k^{-1}} \alpha_{k^{-1}}(a_k)\big) \\ &= \sum_{k\in G} \tau_0\big(\alpha_{k^{-1}}(a_k) b_{k^{-1}}\big) = \sum_{k\in G} \tau_0\big(a_k \alpha_k(b_{k^{-1}})\big) = \tau(ab).\end{aligned}$$

This completes the proof.

Example 11.31. As a special case of Example 11.30, let G be a discrete group, and let X be a compact metric space with an action of G. Then every G-invariant Borel probability measure on X induces a tracial state τ on $C^*_{\mathrm{r}}(G, X)$. On elements of $C_{\mathrm{c}}(G, C(X))$, written as finite sums $\sum_{g\in G} f_g u_g$ with f_g in $C(X)$ for $g \in G$ and $f_g = 0$ for all but finitely $g \in G$, it is given by the formula

$$\tau\left(\sum_{g\in G} f_g u_g\right) = \int_X f_1 \, d\mu.$$

Lemma 11.32 (Remark 6.2.3 of [174])**.** Let A be a unital C*-algebra, and let τ be a tracial state on A. Then the set

$$\big\{a \in A \colon \tau(a^* a) = 0\big\}$$

is a closed ideal in A.

Proof. Set

$$I = \big\{a \in A \colon \tau(a^* a) = 0\big\}.$$

Since τ is a state, it follows from the Gelfand-Naimark-Segal construction that I is a closed left ideal in A. To show that I is in fact a two sided ideal, it suffices to show that $I^* = I$. But $\tau(a a^*) = 0$ if and only if $\tau(a^* a) = 0$ by the trace property. □

Traces are related to Murray-von Neumann equivalence in the following way.

Lemma 11.33. Let A be a unital C*-algebra, let τ be a tracial state on A, and let $p, q \in A$ be projections.

(1) If $p \sim q$, then $\tau(p) = \tau(q)$.
(2) If $p \precsim q$, then $\tau(p) \leq \tau(q)$.
(3) If A is simple and there is a projection $e \in A$ such that

$$p \sim e, \qquad e \leq q, \qquad \text{and} \qquad e \neq q,$$

then $\tau(p) < \tau(q)$.

Proof. For (1), the hypotheses imply that there is $v \in A$ such that $v^* v = p$ and $v v^* = q$. Therefore

$$\tau(p) = \tau(v^* v) = \tau(v v^*) = \tau(q).$$

Part (2) follows from (1) because positivity of τ implies that if e is a subprojection of q, then $\tau(e) \leq \tau(q)$.

For (3), we have $\tau(p) = \tau(e) \leq \tau(q)$ by (2). It remains to show that $\tau(e) \neq \tau(q)$. Now $q - e$ is a nonzero positive element. By Lemma 11.32, if $\tau(q - e)$ were zero, then A would contain the nontrivial ideal

$$I = \big\{ a \in A \colon \tau(a^* a) = 0 \big\}.$$

This completes the proof. □

Under good conditions (some of which we will see later), there is a kind of converse to Lemma 11.33. For simple unital exact C*-algebras, the right notion is given in the following definition. It is a version of Blackadar's Second Fundamental Comparability Question (FCQ2). See 1.3.1 of [21].

Definition 11.34. Let A be a unital C*-algebra. We say that the *order on projections over A is determined by traces* if whenever $p, q \in M_{\infty}(A)$ are projections such that $\tau(p) < \tau(q)$ for every tracial state τ on A, then $p \precsim q$.

In general, one should use quasitraces in place of tracial states. See Definition II.1.1 of [25] or Definition 2.31 of [4] for the definition of a quasitrace. When A is exact, every quasitrace is a trace; see Theorem 5.11 of [102]. For general C*-algebras, it is an open question whether every quasitrace is a trace.

Algebras with this property include M_n, finite factors, and simple unital AF algebras. (The case of simple unital AF algebras is a special case of Theorem 5.2.1 of [21].)

Another useful condition on the relation between traces and K-theory is presented in Remark 16.12, Definition 16.13, Definition 16.14, and Remark 16.15.

We use the following definition of tracial rank zero. Tracial rank was first defined in Definition 3.1 of [151], and tracial rank zero is equivalent (by Theorem 7.1(a) of [151]) to being tracially AF in the sense of Definition 2.1 of [150] (at least for simple C*-algebras). We use the version in Definition 3.6.2 of [152], with k there taken to be zero. (See Definition 2.4.1 of [152], where it is stated that equivalence means Murray-von Neumann equivalence.) The original version (Definition 2.1 of [150]) omitted the requirement that $p \neq 0$, but required unitary equivalence in (3). One warning: the condition $p \neq 0$ was omitted in Proposition 2.3 of [208]. Without this condition, purely infinite simple unital C*-algebras would have tracial rank zero, by taking $p = 0$.

We use the notation $[a, b]$ for the commutator $ab - ba$.

Definition 11.35 (Definition 3.6.2 of [152])**.** Let A be a simple unital C*-algebra. Then A has tracial rank zero if for every finite subset $F \subset A$, every $\varepsilon > 0$, and every nonzero positive element $c \in A$, there exist a nonzero projection $p \in A$ and a unital finite dimensional subalgebra $D \subset pAp$ such that:

(1) $\| [a, p] \| < \varepsilon$ for all $a \in F$.
(2) $\operatorname{dist}(pap, D) < \varepsilon$ for all $a \in F$.
(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{cAc}$.

The word "nonzero" is missing in Proposition 2.3 of [208]. Without this requirement, all purely infinite simple unital C*-algebras would have tracial rank zero.

When checking whether a C*-algebra has tracial rank zero, it is only necessary to use finite subsets of a fixed generating set.

Lemma 11.36. Let A be a simple unital C*-algebra, and let $T \subset A$ be a subset which generates A as a C*-algebra. Assume that for every finite subset $F \subset T$, every $\varepsilon > 0$, and every nonzero positive element $c \in A$, there exists a nonzero projection $p \in A$ and a unital finite dimensional subalgebra $D \subset pAp$ such that:

(1) $\|[a, p]\| < \varepsilon$ for all $a \in F$.
(2) $\mathrm{dist}(pap, D) < \varepsilon$ for all $a \in F$.
(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{cAc}$.

Then A has tracial has rank zero.

The only change from Definition 11.35 is that we only use finite subsets of T.

Exercise 11.37. Prove Lemma 11.36.

The proof is related to the proof of Lemma 22.10, which is given in full, and also to the proofs of similar statements earlier. For example, see the proof of Lemma 3.14, although that proof is easier.

Lemma 16.16 gives another slightly weaker condition which implies tracial rank zero.

Higher values of the tracial rank also exist (Definition 3.6.2 of [152]), and there is a definition for algebras which are not simple. See Definition 3.1 of [151] for both generalizations.

AF algebras have tracial rank zero; indeed, one can always take $p = 1$. Other examples are less obvious. The condition looks hard to check. One of the important points in the theory is that in fact there are a number of cases in which the condition can be checked. (See Theorem 14.17 for the case most relevant here.)

For our purposes, the most important consequence of tracial rank zero is that, together with simplicity, separability, nuclearity, and the Universal Coefficient Theorem, it implies classification. See Theorem 5.2 of [153].

Theorem 11.38. Let A be an infinite dimensional simple unital C*-algebra with tracial rank zero. Then A has real rank zero and stable rank one, and the order on projections over A is determined by traces.

Proof. Real rank zero and stable rank one are part of Theorem 3.4 of [150]. Order on projections over A determined by traces is Theorem 6.8 of [151], which applies by Theorem 6.13 of [151]. $\square$

These results are also found in [152]: Theorem 3.6.11 (for stable and real rank), and Theorem 3.7.2 (for order on projections determined by traces; to get from A to $M_\infty(A)$, see Lemma 11.41 below).

At least the first two parts can fail in the nonsimple case.

Corollary 11.39 (Lemma 3.6.6 of [152])**.** Let A be an infinite dimensional simple unital C*-algebra with tracial rank zero. Then A has property (SP).

Proof. Combine Theorem 11.38 and Proposition 11.13. $\square$

Lemma 11.40 (Lemma 3.6.5 of [152])**.** Let A be an infinite dimensional simple unital C*-algebra with tracial rank zero, and let $e \in A$ be a nonzero projection. Then eAe has tracial has rank zero.

We omit the proof, although it is not hard with what we now have.

Lemma 11.41 (Special case of Theorem 3.7.3 of [152])**.** Let A be an infinite dimensional simple unital C*-algebra with tracial rank zero, and let $n \in \mathbb{Z}_{>0}$. Then $M_n(A)$ tracial has rank zero.

Proof. We use standard matrix unit notation. Let $F \subset M_n \otimes A$ be finite, let $\varepsilon > 0$, and let $c \in (M_n \otimes A)_+ \setminus \{0\}$. Let $S \subset A$ be the set of all matrix entries of all elements of F.

By Lemma 11.22 and Corollary 11.39, $M_n \otimes A$ has property (SP). So there is a nonzero projection $q \in \overline{c(M_n \otimes A)c}$. Use Lemma 11.17 to find a nonzero projection $q_0 \in A$ such that $e_{1,1} \otimes q_0 \precsim q$. Use Lemma 11.21 to find nonzero Murray-von Neumann equivalent mutually orthogonal projections $e_1, e_2, \ldots, e_n \in q_0 A q_0$. Apply Definition 11.35 with ε / n^2 in place of ε, with e_1 in place of c, and with S in place of F, getting a nonzero projection $p_0 \in A$ and a unital finite dimensional subalgebra $D_0 \subset p_0 A p_0$. Set $p = 1 \otimes p_0$ and $D = M_n \otimes D_0$. Then

$$1 - p \precsim \sum_{j=1}^{n} e_{j,j} \otimes e_1 \sim \sum_{j=1}^{n} e_{1,1} \otimes e_j \leq q_0 \precsim q.$$

Also, for $a \in F$ we can find $a_{j,k} \in S$ for $j, k = 1, 2, \ldots, n$ such that $a = \sum_{j,k=1}^{n} e_{j,k} \otimes a_{j,k}$, and $b_{j,k} \in D_0$ for $j, k = 1, 2, \ldots, n$ such that $\| p_0 a_{j,k} p_0 - b_{j,k} \| < \varepsilon / n^2$. Then $b = \sum_{j,k=1}^{n} e_{j,k} \otimes b_{j,k} \in D$ and

$$\| pap - b \| = \left\| \sum_{j,k=1}^{n} e_{j,k} \otimes p_0 a_{j,k} p_0 - \sum_{j,k=1}^{n} e_{j,k} \otimes b_{j,k} \right\| \leq \sum_{j,k=1}^{n} \| p_0 a_{j,k} p_0 - b_{j,k} \| < \varepsilon.$$

Finally,

$$\| pa - ap \| = \left\| \sum_{j,k=1}^{n} e_{j,k} \otimes (p_0 a_{j,k} - a_{j,k} p_0) \right\| \leq \sum_{j,k=1}^{n} \| p_0 a_{j,k} - p_0 a_{j,k} \| < \varepsilon.$$

This completes the proof. ◻

12. Crossed Products by Finite Groups

In this section, we look briefly at some of the general theory of crossed products by finite groups, mostly in the simple case. In Section 13 we will consider the structure of crossed products when the action has the Rokhlin property, and in Section 14 we will consider the structure of crossed products when the action has the tracial Rokhlin property. A version of the tracial Rokhlin property using positive elements instead of projections seems to be the weakest hypothesis for good structure theorems for crossed products, but in these notes we will only consider the version using projections.

In this section, we give a fairly short proof that if G is finite, A is simple, and $\alpha \colon G \to \mathrm{Aut}(A)$ is pointwise outer, then $C^*(G, A, \alpha)$ is simple. From the point of view of these notes, one can't say more without a stronger hypothesis on the action, presumably some version of the tracial Rokhlin property. The various examples and problems we discuss in this section indicate how things can go wrong if one assumes less. See [207] for a much more extensive discussion, with the defect that higher dimensional Rokhlin properties are not mentioned; they were not known at the time that [207] was written.

Recall (Corollary 9.14) that if $\alpha \colon G \to \mathrm{Aut}(A)$ is an action of a finite group G on a C*-algebra A, then the maps

$$C_{\mathrm{c}}(G, A, \alpha) \to C^*(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha) \tag{12.1}$$

are bijective. This means that, unlike all other cases, one can explicitly write down all elements of $C^*(G, A, \alpha)$: as in Remark 8.8, they are the sums $\sum_{g \in G} a_g u_g$ with $a_g \in A$ for $g \in G$.

The next strongest condition after versions of the tracial Rokhlin property without projections is pointwise outerness.

Definition 12.1. Let A be a C*-algebra, let G be a group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of G on A. The action α is called *pointwise outer* if α_g is not inner (Definition 3.3) for all $g \in G \setminus \{1\}$.

Such actions are often simply called outer. This designation can lead to confusion because of the temptation to say that an action is outer if it is not inner (as in Example 3.4). There are many actions α for which α_g is inner for some choices of $g \in G \setminus \{1\}$ but outer for other choices. There are even actions α for which α_g is inner for all $g \in G$ but α itself is not inner. (See Example 3.5.)

Theorem 12.2. Let G be a finite group, let A be a simple unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be a pointwise outer action of G on A. Then $C^*(G, A, \alpha)$ is simple.

The proof we give for Theorem 12.2 is based on that of Theorem 1.1 of [240], but is much simpler, since we prove much less. In fact, what we actually prove was known long before. Apart from a small piece of operator algebra theory (isolated in Lemma 12.3), it is entirely algebraic, and proves that the skew group ring for a pointwise outer action of a finite group on a simple unital ring is again simple. Rieffel was in fact motivated by arguments from algebra, but the algebraic version of the result we prove was already proved in Theorem 4 of [11].

The result generalizes in at least two directions. By Theorem 15.26 below, the reduced crossed product of a simple C*-algebra by a pointwise outer action of a discrete group is simple. Thus, provided we use the reduced crossed product, we can replace "finite" by "discrete" in Theorem 12.2. The proof of Theorem 15.26 is quite different, requiring much more machinery. We do not give it in these notes, although we give a proof of a special case of a theorem which implies this result (not, however, the case needed for this result). The other direction is that taken in [240]. For example, one part of Theorem 4.1 of [240] states that if G is finite and A^G is type I, then A is type I (without assuming that α is pointwise outer, but results on outerness are used in the proof). This is false for both compact and infinite discrete groups. Also see Section 2 of [240], about primeness of crossed products by finite groups.

We isolate the C*-algebraic part as a general lemma.

Lemma 12.3. Let A be a C*-algebra and let $\alpha \in \mathrm{Aut}(A)$. Suppose there is x in the multiplier algebra $M(A)$ such that for all $a \in A$ we have $\alpha(a) = xax^{-1}$. Then α is inner (Definition 3.3), that is, there is a unitary $u \in M(A)$ such that $\alpha(a) = uau^*$ for all $a \in A$.

The proof is essentially the same as part of the proof of Lemma 11.7. It depends (as it must) on the relation $\alpha(a^*) = \alpha(a)^*$ for all $a \in A$.

Proof of Lemma 12.3. We immediately get $xa = \alpha(a)x$ for all $a \in A$. In this equation, take adjoints and replace a by a^*, getting $x^*\alpha(a) = ax^*$ for all $a \in A$. Combine these two equations, getting $x^*xa = ax^*x$ for all $a \in A$. Therefore

$$(x^*x)^{-1/2}a = a(x^*x)^{-1/2} \tag{12.2}$$

for all $a \in A$. Now $u = x(x^*x)^{-1/2}$ is unitary by Lemma 11.6. Combining $\alpha(a) = xax^{-1}$ for all $a \in A$ with (12.2), we get $\alpha(a) = uau^*$ for all $a \in A$. $\square$

Proof of Theorem 12.2. As discussed at the beginning of this section, we have

$$C^*(G, A, \alpha) = \left\{ \sum\nolimits_{g \in G} a_g u_g \colon a_g \in A \text{ for } g \in G \right\}.$$

Thinking of $C^*(G, A, \alpha)$ as $C_{\mathrm{c}}(G, A, \alpha)$ (bijectivity of the maps in (12.1); see Corollary 9.14), we define the support of an element $a = \sum_{g \in G} a_g u_g \in C^*(G, A, \alpha)$ by

$$\operatorname{supp}(a) = \{ g \in G \colon a_g \neq 0 \}.$$

Now let $I \subset C^*(G, A, \alpha)$ be a nonzero ideal. We will eventually show that $I = C^*(G, A, \alpha)$. Choose $b \in I \setminus \{0\}$ such that $\operatorname{card}(\operatorname{supp}(b))$ is minimal among all nonzero elements of I. There is $h \in G$ such that $b_h \neq 0$. Setting $a = b u_h^*$, we get $a = \sum_{g \in G} a_g u_g \in I \setminus \{0\}$ such that $\operatorname{card}(\operatorname{supp}(a))$ is minimal among all nonzero elements of I and such that $a_1 \neq 0$.

We want to show that $a \in A$. Suppose not. Then there is $h \in G \setminus \{1\}$ such that $a_h \neq 0$. We claim that there is a well defined bijective linear map $T \colon A \to A$ such that, whenever $n \in \mathbb{Z}_{>0}$ and $x_j, y_j \in A$ for $j = 1, 2, \ldots, n$, we have

$$T\left(\sum_{j=1}^{n} x_j a_1 y_j \right) = \sum_{j=1}^{n} x_j a_h \alpha_h(y_j). \tag{12.3}$$

To prove this claim, we first observe that

$$\left\{ \sum_{j=1}^{n} x_j a_1 y_j \colon n \in \mathbb{Z}_{>0} \text{ and } x_j, y_j \in A \text{ for } j = 1, 2, \ldots, n \right\}$$

is equal to A because A is simple and unital. So T is defined on all of A.

Next, we show that if $\sum_{j=1}^{n} x_j a_1 y_j = 0$ then $\sum_{j=1}^{n} x_j a_h \alpha_h(y_j) = 0$. So let $n \in \mathbb{Z}_{>0}$ and for $j = 1, 2, \ldots, n$ let $x_j, y_j \in A$. Suppose $\sum_{j=1}^{n} x_j a_1 y_j = 0$. Define

$$s = \sum_{j=1}^{n} x_j a y_j \in C^*(G, A, \alpha).$$

Then $s \in I$. Moreover, we can calculate

$$s = \sum_{j=1}^{n} \sum_{g \in G} x_j a_g u_g y_j = \sum_{j=1}^{n} \sum_{g \in G} x_j a_g \alpha_g(y_j) u_g = \sum_{g \in G} \left(\sum_{j=1}^{n} x_j a_g \alpha_g(y_j) \right) u_g.$$

It is clear from this formula that $\operatorname{supp}(s) \subset \operatorname{supp}(a)$. Moreover, $s_1 = 0$. The fact that $\operatorname{card}(\operatorname{supp}(a))$ is minimal among all nonzero elements of I therefore implies $s = 0$. In particular, $\sum_{j=1}^{n} x_j a_h \alpha_h(y_j) = s_h = 0$. This proves the desired implication.

By considering differences of two expressions of the form $\sum_{j=1}^{n} x_j a_1 y_j$, it follows that T is well defined. With this in hand, T is obviously linear, and it now also follows that T is injective.

We finish the proof of the claim by showing that T is surjective. Let $d \in A$. Since $a_h \neq 0$ and A is simple and unital, there are $n \in \mathbb{Z}_{>0}$ and $x_j, y_j \in A$ for $j = 1, 2, \ldots, n$ such that $\sum_{j=1}^{n} x_j a_h y_j = d$. Set $c = \sum_{j=1}^{n} x_j a_1 \alpha_h^{-1}(y_j)$. Then $T(c) = d$. This completes the proof of the claim.

It is immediate from (12.3) that for all $a, c \in A$ we have

$$T(ca) = cT(a) \qquad \text{and} \qquad T(ac) = T(a)\alpha_h(c). \tag{12.4}$$

Thus $T(c) = cT(1)$ for all $c \in A$, and using surjectivity to choose $c \in A$ such that $T(c) = 1$, we see that $T(1)$ is left invertible. Similarly, $T(c) = T(1)\alpha_h(c)$ for all $c \in A$, and using surjectivity to choose $c \in A$ such that $T(c) = 1$, we see that $T(1)$ is right invertible. So $T(1)$ is invertible. Combining the two parts of (12.4), we get $T(1)\alpha_h(c) = T(c) = cT(1)$ for all $c \in A$. Applying Lemma 12.3 with $x = T(1)^{-1}$ shows that α_h is inner. This contradiction shows that $a \in A$.

We have shown that $I \cap A \neq \{0\}$. Since $I \cap A$ is an ideal in the simple C*-algebra A, it follows that $1 \in I \cap A$. So $1 \in I$, and $I = C^*(G, A, \alpha)$. $\square$

Pointwise outerness is not good enough for the kind of structural results we have in mind for crossed products by finite groups.

Example 12.4. Example 9 of [70] contains a pointwise outer action α of $\mathbb{Z}/2\mathbb{Z}$ on a simple unital AF algebra A such that $C^*(\mathbb{Z}/2\mathbb{Z}, A, \alpha)$ does not have real rank zero. However, AF algebras have real rank zero for fairly trivial reasons.

The first example of an action of a finite group on an AF algebra such that the crossed product is not AF was given in [22]. The action is in Example 3.29. The actions in Exercise 10.23 and Exercise 10.24 are also examples of this phenomenon. Among the known examples, the one that is easiest to construct is in Section VIII.9 of [52]. It is the dual action to an action of $\mathbb{Z}/2\mathbb{Z}$ on a Bunce-Deddens algebra whose crossed product is AF.

Example 12.5. Example 8.2.1 of [22] gives an example of a pointwise outer action α of $\mathbb{Z}/2\mathbb{Z}$ on a separable unital C*-algebra A such that A has stable rank one but $C^*(\mathbb{Z}/2\mathbb{Z}, A, \alpha)$ has stable rank two.

Example 12.4 and Example 12.5 are both accessible via the methods of Section 10 (although we need to appeal to classification theorems). The action in Example 12.5 is the tensor product of the action in Example 3.29 with the trivial action on $C([0, 1])$.

The following is a long standing open problem.

Problem 12.6. Let A be a simple unital C*-algebra with stable rank one. Let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. Does it follow that $C^*(G, A, \alpha)$ has stable rank one?

A positive answer is not known even if $G = \mathbb{Z}/2\mathbb{Z}$ and A is AF.

We do have (using methods not considered here) the following theorem, which improves earlier known estimates.

Theorem 12.7 (Theorem 2.4 of [128])**.** Let A be a C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then

$$\mathrm{tsr}(C^*(G, A, \alpha)) \leq \mathrm{tsr}(A) + \mathrm{card}(G) - 1.$$

Crossed products by finite groups do preserve type I C*-algebras and nuclear C*-algebras. Both statements are true more generally: preservation of type I holds for compact groups, at least when the algebra is separable and the group is second countable (this can be gotten from Theorem 6.1 of [275]) and preservation of nuclearity holds for amenable groups (Theorem 4.2.6 of [37]).

Crossed products by finite groups presumably do not preserve the Universal Coefficient Theorem, although, as far as we know, no example has been published. The idea is as follows. Let A be the C*-algebra in the example in Section 4 of [257].

It is not KK-equivalent to a nuclear C*-algebra, and therefore does not satisfy the Universal Coefficient Theorem. Choose (see below) a contractible nuclear C*-algebra B in the bootstrap class, with an action β of a finite group G on B such that $C^*(G, B, \beta)$ is also in the bootstrap class and $K_*(C^*(G, B, \beta)) \neq 0$. (Preferably $K_*(C^*(G, B, \beta))$ should have a summand isomorphic to $\mathbb{Z}$.) Set $C = A \otimes B$ and define $\gamma\colon G \to \mathrm{Aut}(C)$ by $\gamma_g = \mathrm{id}_A \otimes \beta_g$ for $g \in G$. Then C is contractible, so satisfies the Universal Coefficient Theorem for trivial reasons. However, $C^*(G, C, \gamma) \cong A \otimes C^*(G, B, \beta)$. (See Remark 10.2 and Exercise 10.3.) So the Künneth formula [251] relates $K_*(C^*(G, C, \gamma))$ to $K_*(A)$ and $K_*(C^*(G, B, \beta))$. For example, if

$$K_0(C^*(G, B, \beta)) \cong \mathbb{Z} \qquad \text{and} \qquad K_1(C^*(G, B, \beta)) = 0,$$

then

$$K_*(C^*(G, C, \gamma)) \cong K_*(C^*(G, B, \beta)).$$

This should transfer failure of the Universal Coefficient Theorem for A to failure of the Universal Coefficient Theorem for $C^*(G, C, \gamma)$.

In Section 3 of [201], there are examples of homotopies $t \mapsto \alpha^{(t)}$ of actions of a finite group G on a nuclear C*-algebra D (even a commutative C*-algebra) such that $K_*\big(G, D, \alpha^{(0)}\big) \not\cong K_*\big(G, D, \alpha^{(1)}\big)$. Such a homotopy defines an action on $C([0, 1], D)$, for which the cone $C_0((0, 1], D)$ is invariant, and for which the K-theory of the crossed product of the cone is sometimes nonzero. This can actually happen in at least some of the examples in [201], but it isn't clear whether one can arrange to have the K-theory of the crossed product isomorphic to $\mathbb{Z}$. (It can be made isomorphic to $\mathbb{Z}[\frac{1}{2}]$.)

Despite all that seems to go wrong with crossed products by pointwise outer actions of finite groups without stronger assumptions, there are no examples in which the crossed product of a classifiable C*-algebra by a pointwise outer action of a finite group is known not to be classifiable.

13. The Rokhlin Property for Actions of Finite Groups

What is needed for good results on the structure of crossed products is some notion of freeness of the action. Free actions on spaces are well known; see Definition 2.3. There are many versions of freeness for actions on C*-algebras even when the group is finite. See [207] for an extensive discussion (which, however, makes no mention of higher dimensional Rokhlin properties; these were introduced after [207] was written). Versions of freeness range from free action on the primitive ideal space (impossible when the group is nontrivial and the algebra is simple) to conditions even weaker than pointwise outerness. The conditions which seem to be most useful for theorems on the structure of crossed products are the Rokhlin property, the tracial Rokhlin property, and various higher dimensional Rokhlin properties. In this section, we consider the Rokhlin property, and in the next section we consider the tracial Rokhlin property. Higher dimensional Rokhlin properties, which we don't discuss, were introduced in [114], and generalized (along with the ordinary Rokhlin property) to the nonunital case in [112]. The paper [112] also defines a related property called the "X-Rokhlin property".

Although we will not discuss them in these notes, there are versions of the Rokhlin property and the tracial Rokhlin property (including versions using positive elements instead of projections) for actions of suitable not necessarily finite groups.

In Definition 13.1 (and in Definition 14.1 below), one must use finite subsets of G instead of the whole group; these finite subsets should be approximately invariant under translation by a given finite set of group elements. (They should be Følner sets in the sense used in the Følner condition for amenability. See Theorem 3.6.1 of [100]; Følner sets were used in the proof of the discrete case of Theorem 5.50 and in the proof of Theorem 9.7.) We give only a few references: [123] for Rokhlin actions of $\mathbb{Z}$, [185] for actions of $\mathbb{Z}$ with the tracial Rokhlin property, [111] for a tracial Rokhlin property for finite groups and $\mathbb{Z}$ in terms of positive elements, and [182] for a tracial Rokhlin property for countable amenable groups in terms of positive elements.

The Rokhlin property was first introduced by Rokhlin, in measurable dynamics for an action of $\mathbb{Z}$ of a measure space. See the discussion at the top of page 611 of [294]. The original Rokhlin Lemma is given in Lemma VIII.3.4 of [52]. The Rokhlin property for actions of finite groups was defined for von Neumann algebras before C*-algebras, in [131], but not under that name and in a slightly different formulation.

The Rokhlin property and higher dimensional Rokhlin properties are also useful in the nonsimple case. We don't know how to define the tracial Rokhlin property in the nonsimple case.

At first sight, the Rokhlin property looks strange. We explain how it can be used in Remark 13.9 and in Lemma 13.19 and the discussion before its proof. The interested reader can skip the discussion of examples of actions with (and without) the Rokhlin property and look first at this remark and lemma.

Definition 13.1. Let A be a unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *Rokhlin property* if for every finite set $S \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in S$.
(3) $\sum_{g \in G} e_g = 1$.

We call $(e_g)_{g \in G}$ a *family of Rokhlin projections* for α, S, and ε.

One can strengthen the statement.

Theorem 13.2 (Proposition 5.26 of [211])**.** Let A be a separable unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then α has the Rokhlin property if and only if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

(1) $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
(3) $\sum_{g \in G} e_g = 1$.

The difference is that in (1) we ask for exact rather than approximate equality.

Theorem 13.2 simplifies some proofs by replacing some approximate equalities by equalities. In particular, Lemma 13.16 becomes unnecessary. However, the proof uses methods which are not standard and are not related to those here, and moreover is more complicated than the work it would save here. In the interest of completeness, we therefore give proofs without using this result.

Example 13.3. Let G be a finite group, let B be any unital C*-algebra, set $A = C(G, B)$, and define $\alpha\colon G \to \operatorname{Aut}(A)$ by $\alpha_g(a)(h) = a(h^{-1}g)$ for $a \in A$ and $g, h \in G$. The algebra A is the direct sum of copies of B, indexed by G, and the action permutes the summands. The projections required for the Rokhlin property can be taken to be given by

$$e_g(h) = \begin{cases} 1 & h = g \\ 0 & h \neq g \end{cases}$$

for $g, h \in G$.

The algebra in Example 13.3 is not simple. The Rokhlin property is very rare for actions on simple C*-algebras. We give in the next section some examples of actions with the tracial Rokhlin property but not the Rokhlin property. Here we mention just a few examples of nonexistence of actions with the Rokhlin property, based on elementary K-theoretic obstructions (not all of which require explicit use of K-theory). There is no action of any nontrivial finite group on $\mathcal{O}_\infty$ or any irrational rotation algebra which has the Rokhlin property (Proposition 13.24; Example 13.21), there is no action of any finite group whose order is divisible by 2 on $\mathcal{O}_3$ which has the Rokhlin property (Proposition 13.25), and there is no action of any finite group whose order is divisible by any prime other than 2 on the 2^∞ UHF algebra which has the Rokhlin property (Example 13.22). Even more obviously, there is no action of a nontrivial finite group on the Jiang-Su algebra (briefly described in Example 3.33) which has the Rokhlin property, since the algebra has no nontrivial projections. See Example 3.12 in [207] and the surrounding discussion for more examples. On the other hand, actions with the Rokhlin property do exist on suitable simple C*-algebras, and can be obtained using suitable choices in Example 3.25. We describe the details for $\mathbb{Z}/2\mathbb{Z}$ in Example 13.6. This example is, in slightly different notation, the special case at the end of Example 3.25, whose crossed product is treated in Example 10.22. See Exercise 13.8 for a more general case.

We look at the commutative case first.

Proposition 13.4. Let G be a finite group, and let X be a compact Hausdorff G-space. Then the corresponding action of G on $C(X)$ has the Rokhlin property if and only if there are a compact Hausdorff space Y and an equivariant homeomorphism from X to $G \times Y$, with G acting on G by translation, trivially on Y, and via the product action on $G \times Y$.

Proof. Assume first that there is an equivariant homeomorphism $X \to G \times Y$. We may then assume that $X = G \times Y$. For any finite set $F \subset A$ and any $\varepsilon > 0$, we can take $e_g = \chi_{\{g\} \times Y}$ for $g \in G$.

Now assume that the action on $C(X)$ has the Rokhlin property. Apply Definition 13.1 with $F = \varnothing$ and $\varepsilon = \frac{1}{2}$, obtaining a family $(e_g)_{g \in G}$ of Rokhlin projections. In $C(X)$, if p and q are projections with $\|p - q\| < 1$, then $p = q$. Therefore we get $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$. There is a closed and open subset $Y \subset X$ such that $e_1 = \chi_Y$. Define a continuous function $m\colon G \times Y \to X$ by $m(g, y) = gy$. Since the projections $\alpha_g(\chi_Y)$ for $g \in G$ are orthogonal, the sets gY are disjoint, so m is injective. Since $\sum_{g \in G} \alpha_g(\chi_Y) = 1$, we have $\bigcup_{g \in G} gY = X$, so m is surjective. Therefore m is a homeomorphism. Giving Y the trivial action of G and G the translation action, it is immediate to check that m is equivariant. $\square$

The following result is a restatement of Theorem 1.2 of [207].

Theorem 13.5. Let G be a finite group, and let X be a totally disconnected G-space. Then the corresponding action of G on $C(X)$ has the Rokhlin property if and only if the action of G on X is free.

Proof. If the action of G on $C(X)$ has the Rokhlin property, then freeness of the action of G on X is is immediate from Proposition 13.4.

So assume the action of G on X is free. We first claim that for every $x \in X$, there is a compact open set $L \subset X$ such that $x \in L$ and the sets gL, for $g \in G$, are disjoint. To prove the claim, for $g \in G$ choose disjoint compact open sets L_g and M_g such that $x \in L_g$ and $gx \in M_g$. Then take

$$L = \bigcap_{g \in G \setminus \{1\}} (L_g \cap g^{-1} M_g).$$

This proves the claim.

Since X is compact, we can now find compact open sets $L_1, L_2, \ldots, L_n \subset X$ which cover X and such that, for each m, the sets gL_m, for $g \in G$, are disjoint. Set $K_1 = L_1$ and for $m = 2, 3, \ldots, n$ set

$$K_m = L_m \cap \left(X \setminus \bigcup_{g \in G} g(L_1 \cup L_2 \cup \cdots \cup L_{m-1}) \right).$$

(This set may be empty.) One verifies by induction on m that the sets gK_j, for $g \in G$ and $j = 1, 2, \ldots, m$, are disjoint and cover $\bigcup_{g \in G} g(L_1 \cup L_2 \cup \cdots \cup L_m)$. For $m = n$, these sets form a partition of X. Set $Y = K_1 \cup K_2 \cup \cdots \cup K_n$. Then the sets gY, for $g \in G$, form a partition of X. The conclusion follows. $\square$

The Rokhlin property is a strong form of freeness. Not all free actions of finite groups on compact spaces have the Rokhlin property. The actions of finite subgroups of S^1 on S^1 be translation (given in Example 2.16) are free but don't have the Rokhlin property. (See [207] for an extensive discussion of notions of freeness of actions of finite groups on C*-algebras, but note that higher dimensional Rokhlin properties had not yet been introduced when this article was written.)

Example 13.6. Let α be the action of $\mathbb{Z}/2\mathbb{Z}$ on the 2^∞ UHF algebra A generated by $\bigotimes_{n=1}^{\infty} \operatorname{Ad} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$. (The 2×2 matrix in the above formula is unitarily equivalent to the 2×2 matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$ used in Example 10.22. One checks that this implies that the actions are conjugate. See Exercise 13.7 below.) We simply write α for the automorphism given by the nontrivial group element. In Example 10.22, we wrote $A = \varinjlim M_{2^n}$, with maps $\varphi_n \colon M_{2^n} \to M_{2^{n+1}}$ given by $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix} \right)$. Here, we identify M_{2^n} as the tensor product of n copies of M_2, which we write for short as $(M_2)^{\otimes n}$. We identify the maps of the direct system

$$(M_2)^{\otimes n} \xrightarrow{\varphi_n} (M_2)^{\otimes (n+1)} = (M_2)^{\otimes n} \otimes M_2$$

as $a \mapsto a \otimes 1$. We also identify $(M_2)^{\otimes n}$ with its image in A.

We claim that α has the Rokhlin property.

Let $S \subset A$ be finite and let $\varepsilon > 0$. We have to find orthogonal projections $e_0, e_1 \in A$ such that:

(1) $\|\alpha(e_0) - e_1\| < \varepsilon$ and $\|\alpha(e_1) - e_0\| < \varepsilon$.
(2) $\|e_0 a - a e_0\| < \varepsilon$ and $\|e_1 a - a e_1\| < \varepsilon$ for all $a \in S$.

(3) $e_0 + e_1 = 1$.

Write $S = \{a_1, a_2, \ldots, a_N\}$. Since $\bigcup_{n \in \mathbb{Z}_{\geq 0}} (M_2)^{\otimes n}$ is dense in A, there are n and $b_1, b_2, \ldots, b_N \in (M_2)^{\otimes n} \subset A$ such that

$$\|b_1 - a_1\| < \tfrac{1}{2}\varepsilon, \qquad \|b_2 - a_2\| < \tfrac{1}{2}\varepsilon, \qquad \ldots, \qquad \|b_N - a_N\| < \tfrac{1}{2}\varepsilon.$$

Define

$$e_0, e_1 \in (M_2)^{\otimes n} \otimes M_2 = (M_2)^{\otimes (n+1)} \subset A$$

by

$$e_0 = 1_{(M_2)^{\otimes n}} \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \qquad \text{and} \qquad e_1 = 1_{(M_2)^{\otimes n}} \otimes \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

It is obvious that $e_0 + e_1 = 1$, which is (3), and we easily check that $\alpha(e_0) = e_1$ and $\alpha(e_1) = e_0$, which implies (1).

It remains to check (2). For $k \in \{1, 2, \ldots, N\}$, the element b_k actually commutes with e_0 and e_1, so

$$\|e_0 a_k - a_k e_0\| \leq \|e_0\| \cdot \|a_k - b_k\| + \|a_k - b_k\| \cdot \|e_0\| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

This completes the proof that α has the Rokhlin property.

Exercise 13.7. Let G be a locally compact group, let A be a UHF algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ and $\rho \colon G \to \mathrm{Aut}(A)$ be two infinite tensor product actions as in Example 3.25, using the same infinite tensor product decomposition. That is, let $k_1, k_2, \ldots$ be integers with $k_n \geq 2$ for all $n \in \mathbb{Z}_{>0}$, assume that $A = \bigotimes_{n=1}^{\infty} M_{k_n}$, and that there are actions $\beta^{(n)}, \sigma^{(n)} \colon G \to \mathrm{Aut}(M_{k_n})$ for $n \in \mathbb{Z}_{>0}$ such that for all $g \in G$, we have $\alpha_g = \bigotimes_{n=1}^{\infty} \beta_g^{(n)}$ and $\rho_g = \bigotimes_{n=1}^{\infty} \sigma_g^{(n)}$.

Now suppose that for every $n \in \mathbb{Z}_{>0}$, the actions $\beta^{(n)}$ and $\sigma^{(n)}$ are conjugate. Prove that the actions α and ρ are conjugate.

Exercise 13.8. In Example 3.25, let G be a finite group, for each $n \in \mathbb{Z}_{>0}$ let $g \mapsto u_n(g)$ be a unitary representation of G on $\mathbb{C}^{k_n}$ which is unitarily equivalent to a finite direct sum of copies of the regular representation, and set $\beta_g^{(n)}(a) = u_n(g) a u_n(g)^*$ for $g \in G$ and $a \in M_{k_n}$. Prove that the corresponding action $g \mapsto \bigotimes_{n=1}^{\infty} \beta_g^{(n)}$ of G on $\bigotimes_{n=1}^{\infty} M_{k_n}$ has the Rokhlin property.

One use of the Rokhlin property is to "average" over the group in ways not normally possible. This construction is more related to its use in classification of actions than its use for structural properties of crossed products. Some cases have an interpretation as "cohomology vanishing lemmas", about which we say nothing more here. The next remark gives an example of the method.

Remark 13.9. Let A be a unital C*-algebra and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the Rokhlin property. Let $u \in A$ be a unitary. The usual average

$$\frac{1}{\mathrm{card}(G)} \sum_{g \in G} \alpha_g(u)$$

will almost never be a unitary. Suppose, however, we choose Rokhlin projections $e_g \in A$ for $g \in G$ as in Theorem 13.2. In particular, $\alpha_g(e_h) = e_{gh}$ for $g, h \in G$. Assume, first, that they exactly commute with u. Then the element

$$v = \sum_{g \in G} \alpha_g(e_1 u e_1) = \sum_{g \in G} e_g \alpha_g(u) e_g$$

is a G-invariant unitary in A.

Having $e_g u = u e_g$ for all $g \in G$ is far too much to hope for. If, however, $\|e_g u - u e_g\|$ is small enough, then

$$b = \sum_{g \in G} \alpha_g(e_1 u e_1) = \sum_{g \in G} e_g \alpha_g(u) e_g$$

will be G-invariant and approximately unitary. So $b(b^* b)^{-1/2}$ will be a G-invariant unitary which is close to b, and thus close to $\sum_{g \in G} e_g \alpha_g(u) e_g$.

One doesn't really need the stronger condition in Theorem 13.2; the approximation argument works nearly as well using Definition 13.1 as it stands.

The next exercise gives an example of what one can do with the ideas in Remark 13.9. It does not have much connection with the main ideas in these notes, but is very important elsewhere.

Let A and B be unital C*-algebras. Two homomorphisms $\varphi, \psi \colon A \to B$ are said to be *approximately unitarily equivalent* if for every $\varepsilon > 0$ and every finite set $F \subset A$, there is a unitary $u \in B$ such that $\|u \varphi(a) u^* - \psi(a)\| < \varepsilon$ for all $a \in F$. (This concept is very important in the Elliott classification program. As just one example, if φ and ψ are approximately unitarily equivalent, then $\varphi_*, \psi_* \colon K_*(A) \to K_*(B)$ are equal.)

Suppose now that G is a finite group, and $\alpha \colon G \to \operatorname{Aut}(A)$ and $\beta \colon G \to \operatorname{Aut}(B)$ are actions of G on A and B. One can easily imagine that one would want equivariant homomorphisms $\varphi, \psi \colon A \to B$ to be not just approximately unitarily equivalent but in fact equivariantly approximately unitarily equivalent, that is, the unitaries u above can be chosen to be G-invariant. (If this is true, then, for example, $\varphi_*, \psi_* \colon K_*^G(A) \to K_*^G(B)$ are equal.)

Exercise 13.10. Let G be a finite group, let A and B be unital C*-algebras, and let $\alpha \colon G \to \operatorname{Aut}(A)$ and $\beta \colon G \to \operatorname{Aut}(B)$ be actions of G on A and B. Let $\varphi, \psi \colon A \to B$ be equivariant unital homomorphisms, and assume that φ and ψ are approximately unitarily equivalent (ignoring the group actions).

(1) Suppose that β has the Rokhlin property. Prove that φ and ψ are equivariantly approximately unitarily equivalent.
(2) Suppose that α has the Rokhlin property. Prove that φ and ψ are equivariantly approximately unitarily equivalent.

We will now return to ideas more directly related to the structure of crossed products.

We first give several results whose proofs are more direct than that of Theorem 13.15 (the main result of this section).

Theorem 13.11 (Proposition 4.14 of [187])**.** Let A be a unital C*-algebra, let G be a finite group, and let $\alpha \colon G \to \operatorname{Aut}(A)$ be an action of G on A which has the Rokhlin property. Then the restriction map defines a bijection from $\mathrm{T}(C^*(G, A, \alpha))$ (see Definition 11.23) to the set $\mathrm{T}(A)^G$ of G-invariant tracial states on A.

Proof. For $g \in G$, following Notation 8.7, let $u_g \in C^*(G, A, \alpha)$ be the standard unitary in the crossed product. Let $E \colon C^*(G, A, \alpha) \to A$ be the standard conditional expectation (Definition 9.18), which is given by $E\left(\sum_{g \in G} a_g u_g\right) = a_1$ when $a_g \in A$ for $g \in G$.

We will show that the map $\tau \mapsto \tau \circ E$ is an inverse of the restriction map. First, let $\tau \in \mathrm{T}(A)^G$. Then $\tau \circ E$ is a tracial state on $C^*(G, A, \alpha)$ by Example 11.30. It is immediate that $(\tau \circ E)|_A = \tau$.

Now let $\tau \in \mathrm{T}(C^*(G, A, \alpha))$. We claim that for all $g \in G \setminus \{1\}$ and $a \in A$, we have $\tau(a u_g) = 0$. Let $\varepsilon > 0$. Choose Rokhlin projections $e_h \in A$ for $h \in G$ according to Definition 13.1, using

$$\delta = \frac{\varepsilon}{(1 + \|a\|)\mathrm{card}(G)}$$

in place of ε and with $F = \{a\}$. For $h \in G$, using at the second step $g \neq 1$ (so that $e_h e_{gh} = 0$), we get

$$e_h u_g e_h = e_h (u_g e_h u_g^* - e_{gh}) u_g + e_h e_{gh} u_g = e_h (u_g e_h u_g^* - e_{gh}) u_g.$$

Therefore

$$\|e_h u_g e_h\| \leq \|e_h\| \|\alpha_h(e_h) - e_{gh}\| \|u_g\| < \delta.$$

So, using $\sum_{h \in G} e_h = 1$ at the first step and the trace property at the second step,

$$\begin{aligned} |\tau(a u_g)| &\leq \sum_{h \in G} |\tau(a u_g e_h^2)| = \sum_{h \in G} |\tau(e_h a u_g e_h)| \leq \sum_{h \in G} \big(\|e_h a - a e_h\| + |\tau(a e_h u_g e_h)| \big) \\ &\leq \sum_{h \in G} \big(\|e_h a - a e_h\| + \|a\| \cdot \|e_h u_g e_h\| \big) < \mathrm{card}(G)(1 + \|a\|)\delta = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the claim follows.

Now let $a \in C^*(G, A, \alpha)$, and choose $a_g \in A$ for $g \in G$ such that $a = \sum_{g \in G} a_g u_g$. Then $E(a) = a_1$, so, remembering that $u_1 = 1$ and $\tau(a u_g) = 0$ for $g \neq 1$, we get

$$(\tau|_A) \circ E = \tau(a_1) = \sum_{g \in G} \tau(a u_g) = \tau(a).$$

This completes the proof. $\square$

Proposition 13.12. Let A be a unital C*-algebra, let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of G on A which has the Rokhlin property. Then α_g is outer for every $g \in G \setminus \{1\}$.

Proposition 4.16 of [207] has a stronger statement, with a closely related but more complicated proof: α is strongly pointwise outer in the sense of Definition 4.11 of [207].

Proof of Proposition 13.12. Let $h \in G \setminus \{1\}$, and suppose that α_h is inner. Thus, there is a unitary $u \in A$ such that such that $\alpha_h(a) = u a u^*$ for all $a \in A$. Choose projections $e_g \in A$ for $g \in G$ as in Definition 13.1, with $S = \{u\}$ and $\varepsilon = \frac{1}{3}$. Then calculate as follows, using orthogonality of e_1 and e_h at the first step and $\alpha_h(e_1) = u e_1 u^*$ at the second step:

$$\begin{aligned} 1 = \|e_1 - e_h\| &\leq \|e_1 - u e_1 u^*\| + \|\alpha_h(e_1) - e_h\| \\ &= \|e_1 u - u e_1\| + \|\alpha_h(e_1) - e_h\| < \frac{1}{3} + \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

This is a contradiction. $\square$

Thus, if A is simple, then Theorem 15.26 below implies that $C^*(G, A, \alpha)$ is simple. Actually, more can be proved directly, although with a bit of work.

Proposition 13.13 (Corollary 2.5 of [194])**.** Let A be a unital C*-algebra, let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of G on A which has the Rokhlin property. Let J be an ideal in $C^*(G, A, \alpha)$. Then there is a G-invariant ideal $I \subset A$ such that $J = C^*(G, I, \alpha)$.

The proof in [194] uses other results not directly related to the Rokhlin property. We give a direct proof. A direct proof for the same result for integer actions with the Rokhlin property is given for Theorem 2.2 of [195]. We state a lemma separately, which is the finite group version of Lemma 2.1 of [195]. It is in the proof of the lemma that the Rokhlin property is actually used.

Lemma 13.14. Let A be a unital C*-algebra, let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of G on A which has the Rokhlin property. Let $E \colon C^*(G, A, \alpha) \to A$ be the standard conditional expectation (Definition 9.18). Then for every finite set $F \subset C^*(G, A, \alpha)$ and every $\varepsilon > 0$, there exist mutually orthogonal projections $e_g \in A$ for $g \in G$ such that $\sum_{g \in G} e_g = 1$ and

$$\left\| E(a) - \sum\nolimits_{g \in G} e_g a e_g \right\| < \varepsilon$$

for all $a \in F$.

Proof. Write $F = \{b_1, b_2, \ldots, b_n\}$. For $j = 1, 2, \ldots, n$ write $b_j = \sum_{h \in G} a_{j,h} u_h$ with $a_{j,h} \in A$ for $h \in G$. Set

$$F_0 = \big\{ a_{j,h} \colon h \in G \text{ and } j \in \{1, 2, \ldots, n\} \big\} \qquad \text{and} \qquad M = \sup_{a \in F_0} \|a\|.$$

Choose projections $e_g \in A$ for $g \in G$ according to Definition 13.1, with

$$\delta = \frac{\varepsilon}{(1 + M)\mathrm{card}(G)^2}$$

in place of ε and with F_0 in place of F.

Let $j \in \{1, 2, \ldots, n\}$. The key estimate is as follows: for $g, h \in G$, we have

$$\begin{aligned} &\big\| e_g a_{j,h} u_h e_g - e_g e_{hg} a_{j,h} u_h \big\| \\ &\qquad \leq \|e_g\| \|a_{j,h}\| \big\| u_h e_g - e_{hg} u_h \big\| + \|e_g\| \big\| a_{j,h} e_{hg} - e_{hg} a_{j,h} \big\| \|u_h\| \\ &\qquad < M\delta + \delta. \end{aligned}$$

For $h \neq 1$ we have $e_g e_{hg} = 0$, so $\|e_g a_{j,h} u_h e_g\| < (M + 1)\delta$. For $h = 1$ we have $e_g e_{hg} = e_g$, so $\big\| e_g a_{j,h} u_h e_g - e_g a_{j,h} u_h \big\| < (M+1)\delta$. (Since $u_h = 1$ here, one actually gets the estimate δ.) Summing over $g, h \in G$, we get

$$\left\| \sum\nolimits_{g,h \in G} e_g a_{j,h} u_h e_g - \sum_{g \in G} e_g a_{j,1} u_1 \right\| < \mathrm{card}(G)^2 (M + 1)\delta = \varepsilon.$$

Using

$$b_j = \sum_{h \in G} a_{j,h} u_h, \qquad \sum_{g \in G} e_g = 1, \qquad u_1 = 1, \qquad \text{and} \qquad E(b_j) = a_{j,1},$$

we can rewrite this inequality as

$$\left\| \sum\nolimits_{g \in G} e_g b_j e_g - E(b_j) \right\| < \varepsilon,$$

which is the desired estimate. ∎

Proof of Proposition 13.13. For $g \in G$, following Notation 8.7, let $u_g \in C^*(G, A, \alpha)$ be the standard unitary in the crossed product.

Let $J \subset C^*(G, A, \alpha)$ be an ideal. Set $I = J \cap A$.

We first claim that I is a G-invariant ideal in A. It is obvious that I is an ideal in A. Let $g \in G$ and $a \in I$. Certainly $\alpha_g(a) \in A$, and $\alpha_g(a) = u_g a u_g^*$, which is in J since $a \in J$. The claim is proved.

By Theorem 8.32, we can identify $C^*(G, I, \alpha)$ with an ideal in $C^*(G, A, \alpha)$. We next claim that $C^*(G, I, \alpha) \subset J$. So let $a \in C^*(G, I, \alpha)$. Since G is finite, there are $a_g \in I$ for $g \in G$ such that $a = \sum_{g \in G} a_g u_g$. Since the elements a_g are in J and J is an ideal in $C^*(G, A, \alpha)$, it follows that $a = \sum_{g \in G} a_g u_g \in J$. The claim is proved.

Let $E \colon C^*(G, A, \alpha) \to A$ be the standard conditional expectation. We claim that $E(J) \subset I$. So let $a \in J$, and let $\varepsilon > 0$. Lemma 13.14 provides mutually orthogonal projections $e_g \in A$ for $g \in G$ such that $\sum_{g \in G} e_g = 1$ and

$$\left\| E(a) - \sum_{g \in G} e_g a e_g \right\| < \varepsilon.$$

Since $\sum_{g \in G} e_g a e_g \in J$ and $\varepsilon > 0$ is arbitrary, it follows that $E(a) \in J$. The claim follows.

We finish the proof by showing that $J \subset C^*(G, I, \alpha)$. Let $a \in J$. Choose $a_g \in A$ for $g \in G$ such that $a = \sum_{g \in G} a_g u_g$. For $g \in G$, we have $a u_g^* \in I$, so $a_g = E(a u_g^*) \in I$ by the previous claim. Therefore $a = \sum_{g \in G} a_g u_g \in C^*(G, I, \alpha)$. □

The Rokhlin property for finite groups was used in noncommutative von Neumann algebras before it was used in noncommutative C*-algebras. It was first used there for the purpose of classification of actions on the hyperfinite factor of type II_1 [131]. In that situation, pointwise outerness implies the Rokhlin property, which is far from the case for C*-algebras. The Rokhlin property has been used for classification of actions on C*-algebras; for a brief survey, in [207] see Theorems 2.10–2.13 and the preceding discussion. That it has strong consequences for classification of crossed products was only realized very late. We show (Theorem 2.2 of [208]) that if G is finite, A is a unital AF algebra, and $\alpha \colon G \to \mathrm{Aut}(A)$ has the Rokhlin property, then $C^*(G, A, \alpha)$ is AF. Thus, crossed products by actions with the Rokhlin property preserve classifiability in the sense of Elliott's original AF algebra classification theorem [70].

Theorem 13.15 (Theorem 2.2 of [208])**.** *Let A be a unital AF algebra. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the Rokhlin property. Then $C^*(G, A, \alpha)$ is an AF algebra.*

The basic idea is as follows. Let $e_g \in A$, for $g \in G$, be Rokhlin projections. Let $u_g \in C^*(G, A, \alpha)$ be the canonical unitary implementing the automorphism α_g (Notation 8.7). Then $w_{g,h} = u_{gh^{-1}} e_h$ defines an approximate system of matrix units in $C^*(G, A, \alpha)$. (This formula is derived from the formula (10.1) for $v_{g,h}$ in Example 10.8.) Let $(v_{g,h})_{g,h \in G}$ be a nearby true system of matrix units. Using the homomorphism $M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ given by $v_{g,h} \otimes d \mapsto v_{g,1} d v_{1,h}$, one can approximate $C^*(G, A, \alpha)$ by matrix algebras over corners of A. A more detailed discussion is given after the statement of Lemma 13.19.

We begin with a semiprojectivity lemma (Lemma 2.1 of [208]), whose proof we omit. The proof uses the kinds of methods (functional calculus) that go into the proof of Lemma 11.7, but is more work.

Lemma 13.16. Let $n \in \mathbb{Z}_{>0}$. For every $\varepsilon > 0$ there is $\delta > 0$ such that, whenever $(e_{j,k})_{1 \leq j,k \leq n}$ is a system of matrix units for M_n, whenever B is a unital C*-algebra, and whenever $w_{j,k}$, for $1 \leq j,k \leq n$, are elements of B such that $\| w_{j,k}^* - w_{k,j} \| < \delta$ for $1 \leq j,k \leq n$, such that $\| w_{j_1,k_1} w_{j_2,k_2} - \delta_{j_2,k_1} w_{j_1,k_2} \| < \delta$ for $1 \leq j_1, j_2, k_1, k_2 \leq n$, and such that the $w_{j,j}$ are orthogonal projections with $\sum_{j=1}^{n} w_{j,j} = 1$, then there exists a unital homomorphism $\varphi \colon M_n \to B$ such that $\varphi(e_{j,j}) = w_{j,j}$ for $1 \leq j \leq n$ and $\| \varphi(e_{j,k}) - w_{j,k} \| < \varepsilon$ for $1 \leq j,k \leq n$.

Exercise 13.17. Prove Lemma 13.16.

Whenever we have a unital homomorphism $\psi \colon M_n \to A$, then A has a tensor factorization as $M_n \otimes B$, in which B is the corner of A corresponding to the image under ψ of a rank one projection in M_n.

Lemma 13.18. Let A be a unital C*-algebra, let S be a finite set, and let $\varphi_0 \colon L(l^2(S)) \to A$ be a unital homomorphism. Let $(v_{s,t})_{s,t \in S}$ be the standard system of matrix units in $L(l^2(S))$ (as in Notation 10.7, except that they were called $e_{j,k}$ there). Let $s_0 \in S$, and set $e = \varphi(v_{s_0,s_0})$. Then there is an isomorphism $\varphi \colon L(l^2(S)) \otimes eAe \to A$ such that for all $s,t \in S$ and $a \in eAe$, we have

$$\varphi(v_{s,t} \otimes a) = \varphi_0(v_{s,s_0}) a \varphi_0(v_{s_0,t}).$$

Proof. To check that there is such a homomorphism, it suffices to show that

$$\left[\varphi_0(v_{s,s_0}) a \varphi_0(v_{s_0,t}) \right]^* = \varphi_0(v_{t,s_0}) a^* \varphi_0(v_{s_0,s})$$

for $s,t \in S$ and $a \in eAe$, and that

$$\begin{aligned} &\left[\varphi_0(v_{s_1,s_0}) a_1 \varphi_0(v_{s_0,t_1}) \right] \left[\varphi_0(v_{s_2,s_0}) a_2 \varphi_0(v_{s_0,t_2}) \right] \\ &\qquad\qquad = \begin{cases} \left[\varphi_0(v_{s_1,s_0}) a_1 a_2 \varphi_0(v_{s_0,t_2}) \right] & s_2 = t_1 \\ 0 & s_2 \neq t_1 \end{cases} \end{aligned}$$

for $s_1, s_2, t_1, t_2 \in S$ and $a_1, a_2 \in eAe$. Both these are immediate.

For surjectivity, let $a \in A$. Then one easily checks that

$$\varphi \left(\sum\nolimits_{s,t \in S} v_{s,t} \otimes \varphi_0(v_{s_0,s}) a \varphi_0(v_{t,s_0}) \right) = a.$$

For injectivity, suppose that $a_{s,t} \in eAe$ for $s,t \in S$, and that

$$\varphi \left(\sum\nolimits_{s,t \in S} v_{s,t} \otimes a_{s,t} \right) = 0.$$

For $s,t \in S$, multiply this equation on the left by $\varphi_0(v_{s_0,s})$ and on the right by $\varphi_0(v_{t,s_0})$ to get

$$0 = \varphi(v_{s_0,s_0}) a_{s,t} \varphi(v_{s_0,s_0}) = a_{s,t}.$$

Since this is true for all $s,t \in S$, injectivity of φ follows. □

Lemma 13.19. Let G be a finite group, and set $n = \operatorname{card}(G)$. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that the following holds. Let (G, A, α) be a unital G-algebra, let $(e_g)_{g \in G}$ be a family of orthogonal projections in A, and let $F \subset A$ be a finite set such that $\|a\| \leq 1$ for all $a \in F$. Suppose that:

(1) $\| \alpha_g(e_h) - e_{gh} \| < \delta$ for all $g,h \in G$.
(2) $\| e_g a - a e_g \| < \delta$ for all $g \in G$ and all $a \in F$.
(3) $\sum_{g \in G} e_g = 1$.

For $g \in G$ let u_g be the standard unitary of Notation 8.7. Then there exists a unital homomorphism $\varphi \colon M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ such that for every $a \in F \cup \{u_g \colon g \in G\}$, we have $\operatorname{dist}\big(a, \, \varphi\big(L(l^2(G)) \otimes e_1 A e_1\big)\big) < \varepsilon$, and such that (using standard matrix unit notation) for every $a \in e_1 A e_1$ we have $\varphi(e_{1,1} \otimes a) = a$.

To make clear what is happening, suppose that in the hypotheses of Lemma 13.19 we actually had:

(1) $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
(2) $e_g a = a e_g$ for all $g \in G$ and all $a \in F$.
(3) $\sum_{g \in G} e_g = 1$.

We use $L(l^2(G))$ instead of M_n. The same computation as in Example 10.8 (where we showed that if G is discrete then $C^*(G, \, C_0(G)) \cong K(l^2(G)))$ shows that if we define $w_{g,h} \in C^*(G, A, \alpha)$ by $w_{g,h} = e_g u_{gh^{-1}}$ for $g, h \in G$ (compare with equation (10.1)), then the $w_{g,h}$ form a system of matrix units in $C^*(G, A, \alpha)$. That is, letting $(v_{g,h})_{g,h \in G}$ be the standard system of matrix units in $L(l^2(G))$ (as in Notation 10.7, except that they were called $e_{j,k}$ there), there is a unital homomorphism $\varphi_0 \colon L(l^2(G)) \to C^*(G, A, \alpha)$ such that $\varphi_0(v_{g,h}) = w_{g,h}$ for all $g, h \in G$.

The elements u_g are already in the range of φ_0. Indeed, we have $w_{h, \, g^{-1}h} = e_h u_g$, so

$$u_g = \sum_{h \in G} e_h u_g = \varphi_0 \left(\textstyle\sum_{h \in G} v_{h, \, g^{-1} h} \right).$$

Since $\varphi_0(v_{1,1}) = e_1 \in A$, we have $e_1 A e_1 \subset e_1 C^*(G, A, \alpha) e_1$, and we can apply Lemma 13.18 to get a unital homomorphism $\varphi \colon L(l^2(G)) \otimes e_1 A e_1 \to C^*(G, A, \alpha)$. Suppose now that $a \in A$ commutes with e_g for all $g \in G$. Then

$$a = \sum_{g \in G} e_g a e_g = \sum_{g \in G} \alpha_g \big(e_1 \alpha_g^{-1}(a) e_1 \big).$$

Applying the formula for φ in Lemma 13.18, we get

$$\begin{aligned} \varphi \left(\textstyle\sum_{g \in G} v_{g,g} \otimes e_1 \alpha_g^{-1}(a) e_1 \right) & = \sum_{g \in G} e_g u_g e_1 \alpha_g^{-1}(a) e_1 u_g^* e_g \\ & = \sum_{g \in G} e_g \alpha_g \big(e_1 \alpha_g^{-1}(a) e_1 \big) e_g \\ & = \sum_{g \in G} \alpha_g \big(e_1 \alpha_g^{-1}(a) e_1 \big) = a. \end{aligned}$$

In the actual proof, many of the equations in the computations above become statements that the norm of the difference between the two sides is small.

Proof of Lemma 13.19. We will use $L(l^2(G))$ instead of M_n; the lemma as stated will follow by choosing a bijection from G to $\{1, 2, \ldots, n\}$. As will be seen later, we require that this bijection send the identity of G to 1.

Set $\varepsilon_0 = \varepsilon/(4n)$. Choose $\delta > 0$ according to Lemma 13.16 for n as given and for ε_0 in place of ε. Also require $\delta \leq \varepsilon/[2n(n+1)]$. Assume that $(e_g)_{g \in G}$ is a family of orthogonal projections in A and that $F \subset A$ is a finite set such that the hypotheses (1), (2), and (3) of the lemma hold for this value of δ. Define $w_{g,h} = e_g u_{gh^{-1}}$ for $g, h \in G$.

We claim that the $w_{g,h}$ form a δ-approximate system of $n \times n$ matrix units in $C^*(G, A, \alpha)$. We estimate:

$$\|w_{g,h}^* - w_{h,g}\| = \|u_{gh^{-1}}^* e_g - e_h u_{hg^{-1}}\| = \|e_g - u_{gh^{-1}} e_h u_{gh^{-1}}^*\| = \|e_g - \alpha_{gh^{-1}}(e_h)\| < \delta.$$

Also, using $e_g e_h = \delta_{g,h} e_h$ for $g, h \in G$ at the second step,

$$\begin{aligned}\big\|w_{g_1,h_1} w_{g_2,h_2} - \delta_{g_2,h_1} w_{g_1,h_2}\big\| &= \big\|e_{g_1} u_{g_1 h_1^{-1}} e_{g_2} u_{g_2 h_2^{-1}} - \delta_{g_2,h_1} e_{g_1} u_{g_1 h_2^{-1}}\big\| \\ &= \big\|e_{g_1} u_{g_1 h_1^{-1}} e_{g_2} u_{g_2 h_2^{-1}} - e_{g_1} e_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}\big\| \\ &= \big\|e_{g_1} \big(u_{g_1 h_1^{-1}} e_{g_2} u_{g_1 h_1^{-1}}^* - e_{g_1 h_1^{-1} g_2}\big) u_{g_1 h_1^{-1} g_2 h_2^{-1}}\big\| \\ &= \big\|e_{g_1} \big(\alpha_{g_1 h_1^{-1}}(e_{g_2}) - e_{g_1 h_1^{-1} g_2}\big) u_{g_1 h_1^{-1} g_2 h_2^{-1}}\big\| < \delta.\end{aligned}$$

Finally, $\sum_{g \in G} w_{g,g} = \sum_{g \in G} e_g = 1$. This proves the claim.

Let $(v_{g,h})_{g,h \in G}$ be the standard system of matrix units in $L(l^2(G))$ (as in Notation 10.7, except that they were called $e_{j,k}$ there). By the choice of δ, there exists a unital homomorphism $\varphi_0 \colon L(l^2(G)) \to C^*(G, A, \alpha)$ such that $\|\varphi_0(v_{g,h}) - w_{g,h}\| < \varepsilon_0$ for all $g, h \in G$, and $\varphi_0(v_{g,g}) = e_g$ for all $g \in G$. Since $\varphi_0(v_{1,1}) = e_1 \in A \subset C^*(G, A, \alpha)$, we can restrict the homomorphism of Lemma 13.18 from $L(l^2(G)) \otimes e_1 C^*(G, A, \alpha) e_1$ to the subalgebra $L(l^2(G)) \otimes e_1 A e_1$. We get a unital homomorphism $\varphi \colon L(l^2(G)) \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ such that $\varphi(v_{g,h} \otimes a) = \varphi_0(v_{g,1}) a \varphi_0(v_{1,h})$ for $g, h \in G$ and $a \in e_1 A e_1$. Since $\varphi_0(v_{1,1}) = e_1$ and we identify $1 \in G$ with $1 \in \{1, 2, \ldots, n\}$, the relation $\varphi\big(e_{1,1} \otimes a\big) = a$ for $a \in e_1 A e_1$ is immediate.

We complete the proof by showing that every element of S is within ε of an element of the algebra $D = \varphi\big(L(l^2(G)) \otimes e_1 A e_1\big)$.

For $g \in G$ we have $\sum_{h \in G} \varphi_0(v_{gh,h}) \in D$ and, using $\sum_{h \in G} e_h = 1$,

$$\begin{aligned}\Big\|u_g - \textstyle\sum_{h \in G} \varphi_0(v_{h,g^{-1}h})\Big\| &\leq \sum_{h \in G} \|e_h u_g - \varphi_0(v_{h,g^{-1}h})\| \\ &= \sum_{h \in G} \|w_{h,g^{-1}h} - \varphi_0(v_{h,g^{-1}h})\| < n\varepsilon_0 \leq \varepsilon.\end{aligned}$$

Now let $a \in F$. Set

$$b = \sum_{g \in G} v_{g,g} \otimes e_1 \alpha_g^{-1}(a) e_1 \in M_n \otimes e_1 A e_1.$$

Using $\|e_g a e_h\| \leq \|e_g a - a e_g\| + \|a e_g e_h\| = \|e_g a - a e_g\|$ at the third step, we get

$$\begin{aligned}\Big\|a - \textstyle\sum_{g \in G} e_g a e_g\Big\| &= \Big\|\textstyle\sum_{g,h \in G} e_g a e_h - \textstyle\sum_{g \in G} e_g a e_g\Big\| \\ &\leq \sum_{g \neq h} \|e_g a e_h\| < n(n-1)\delta.\end{aligned} \tag{13.1}$$

Combining $\|e_1 - \alpha_g^{-1}(e_g)\| < \delta$ for all $g \in G$ and $\|a\| \leq 1$ for all $a \in F$, we get

$$\big\|e_1 \alpha_g^{-1}(a) e_1 - \alpha_g^{-1}(e_g a e_g)\big\| < 2\delta. \tag{13.2}$$

Also, for $g \in G$ we have, taking adjoints at the first step,

$$\|\varphi_0(v_{g,1}) e_1 - u_g e_1\| = \|e_1 \varphi_0(v_{1,g}) - e_1^2 u_g^*\| \leq \|e_1\| \|\varphi_0(v_{1,g}) - w_{1,g}\| < \varepsilon_0. \tag{13.3}$$

For $a \in F$, we use (13.3) and $\|a\| \leq 1$ at the second step, (13.2) at the third step, and (13.1) at the fifth step, to get

$$\begin{aligned}
\|a - \varphi(b)\| &= \left\| a - \sum\nolimits_{g \in G} \varphi_0(v_{g,1}) e_1 \alpha_g^{-1}(a) e_1 \varphi_0(v_{1,g}) \right\| \\
&< 2n\varepsilon_0 + \left\| a - \sum\nolimits_{g \in G} u_g e_1 \alpha_g^{-1}(a) e_1 u_g^* \right\| \\
&< 2n\varepsilon_0 + 2n\delta + \left\| a - \sum\nolimits_{g \in G} u_g \alpha_g^{-1}(e_g a e_g) u_g^* \right\| \\
&= 2n\varepsilon_0 + 2n\delta + \left\| a - \sum\nolimits_{g \in G} e_g a e_g \right\| \\
&< 2n\varepsilon_0 + 2n\delta + n(n-1)\delta \leq \varepsilon.
\end{aligned}$$

This completes the proof. ∎

Proof of Theorem 13.15. We prove that for every finite set $S \subset C^*(G, A, \alpha)$ and every $\varepsilon > 0$, there is an AF subalgebra $D \subset C^*(G, A, \alpha)$ such that every element of S is within ε of an element of D. It is then easy to use Theorem 2.2 of [30] to show that $C^*(G, A, \alpha)$ is AF.

It suffices to fix a set T which generates $C^*(G, A, \alpha)$ as a C*-algebra, and to consider only finite subsets $S \subset T$. Thus, we need only consider S of the form $S = F \cup \{u_g \colon g \in G\}$, where F is a finite subset of the unit ball of A and $u_g \in C^*(G, A, \alpha)$ is the canonical unitary implementing the automorphism α_g (Notation 8.7). So let $F \subset A$ be a finite subset with $\|a\| \leq 1$ for all $a \in F$ and let $\varepsilon > 0$. Choose $\delta > 0$ as in Lemma 13.19 for ε as given. Apply the Rokhlin property to α with F as given and with δ in place of ε, obtaining projections $e_g \in A$ for $g \in G$. Define $w_{g,h} = e_g u_{gh^{-1}}$ for $g, h \in G$. Set $n = \operatorname{card}(G)$, and let $\varphi \colon M_n \otimes e_1 A e_1 \to A$ be the homomorphism of Lemma 13.19. It is well known that a corner of an AF algebra is AF, and that a quotient of an AF algebra is AF, so $D = \varphi(M_n \otimes e_1 A e_1)$ is an AF subalgebra of $C^*(G, A, \alpha)$ such that for every $a \in F \cup \{u_g \colon g \in G\}$, we have $\operatorname{dist}\big(a, \, \varphi(M_n \otimes e_1 A e_1)\big) < \varepsilon$. This completes the proof. ∎

We summarize a number of other theorems related to Theorem 13.15 which have been proved in [187] and elsewhere.

Crossed products by actions of finite groups with the Rokhlin property preserve the following classes of C*-algebras:

(1) Various other classes of unital but not necessarily simple countable direct limit C*-algebras using semiprojective building blocks, and in which the maps of the direct system need not be injective:
  (a) AI algebras (Corollary 3.6(1) of [187]).
  (b) AT algebras (Corollary 3.6(2) of [187]).
  (c) Unital direct limits of one dimensional noncommutative CW complexes (Corollary 3.6(4) of [187]).
  (d) Unital direct limits of Toeplitz algebras, a special case of the sort studied in [158] except not necessarily of real rank zero (Example 2.10 and Theorem 3.5 of [187]).
  (e) Various other classes; see Section 2 and Theorem 3.5 of [187] for details.
(2) Simple unital AH algebras with slow dimension growth and real rank zero (Theorem 3.10 of [187]).

(3) D-absorbing separable unital C*-algebras for a strongly self-absorbing C*-algebra D (Theorem 1.1(1) and Corollary 3.4(i) of [113]). (See [113] for the definition of a strongly self-absorbing C*-algebra.)
(4) Unital C*-algebras with real rank zero (Proposition 4.1(1) of [187]).
(5) Unital C*-algebras with stable rank one (Proposition 4.1(2) of [187]).
(6) Separable nuclear unital C*-algebras whose quotients all satisfy the Universal Coefficient Theorem (Proposition 3.7 of [187]).
(7) Unital Kirchberg algebras satisfying the Universal Coefficient Theorem (Corollary 3.11 of [187]).
(8) Separable unital approximately divisible C*-algebras (Corollary 3.4(2) of [113], which also covers actions of compact groups; also see Proposition 4.5 of [187]).
(9) Unital C*-algebras with the ideal property and unital C*-algebras with the projection property ([194]; also see [194] for the definitions of these properties).
(10) Simple unital C*-algebras whose K-theory:
    (a) Is torsion free.
    (b) Is a torsion group.
    (c) Is zero.

    (Theorem 2.6(11) of [207]; the proof of this part is in the discussion after Theorem 2.7 of [207]. The main part of the proof comes from Theorem 3.13 of [124].)

We now give some examples to show that the Rokhlin property is rare. The proofs depend implicitly or explicitly on K-theory. For the first three, we use the restriction in the next lemma.

Lemma 13.20. Let A be a unital C*-algebra with a unique tracial state τ, let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A which has the Rokhlin property. Then there exists a projection $p \in A$ such that $\tau(p) = \mathrm{card}(G)^{-1}$.

Proof. In Definition 13.1, take $\varepsilon = 1$ and $F = \varnothing$. We get projections e_g for $g \in G$ such that, in particular:

(1) $\|\alpha_g(e_1) - e_g\| < 1$ for all $g \in G$.
(2) $\sum_{g\in G} e_g = 1$.

It follows from (1) and Lemma 11.7 that $\alpha_g(e_1)$ is Murray-von Neumann equivalent to e_g for all $g \in G$. Therefore $\tau(e_g) = \tau(\alpha_g(e_1))$ by Lemma 11.33(1). Since τ is unique, we have $\tau \circ \alpha_g = \tau$ for all $g \in G$. So $\tau(e_g) = \tau(e_1)$ for all $g \in G$. It follows that

$$\tau(e_1) = \frac{1}{\mathrm{card}(G)}.$$

This completes the proof. □

Existence of an action of G with the Rokhlin property implies much stronger restrictions on the K-theory than are suggested by this result, or by the methods used in Proposition 13.24 and Proposition 13.25 below. See Theorem 3.2 of [125].

Example 13.21. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let A_θ be the rotation algebra, as in Example 3.10. It is known (Proposition VI.1.3 of [52]) that A_θ has a unique tracial state τ. Moreover, for every projection $p \in A_\theta$, one has $\tau(p) \in \mathbb{Z} + \theta\mathbb{Z} \subset \mathbb{R}$. (See Theorem VI.5.2 of [52]; for less technical proofs relying on the Pimsner-Voiculescu exact sequence in K-theory, see the Appendix in [221] and the general theory developed in [80], specifically Example IX.12 there.) In particular, there is no $n \in \mathbb{Z}_{>0}$ with $n \geq 2$ such that there is a projection $p \in A_\theta$ with $\tau(p) = \frac{1}{n}$. It follows from Lemma 13.20 that there is no action of any nontrivial finite group on A_θ which has the Rokhlin property.

Example 13.22. Let A be the 2^∞ UHF algebra. Then A has a unique tracial state τ. Moreover, for every projection $p \in A$, one has $\tau(p) \in \mathbb{Z}\left[\frac{1}{2}\right] \subset \mathbb{R}$. (This is really a statement in K-theory, but there is enough in Section 11 to give a direct proof. See below.) It follows from Lemma 13.20 that the only finite groups G which can possibly have actions on A with the Rokhlin property are groups whose order is a power of 2. In particular, there is no action of $\mathbb{Z}/3\mathbb{Z}$ on A which has the Rokhlin property.

We prove the statement about traces of projections. Let $p \in A$ be a projection. Choose $\delta > 0$ as in Lemma 11.12 for $\varepsilon = 1$. By the direct limit description of A, there are $n \in \mathbb{Z}_{\geq 0}$ and a unital subalgebra $B \subset A$ with $B \cong M_{2^n}$ such that $\operatorname{dist}(p, B) < \delta$. By the choice of δ using Lemma 11.12, there is a projection $q \in B$ such that $\|p - q\| < 1$. We have $p \sim q$ by Lemma 11.7, so $\tau(p) = \tau(q)$ by Lemma 11.33(1). The restriction of τ to B must be the normalized trace on M_{2^n}, so $\tau(q)$ is an integer multiple of $\frac{1}{2^n}$. Therefore so is $\tau(p)$, as desired.

Example 13.23. The same reasoning as in Example 13.22 shows that there is no action of $\mathbb{Z}/2\mathbb{Z}$ on the 3^∞ UHF algebra which has the Rokhlin property.

The next two examples depend much more heavily on K-theory, and we therefore assume basic knowledge of K-theory.

Proposition 13.24. There is no action of any nontrivial finite group on $\mathcal{O}_\infty$ which has the Rokhlin property.

Proof. Let G be a nontrivial finite group, and let $\alpha \colon G \to \operatorname{Aut}(\mathcal{O}_\infty)$ be an action with the Rokhlin property. The computation $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ is Corollary 3.11 of [48]. The fact that $[1]$ is a generator can be read from the proof there and the proof of Proposition 3.9 of [48]. It follows that every automorphism of $\mathcal{O}_\infty$ is the identity on $K_0(\mathcal{O}_\infty)$. Now apply Definition 13.1 with $F = \varnothing$ and $\varepsilon = \frac{1}{2}$. We get projections $e_g \in \mathcal{O}_\infty$ for $g \in G$ such that $\|e_g - \alpha_g(e_1)\| < \frac{1}{2}$ for $g \in G$ and $\sum_{g \in G} e_g = 1$. By Lemma 11.7, the inequality implies $[e_g] = [\alpha_g(e_1)]$ in $K_0(\mathcal{O}_\infty)$; since α_g is the identity on $K_0(\mathcal{O}_\infty)$, it follows that $[e_g] = [e_1]$. From $\sum_{g \in G} e_g = 1$ we therefore get $\operatorname{card}(G)[e_1] = [1]$ in $K_0(\mathcal{O}_\infty)$. Since $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ via $n \mapsto n[1]$, this is a contradiction. $\square$

Proposition 13.25. The only finite groups G which can possibly have actions on $\mathcal{O}_3$ with the Rokhlin property are groups of odd order.

Proof. The proof is similar to that of Proposition 13.24. Let G be a finite group with even order, and let $\alpha \colon G \to \operatorname{Aut}(\mathcal{O}_3)$ be an action with the Rokhlin property. The computation $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ is Theorem 3.7 of [48]; in the proof it is shown that $[1]$ is a generator. It follows that every automorphism of $\mathcal{O}_n$ is the identity on $K_0(\mathcal{O}_n)$. In particular, $K_0(\mathcal{O}_3) \cong \mathbb{Z}/2\mathbb{Z}$ and $(\alpha_g)_*$ is the identity on $K_0(\mathcal{O}_3)$ for all $g \in G$. Apply Definition 13.1 with $F = \varnothing$ and $\varepsilon = \frac{1}{2}$. As in the proof of Proposition 13.24, one gets $\operatorname{card}(G)[e_1] = [1]$ in $K_0(\mathcal{O}_3)$. Since $\operatorname{card}(G)$ is even, this implies $[1] = 0$ in $K_0(\mathcal{O}_3)$, a contradiction. $\square$

14. The Tracial Rokhlin Property for Actions of Finite Groups

As was discussed in Section 13, there are very few actions of finite groups which have the Rokhlin property. The tracial Rokhlin property (Definition 14.1 below) is much more common. The differences are discussed in several places in Section 3 of [207], and an illuminating example is given in Exercise 14.11.

The tracial Rokhlin property is still very useful in classification. Indeed, if G is finite, A is a simple unital C*-algebra with tracial rank zero in the sense of Lin (originally called "tracially AF"; see Definition 11.35 above), and $\alpha\colon G \to \mathrm{Aut}(A)$ has the tracial Rokhlin property, then $C^*(G, A, \alpha)$ has tracial rank zero. This is Theorem 2.6 of [208] (Theorem 14.17 below). Lin has proved (Theorem 5.2 of [153]) that simple separable unital nuclear C*-algebras with tracial rank zero and which satisfy the Universal Coefficient Theorem are classifiable. Thus, this result can be used for classification purposes. For example, it played a key role in the proof [65] that the crossed products by the actions of Example 3.12 are AF (except that this was known earlier for the action of $\mathbb{Z}/2\mathbb{Z}$).

It is not true the crossed products of simple unital AF algebras by actions of finite groups with the tracial Rokhlin property are AF. See Sections 3 and 4 of [209]. The actions used are those in Example 3.29, Exercise 10.23, and Exercise 10.24.

Ironically, Theorem 2.6 of [208] was proved *before* Theorem 2.2 of [208] (the AF algebra and Rokhlin property version).

It is presumably not sufficient for classification purposes to just consider pointwise outer actions. Example 12.4 shows that the crossed product by a pointwise outer action of a finite group on a simple unital AF algebra need not have real rank zero; in particular, by Theorem 11.38, it need not have tracial rank zero. Example 12.5 shows that the crossed product by a pointwise outer action of a finite group on a nonsimple unital C*-algebra with stable rank one need not have stable rank one. These results suggest that crossed products by pointwise outer actions of finite groups might well not respect classifiability, although no examples are known. Although we will not pursue this direction in these notes, there are useful weakenings of the tracial Rokhlin property which are stronger than pointwise outerness. For example, see Definition 5.2 of [111] (and also Definition 6.1 of [111] for actions of $\mathbb{Z}$ and [182] for actions of countable amenable groups).

Definition 14.1 (Definition 1.2 of [208])**.** Let G be a finite group, let A be an infinite dimensional simple unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. We say that α has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are nonzero mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
(3) With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.
(4) With e as in (3), we have $\|exe\| > 1 - \varepsilon$.

When A is finite, the last condition is redundant. (See Lemma 1.16 of [208], which is Lemma 14.14 below.) However, without it, the trivial action on $\mathcal{O}_2$ would have the tracial Rokhlin property. (It is, however, not clear that this condition is really the right extra condition to impose.) Without the requirement that the algebra be infinite dimensional, the trivial action on $\mathbb{C}$ would have the tracial Rokhlin property (except for the condition (4)), for the rather silly reason that the hereditary subalgebra in Condition (3) can't be "small".

Remark 14.2 (Remark 1.4 of [208])**.** Let G be a finite group, let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. If α has the Rokhlin property, then α has the tracial Rokhlin property.

Lemma 14.3 (Lemma 1.13 of [208])**.** Let G be a finite group, let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action which has the tracial Rokhlin property. If A does not have property (SP), then α has the Rokhlin property.

Proof. Suppose A does not have property (SP). Then there is $x \in A_{+} \setminus \{0\}$ which generates a hereditary subalgebra which contains no nonzero projections. So the projection e in condition (3) must be equal to 1. □

As with the Rokhlin property (see Theorem 13.2), one can strengthen the statement.

Theorem 14.4 (Proposition 5.27 of [211])**.** Let G be a finite group, let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. Then α has the tracial Rokhlin property if and only if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

(1) $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
(3) With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.
(4) With e as in (3), we have $\|exe\| > 1 - \varepsilon$.

The proof uses the same methods as that of Theorem 13.2, and, for the same reasons as given in the discussion there, we do not use the stronger condition in these notes.

We give two other conditions for the tracial Rokhlin property. We omit both proofs.

The first uses an an assumption on comparison of projections using traces to substitute an estimate on the trace of the error projection for condition (3) in Definition 14.1, and finiteness and Lemma 14.14 below to omit condition (4). It is the motivation for the term "tracial Rokhlin property".

Proposition 14.5 (Lemma 5.2 of [65])**.** Let G be a finite group, let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. Assume that A is finite, that A has property (SP), and that the order on projections over A is determined by traces. Then α has the tracial Rokhlin property if and only if for every finite set $S \subset A$ and every $\varepsilon > 0$, there exist orthogonal projections $e_g \in A$ for $g \in G$ such that:

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in S$.
(3) With $e = \sum_{g \in G} e_g$, we have $\tau(1 - e) < \varepsilon$ for all $\tau \in \mathrm{T}(A)$.

The second applies to C*-algebras with tracial rank zero (Definition 11.35) and a unique tracial state. (There should be an analog without requiring uniqueness of the tracial state, but we don't know what it is.) It relates the tracial Rokhlin property for α to the corresponding action α'' on the factor of type II_1 gotten by applying the Gelfand-Naimark-Segal construction to the tracial state. Since this factor is hyperfinite, and since pointwise outer actions of finite groups on the hyperfinite type II_1 factor necessarily have the von Neumann algebra version of the Rokhlin property, it says that α has the tracial Rokhlin property if and only if α'' has the Rokhlin property. Its proof proceeds via Theorem 5.3 of [65], a criterion for the tracial Rokhlin property for an action of a finite group on a simple C*-algebra with tracial rank zero which looks very similar to Definition 14.1 except that it uses trace norms in place of the usual norm. This criterion does not depend on uniqueness of the tracial state.

Theorem 14.6 (Theorem 5.5 of [65])**.** Let G be a finite group, let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A. Assume that A has tracial rank zero and has a unique tracial state τ. Let $\pi_\tau\colon A \to B(H_\tau)$ be the Gelfand-Naimark-Segal representation associated with τ, and for $\beta \in \mathrm{Aut}(A)$ let β'' denote the automorphism of $\pi_\tau(A)''$ determined by β. Then α has the tracial Rokhlin property if and only if α''_g is an outer automorphism of $\pi_\tau(A)''$ for every $g \in G \setminus \{1\}$.

Proposition 13.12 is also valid for actions with the tracial Rokhlin property, with essentially the same proof.

Lemma 14.7 (Lemma 1.5 of [208])**.** Let G be a finite group, let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action which has the tracial Rokhlin property. Then α is pointwise outer (Definition 12.1).

Proof. Let $g \in G \setminus \{1\}$; we prove that α_g is outer. So let $u \in A$ be unitary. Apply Definition 14.1 with $F = \{u\}$, with $\varepsilon = \frac{1}{2}$, and with $x = 1$. Then e_1 and e_g are orthogonal nonzero projections, so

$$\|\alpha_g(e_1) - ue_1u^*\| \geq \|e_g - e_1\| - \|\alpha_g(e_1) - e_g\| - \|ue_1u^* - e_1\| > 1 - \frac{1}{2} - \frac{1}{2} = 0.$$

Therefore $\alpha_g \neq \mathrm{Ad}(u)$. Since u is arbitrary, this shows that α_g is outer. $\square$

We will prove results about the tracial Rokhlin property below. First, we give an example for which it is easy to see (using several of the results below) that the tracial Rokhlin property holds, but where the Rokhlin property fails.

Example 14.8. For $k \in \mathbb{Z}_{>0}$, define $v_k \in M_{3^k}$ to be the unitary

$$v_k = \mathrm{diag}\big(1, 1, \ldots, 1, -1, -1, \ldots, -1\big) \in M_{3^k},$$

in which the diagonal entry 1 occurs $\frac{1}{2}(3^k + 1)$ times and the diagonal entry -1 occurs $\frac{1}{2}(3^k - 1)$ times. Set $A = \bigotimes_{k=1}^{\infty} M_{3^k}$, which is just a somewhat different expression for the 3^∞ UHF algebra. Define

$$\mu = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(v_k) \in \mathrm{Aut}(A).$$

Then μ is an automorphism of order 2. Let $\alpha\colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(A)$ be the action generated by μ. Then α is a product type action, as in Example 3.25. It follows from Example 13.23 that α does not have the Rokhlin property. However, we will show that α does have the tracial Rokhlin property.

Set $r(k) = \frac{1}{2}(3^k - 1)$. It is easy to check that v_k is unitarily equivalent to the block unitary

$$w_k = \begin{pmatrix} 0 & 1_{M_{r(k)}} & 0 \\ 1_{M_{r(k)}} & 0 & 0 \\ 0 & 0 & 1_{\mathbb{C}} \end{pmatrix} \in M_{3^k}.$$

It follows (Exercise 13.7) that μ is conjugate to the automorphism

$$\nu = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(w_k),$$

and therefore that α is conjugate to the action $\beta\colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(A)$ generated by ν.

We claim that β has the tracial Rokhlin property. It will follow that α does too.

Let $S \subset A$ be finite and let $\varepsilon > 0$. Let τ be the unique tracial state on A. Appealing to Proposition 14.5, we have to find orthogonal projections $e_0, e_1 \in A$ such that:

(1) $\|\nu(e_0) - e_1\| < \varepsilon$ and $\|\nu(e_1) - e_0\| < \varepsilon$.
(2) $\|e_0 a - a e_0\| < \varepsilon$ and $\|e_1 a - a e_1\| < \varepsilon$ for all $a \in S$.
(3) $\tau(1 - e_0 - e_1) < \varepsilon$.

Write $S = \{a_1, a_2, \ldots, a_N\}$. For $n \in \mathbb{Z}_{>0}$ set $A_n = \bigotimes_{k=1}^{n} M_{3^k}$ and identify A_n with its image in A. Since $\bigcup_{n \in \mathbb{Z}_{\geq 0}} A_n$ is dense in A, there are n and $b_1, b_2, \ldots, b_N \in A_n$ such that

$$\|b_1 - a_1\| < \frac{\varepsilon}{2}, \quad \|b_2 - a_2\| < \frac{\varepsilon}{2}, \quad \ldots, \quad \|b_N - a_N\| < \frac{\varepsilon}{2}.$$

We can increase n, so we may also assume that $3^{-n-1} < \varepsilon$.

Using subscripts to indicate block sizes on the diagonals, set

$$p_0 = \begin{pmatrix} 1_{r(n+1)} & 0 & 0 \\ 0 & 0_{r(n+1)} & 0 \\ 0 & 0 & 0_1 \end{pmatrix} \in M_{3^{n+1}}$$

and

$$p_1 = \begin{pmatrix} 0_{r(n+1)} & 0 & 0 \\ 0 & 1_{r(n+1)} & 0 \\ 0 & 0 & 0_1 \end{pmatrix} \in M_{3^{n+1}}.$$

Then

$$w_{n+1} p_0 w_{n+1}^* = p_1 \qquad \text{and} \qquad w_{n+1} p_1 w_{n+1}^* = p_0,$$

and the normalized trace of $1 - p_0 - p_1$ is

$$\frac{1}{2r(n+1) + 1} = \frac{1}{3^{n+1}} < \varepsilon.$$

Set

$$e_0 = 1_{A_n} \otimes p_0 \qquad \text{and} \qquad e_1 = 1_{A_n} \otimes p_1,$$

so

$$e_0, e_1 \in A_n \otimes M_{3^{n+1}} = A_{n+1} \subset A.$$

On $A_n \otimes M_{3^{n+1}}$, the automorphism ν has the form $\mathrm{Ad}\big((\bigotimes_{k=1}^{n} w_k) \otimes w_{n+1} \big)$, so

$$\nu(e_0) = e_1 \qquad \text{and} \qquad \nu(e_1) = e_0.$$

Condition (1) follows trivially.

The tracial state τ restricts to the unique tracial state on $M_{3^{n+1}}$, so $\tau(1-e_0-e_1)$ is the normalized trace of $1-p_0-p_1$, and is thus equal to $3^{-n-1}<\varepsilon$. This is condition (3).

It remains to check (2). For $k \in \{1, 2, \ldots, N\}$, the element b_k actually commutes with e_0 and e_1, so

$$\|e_0 a_k - a_k e_0\| \leq \|e_0\| \|a_k - b_k\| + \|a_k - b_k\| \|e_0\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof that α has the tracial Rokhlin property.

Remark 14.9. The matrix sizes in Example 14.8 grow rapidly, and the number of diagonal entries equal to 1 and the number of diagonal entries equal to -1 are very close. This looks special. It actually isn't. It turns out that the action α of Example 14.8 is conjugate to the action generated by

$$\rho = \bigotimes_{n=1}^{\infty} \operatorname{Ad} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \text{on} \qquad \bigotimes_{n=1}^{\infty} M_3.$$

So the action generated by ρ has the tracial Rokhlin property.

Exercise 14.10. Prove the conjugacy statement in Remark 14.9 by combining suitable finite collections of tensor factors in the definition of ρ.

The following exercise (which requires work, and also requires some of the results below) gives some idea of the differences between the Rokhlin property, the tracial Rokhlin property, and pointwise outerness.

Exercise 14.11 (Section 2 of [209])**.** Let D be a UHF algebra and let $\alpha \in \operatorname{Aut}(D)$ be an automorphism of order two, of the form

$$D = \bigotimes_{n=1}^{\infty} M_{k(n)} \qquad \text{and} \qquad \alpha = \bigotimes_{n=1}^{\infty} \operatorname{Ad}(p_n - q_n),$$

with $k(n) \in \mathbb{Z}_{>0}$ and where $p_n, q_n \in M_{k(n)}$ are projections with $p_n + q_n = 1$ and $\operatorname{rank}(p_n) \geq \operatorname{rank}(q_n)$. Define

$$\lambda_n = \frac{\operatorname{rank}(p_n) - \operatorname{rank}(q_n)}{\operatorname{rank}(p_n) + \operatorname{rank}(q_n)} \geq 0$$

for $n \in \mathbb{Z}_{>0}$, and, for $m \leq n$, define

$$\Lambda(m, n) = \lambda_{m+1} \lambda_{m+2} \cdots \lambda_n \qquad \text{and} \qquad \Lambda(m, \infty) = \lim_{n \to \infty} \Lambda(m, n).$$

Prove the following:

(1) The action α has the Rokhlin property if and only if there are infinitely many $n \in \mathbb{Z}_{>0}$ such that $\operatorname{rank}(p_n) = \operatorname{rank}(q_n)$ (that is, $\lambda_n = 0$).
(2) The action α has the tracial Rokhlin property if and only if $\Lambda(m, \infty) = 0$ for all m.
(3) The action α is pointwise outer if and only if there are infinitely many $n \in \mathbb{Z}_{>0}$ such that $\lambda_n < 1$ (that is, $q_n \neq 0$).

Further prove that if $(k(n))_{n \in \mathbb{Z}_{>0}}$ is bounded and $q_n \neq 0$ for all $n \in \mathbb{Z}_{>0}$, then α has the tracial Rokhlin property.

In Section 2 of [209], in each case various other equivalent conditions are proved, involving tracial states, K-theory, or the dual action.

Example 3.12 of [207] contains a list of other interesting actions which have the tracial Rokhlin property but not the Rokhlin property. They include the actions in Example 3.29, Exercise 10.23, and Exercise 10.24.

We now show that when A is finite, the last condition in Definition 14.1 (part (4), the requirement that $\|exe\| > 1 - \varepsilon$) can be omitted. The next lemma comes from an argument that goes back to Cuntz, in the proof of Lemma 1.7 of [48].

Lemma 14.12 (Lemma 1.14 of [208])**.** Let A be a C*-algebra with property (SP), let $x \in A_+ \setminus \{0\}$ satisfy $\|x\| = 1$, and let $\varepsilon > 0$. Then there is a nonzero projection $p \in \overline{xAx}$ such that, for every nonzero projection q satisfying $q \leq p$, we have

$$\|qx - xq\| < \varepsilon, \qquad \|qxq - q\| < \varepsilon, \qquad \text{and} \qquad \|qxq\| > 1 - \varepsilon.$$

Proof. Choose continuous functions $g_1, g_2 \colon [0,1] \to [0,1]$ satisfying $g_1(0) = 0$, $g_1(t) = 1$ for $t \geq 1 - \frac{1}{4}\varepsilon$, and $|g_1(t) - t| \leq \frac{\varepsilon}{4}$ for all t, and such that $g_2(1) = 1$ and $g_1 g_2 = g_2$. Define $y = g_1(x)$ and $z = g_2(x)$. These elements satisfy $\|x - y\| \leq \frac{\varepsilon}{4}$ and $yz = z$. Since $1 \in \mathrm{sp}(x)$, we have $z \neq 0$. Property (SP) provides a nonzero projection $p \in \overline{zAz}$. Now suppose that q is a nonzero projection such that $q \leq p$. Since $q \in \overline{zAz}$, we have $yq = qy = q$. So

$$\|qx - xq\| \leq 2\|x - y\| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Moreover,

$$\|qxq - q\| = \|qxq - qyq\| \leq \|x - y\| \leq \frac{\varepsilon}{4} < \varepsilon,$$

whence also $\|qxq\| > 1 - \varepsilon$. This completes the proof. □

Lemma 14.13 (Lemma 1.15 of [208])**.** Let A be an infinite dimensional finite unital C*-algebra with property (SP). Let $x \in A_+$ satisfy $\|x\| = 1$, and let $\varepsilon > 0$. Then there is a nonzero projection $q \in \overline{xAx}$ such that, for every projection $e \in A$ satisfying $1 - e \precsim q$, we have $\|exe\| > 1 - \varepsilon$.

Proof. We apply Lemma 14.12 with $x^{1/2}$ in place of x and with $\frac{\varepsilon}{5}$ in place of ε. Since $\overline{x^{1/2} A x^{1/2}} = \overline{xAx}$, this gives a nonzero projection $p \in \overline{xAx}$ such that for every nonzero projection $q \leq p$ we have, in particular,

$$\left\| q x^{1/2} q - q \right\| < \frac{\varepsilon}{5} \qquad \text{and} \qquad \left\| q x^{1/2} - x^{1/2} q \right\| < \frac{\varepsilon}{5}.$$

Combining these estimates gives

$$\left\| q x^{1/2} - q \right\| < \frac{2\varepsilon}{5}. \tag{14.1}$$

Using Lemma 11.21, choose a nonzero projection $q \leq p$ such that $p - q \neq 0$. Now let $e \in A$ be a projection satisfying $1 - e \precsim q$ and $\|exe\| \leq 1 - \varepsilon$. Using $\|a^* a\| = \|a a^*\|$ at the first and fourth steps and (14.1) at the second step, we get

$$\|epe\| = \|pep\| < \left\| p x^{1/2} e x^{1/2} p \right\| + \frac{4\varepsilon}{5} \leq \left\| x^{1/2} e x^{1/2} \right\| + \frac{4\varepsilon}{5} = \|exe\| + \frac{4\varepsilon}{5} < 1 - \frac{\varepsilon}{5}.$$

So

$$\|e - e(1 - p)\| = \|ep\| = \|epe\|^{1/2} < \left(1 - \tfrac{\varepsilon}{5}\right)^{1/2} < 1.$$

By Lemma 11.8, we now get $e \precsim 1 - p$. We have $1 - e \precsim q$ by assumption, so it follows that $1 \precsim 1 - (p - q)$. We have contradicted finiteness of A. □

Lemma 14.14 (Lemma 1.16 of [208])**.** Let A be an infinite dimensional finite simple separable unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then α has the tracial Rokhlin property if and only if for every finite set $F \subset A$, every $\varepsilon > 0$, and every $x \in A_+ \setminus \{0\}$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
(3) With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.

Proof. Since (1), (2), and (3) are all part of Definition 14.1, the tracial Rokhlin property certainly implies the condition in the lemma. So assume the condition in the lemma holds.

If A does not have property (SP), we can choose $x \in A_+ \setminus \{0\}$ so that the hereditary subalgebra it generates contains no nonzero projections. Then the projection e in condition (3) must be equal to 1. This shows that α has the Rokhlin property. (This is the same proof as for Lemma 14.3.) Accordingly, we assume that A has property (SP).

Let $F \subset A$ be finite, let $\varepsilon > 0$, and let $x \in A_+$ satisfy $\|x\| = 1$. Lemma 14.13 gives us a nonzero projection $q \in \overline{xAx}$ such that for all projections $e \in A$ with $1 - e \precsim q$, we have $\|exe\| > 1 - \varepsilon$. Apply the hypothesis of the lemma with F and ε as given and with q in place of x. We get projections $e_g \in A$ for $g \in G$. As in (3), define $e = \sum_{g \in G} e_g$. Then $\|exe\| > 1 - \varepsilon$ by the relation $1 - e \precsim q$ and the choice of q using Lemma 14.13. This completes the proof. $\square$

It is convenient to have a formally stronger version of the tracial Rokhlin property, in which the defect projection is α-invariant. This is a weaker statement than Theorem 14.4, but is much easier to prove.

Lemma 14.15 (Lemma 1.17 of [208])**.** Let G be a finite group, let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on A which has the tracial Rokhlin property. Let $F \subset A$ be finite, let $\varepsilon > 0$, and let $x \in A$ be a positive element with $\|x\| = 1$. Then there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
(3) With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.
(4) With e as in (3), we have $\|exe\| > 1 - \varepsilon$.
(5) The projection e of (3) is α-invariant.

Proof. Without loss of generality $\|a\| \leq 1$ for all $a \in F$. Set

$$\varepsilon_0 = \min\left(\frac{\varepsilon}{41}, \frac{1}{20}\right).$$

Choose δ as in Lemma 11.12 with ε_0 in place of ε, and also require $\delta \leq \frac{\varepsilon}{2}$. Set

$$\delta_0 = \frac{\delta}{\mathrm{card}(G)}.$$

Apply Definition 14.1 to α, with F and x as given, and with δ_0 in place of ε. Let $(p_g)_{g \in G}$ be the resulting family of projections. Define $p = \sum_{h \in G} p_h$. For $g \in G$ we have

$$\|\alpha_g(p) - p\| \leq \sum_{h \in G} \|\alpha_g(p_h) - p_{gh}\| < \mathrm{card}(G)\delta_0 \leq \delta.$$

Set

$$b = \frac{1}{\mathrm{card}(G)} \sum_{g \in G} \alpha_g(p).$$

Then b is in the fixed point algebra A^G and

$$\|b - p\| \leq \frac{1}{\mathrm{card}(G)} \sum_{g \in G} \|\alpha_g(p) - p\| < \delta.$$

The choice of δ using Lemma 11.12 means that there is a projection $e \in A^G$ such that $\|e - p\| < \varepsilon_0$.

Since $\varepsilon_0 \leq \frac{1}{20}$, Lemma 11.9 provides a unitary $v \in A$ such that $\|v - 1\| \leq 10\|e - p\| < 10\varepsilon_0$ and $vpv^* = e$. Now define $e_g = vp_gv^*$ for $g \in G$. Clearly $\|e_g - p_g\| < 20\varepsilon_0$. So, for $g, h \in G$,

$$\|\alpha_g(e_h) - e_{gh}\| \leq \|e_h - p_h\| + \|e_{gh} - p_{gh}\| + \|\alpha_g(p_h) - p_{gh}\| < 20\varepsilon_0 + 20\varepsilon_0 + \delta_0 \leq \varepsilon.$$

For $g \in G$ and $a \in F$, and using $\|a\| \leq 1$, we similarly get

$$\|e_g a - a e_g\| < 20\varepsilon_0 + 20\varepsilon_0 + \delta_0 \leq \varepsilon.$$

We have

$$\|(1 - e) - (1 - p)\| < 20\varepsilon_0 \leq 1,$$

so $1 - e \sim 1 - p$, and is hence Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x. Finally,

$$\|exe\| \geq \|pxp\| - 2\|e - p\| > 1 - \delta_0 - 2\varepsilon_0 \geq 1 - \frac{\varepsilon}{41} - \frac{2\varepsilon}{41} > 1 - \varepsilon.$$

This completes the proof. □

We adapt Lemma 13.19, the key step in the proof that crossed products of AF algebras by Rokhlin actions are AF (Theorem 13.15), to the tracial Rokhlin property.

Lemma 14.16. Let G be a finite group, and set $n = \mathrm{card}(G)$. Identify M_n with $L(l^2(G))$, and let for $g, h \in G$ let $e_{g,h}$ be the rank one operator on $l^2(G)$ given by $e_{g,h}\xi = \langle \xi, \delta_h \rangle \delta_g$, as in Notation 10.7. Also, for $g \in G$ let u_g be the standard unitary of Notation 8.7. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that the following holds. Let (G, A, α) be a G-algebra, let $(e_g)_{g \in G}$ be a family of orthogonal projections, and let $F \subset A$ be a finite set such that $\|a\| \leq 1$ for all $a \in F$. Suppose that:

(1) $\|\alpha_g(e_h) - e_{gh}\| < \delta$ for all $g, h \in G$.
(2) $\|e_g a - a e_g\| < \delta$ for all $g \in G$ and all $a \in F$.
(3) The projection $e = \sum_{g \in G} e_g$ is α-invariant.

Then there exists a unital homomorphism $\varphi\colon M_n \otimes e_1 A e_1 \to eAe$ such that for every $a \in F \cup \{u_g : g \in G\}$ there are

$$x \in M_n \otimes e_1 A e_1 \qquad \text{and} \qquad y \in (1 - e)A(1 - e)$$

with $\|[\varphi(x) + y] - a\| < \varepsilon$, and such that (using standard matrix unit notation) for every $a \in e_1 A e_1$ we have $\varphi(e_{1,1} \otimes a) = a$.

Proof. Apply Lemma 13.19 with $\frac{\varepsilon}{2}$ in place of ε, getting a number $\delta > 0$, and further require that $\delta \leq \varepsilon/(4n)$. Now let (G, A, α) be a G-algebra, let $(e_g)_{g\in G}$ be a family of orthogonal projections, let $F \subset A$ is a finite set such that $\|a\| \leq 1$ for all $a \in F$, and suppose that the conditions (1), (2), and (3) hold. Define $e = \sum_{g\in G} e_g$. Using $e_g e - e e_g = e_g$ for $g \in G$, it is easy to check that $\|e_g eae - eaee_g\| < \delta$ for all $g \in G$ and all $a \in F$. Since e is α-invariant, G acts on the algebra eAe. Call this action β, and for $g \in G$ let $v_g \in C^*(G, eAe, \beta)$ be the standard unitary of Notation 8.7. As usual, we let $u_g \in C^*(G, A, \alpha)$ be the standard unitary in this crossed product. One immediately checks that $C^*(G, eAe, \beta)$ is a subalgebra of $C^*(G, A, \alpha)$, in fact, that $C^*(G, eAe, \beta) = eC^*(G, A, \alpha)e$, that u_g commutes with e for all $g \in G$, and that $v_g = eu_g e$.

With this in mind, apply the choice of δ using Lemma 13.19 to the algebra eAe and the finite set $\{eae \colon a \in F\}$. The result is a unital homomorphism

$$\varphi\colon M_n \otimes e_1Ae_1 \to C^*(G, eAe, \beta)$$

such that for every $a \in F \cup \{u_g \colon g \in G\}$, we have

$$\operatorname{dist}\big(eae,\, \varphi(M_n \otimes e_1Ae_1)\big) < \frac{\varepsilon}{2}, \tag{14.2}$$

and $\varphi(e_{1,1} \otimes a) = a$ for all $a \in e_1Ae_1$. This last condition is the last part of the conclusion of the lemma.

We next claim that for all $a \in F \cup \{u_g \colon g \in G\}$, we have

$$\big\|a - [eae + (1-e)a(1-e)]\big\| < \frac{\varepsilon}{2}. \tag{14.3}$$

For $a = u_g$ with $g \in G$, this is immediate since e commutes with u_g. To prove the claim for $a \in F$, first estimate

$$\|ea - ae\| \leq \sum_{g\in G} \|e_g a - a e_g\| < \operatorname{card}(G)\delta \leq \frac{\varepsilon}{4}.$$

Therefore

$$\|ea(1-e)\| < \frac{\varepsilon}{4} \qquad \text{and} \qquad \|(1-e)ae\| < \frac{\varepsilon}{4},$$

so

$$\big\|a - [eae + (1-e)a(1-e)]\big\| \leq \|ea(1-e)\| + \|(1-e)ae\| < \frac{\varepsilon}{2}.$$

Now let $a \in F \cup \{u_g \colon g \in G\}$. Use (14.2) to choose $x \in M_n \otimes e_1Ae_1$ such that $\|\varphi(x) - eae\| < \frac{\varepsilon}{2}$. Set $y = (1-e)a(1-e)$. Then, using (14.3) at the second step, we get

$$\|[\varphi(x) + y] - a\| \leq \|\varphi(x) - eae\| + \big\|[eae + (1-e)a(1-e)] - a\big\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as desired. This completes the proof. $\square$

Theorem 14.17 (Theorem 2.6 of [208])**.** Let G be a finite group, and let A be an infinite dimensional simple separable unital C*-algebra with tracial rank zero. Let $\alpha\colon G \to \operatorname{Aut}(A)$ be an action of G on A which has the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has tracial rank zero.

The proof will be given at the end of this section. As mentioned above, in [208] the condition $p \neq 0$ was omitted in one of the ingredients, Proposition 2.3 of [208].

The basic idea is the same as that of the proof of Theorem 13.15. The main difference is that there is a small "error projection" in both the definition of the tracial Rokhlin property and the definition of tracial rank zero. The main technical complication is that when one carries out the obvious modification of the proof of Theorem 13.15, what one gets is that the "error projection" in the definition of tracial rank zero for the crossed product, which is supposed to be Murray-von Neumann equivalent to a projection in a previously specified hereditary subalgebra of $C^*(G, A, \alpha)$, actually comes out to be Murray-von Neumann equivalent to a projection in a previously specified hereditary subalgebra of A. A priori, this is not good enough. The day is saved by the following theorem, which is a special case of Theorem 4.2 of [127].

Theorem 14.18 (see Theorem 4.2 of [127])**.** Let G be a finite group. Let A be a simple unital C*-algebra with property (SP). Let $\alpha \colon G \to \mathrm{Aut}(A)$ be a pointwise outer action. Let $B \subset C^*(G, A, \alpha)$ be a nonzero hereditary subalgebra. Then there exists a nonzero projection $p \in B$ which is Murray-von Neumann equivalent to a projection in A.

We omit the proof of Theorem 14.18. Instead, we give a proof of a special case which is good enough for the purposes of this section, with some of the lemmas given in greater generality. Our proof requires less work and uses methods closer to those of these notes.

Definition 14.19. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. We say that α is *minimal*, or that A is *G-simple*, if the only G-invariant (closed) ideals in A are $\{0\}$ and A.

This definition generalizes the usual definition of minimality of a group action on a locally compact Hausdorff space, which is given in Definition 2.1. We make three brief comments. First, if A is simple (the case of most interest to us now), then clearly A is G-simple. However, in Theorem 14.22, where we assume G-simplicity, there is no simplification in the proof by assuming simplicity instead. Second, G-simplicity is an elementary necessary condition for simplicity of the reduced crossed product $C^*_{\mathrm{r}}(G, A, \alpha)$, since if I is a proper G-invariant ideal in A, then $C^*_{\mathrm{r}}(G, I, \alpha)$ is a proper ideal in $C^*_{\mathrm{r}}(G, A, \alpha)$. (See Theorem 9.24(4).) Third, G-simplicity is not a sufficient condition for simplicity of the reduced crossed product. Indeed, the trivial action of a locally compact group G on $\mathbb{C}$ is obviously minimal, but the reduced crossed product is $C^*_{\mathrm{r}}(G)$, which is usually not simple (in particular, never simple if G is amenable and nontrivial).

We introduce a property of actions, not previously named, which we call Kishimoto's condition after the paper [142] in which it appeared in close to this form. We proceed via Kishimoto's condition in this section for two reasons. First, it is the first step in the proofs of two different results which we need here. Second, we will want it again later, for proofs of these same results under weaker hypotheses.

Definition 14.20. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. We say that α satisfies *Kishimoto's condition* if for every positive element $x \in A$ with $\|x\| = 1$, every finite set $F \subset G \setminus \{1\}$, every finite set $S \subset A$, and every $\varepsilon > 0$, there is a positive element $c \in A$ with $\|c\| = 1$ such that:

(1) $\|cxc\| > 1 - \varepsilon$.
(2) $\|cb\alpha_g(c)\| < \varepsilon$ for all $g \in F$ and $b \in S$.

This condition is essentially the conclusion of Lemma 3.2 of [142]. It is a kind of freeness condition. For example, if $A = C(X)$ and $f \in C(X)$ is a function such that $\mathrm{supp}(f) \cap g \cdot \mathrm{supp}(f) = \varnothing$, then $fb\alpha_g(f) = 0$ for every $y \in C(X)$. It is shown in [142] that if A is simple and α is pointwise outer, then α satisfies Kishimoto's condition. In fact, as discussed there, weaker hypotheses suffice. (In [142], see Lemma 3.2 and the second part of Remark 2.2.) We give here a much easier proof of a special case of this fact, strengthening the hypotheses to the tracial Rokhlin property. The fact that our hypotheses are unnecessarily strong is suggested by the fact that we never use the condition in the tracial Rokhlin property which requires that $1 - e$ be "small".

We point out that a condition related to Kishimoto's condition has been generalized in [184] to conditional expectations on unital C*-algebras, with the reduced crossed product situation corresponding to the standard conditional expectation (Definition 9.18). The definition is near the beginning of Section 2 of [184]. In general, outerness of the conditional expectation is stronger than pointwise outerness of the action.

Lemma 14.21. Let G be a finite group, let A be an infinite dimensional simple unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of G on A. Assume that α has the tracial Rokhlin property (Definition 14.1). Then α satisfies Kishimoto's condition (Definition 14.20). In fact, the element a in the conclusion can be taken to be a projection.

Proof. Let $x \in A$ be a positive element with $\|x\| = 1$, let $S \subset A$ be finite, and let $\varepsilon > 0$. We may as well take the finite set $F \subset G \setminus \{1\}$ in Kishimoto's condition to be $G \setminus \{1\}$ itself.

Without loss of generality $\varepsilon < 1$. Set

$$n = \mathrm{card}(G), \qquad M = \max\left(1, \sup_{b \in S} \|b\|\right), \qquad \text{and} \qquad \varepsilon_0 = \min\left(\frac{\varepsilon}{M+1}, \frac{\varepsilon}{n^2}\right).$$

Apply the tracial Rokhlin property (Definition 14.1) with $S \cup \{x\}$ in place of F, with ε_0 in place of ε, and with x as given. Call the resulting family of projections $(e_g)_{g \in G}$, and set $e = \sum_{g \in G} e_g$.

We have

$$\left\| exe - \sum\nolimits_{g \in G} e_g x e_g \right\| \leq \sum\nolimits_{g \neq h} \|e_g x e_h\|.$$

Since $e_g e_h = 0$ for $g \neq h$, the term $\|e_g x e_h\|$ on the right is dominated by $\|x e_h - e_h x\| < \varepsilon_0$. Therefore

$$\left\| exe - \sum\nolimits_{g \in G} e_g x e_g \right\| < n(n-1)\varepsilon_0,$$

and

$$\left\| \sum\nolimits_{g \in G} e_g x e_g \right\| \geq \|exe\| - \left\| exe - \sum\nolimits_{g \in G} e_g x e_g \right\| > 1 - \varepsilon_0 - n(n-1)\varepsilon_0 \geq 1 - \varepsilon.$$

Since the elements $e_g x e_g$, for $g \in G$, are orthogonal, it follows that there is $g_0 \in G$ such that $\|e_{g_0} x e_{g_0}\| > 1 - \varepsilon$. Set $a = e_{g_0}$. Since $\varepsilon < 1$, we have $e_{g_0} \neq 0$, so $\|a\| = 1$.

Now let $b \in S$ and let $h \in G \setminus \{1\}$. Then $e_{g_0} e_{h g_0} = 0$, so

$$\begin{aligned}
\|a b \alpha_h(a)\| &= \|e_{g_0} b \alpha_h(e_{g_0})\| = \|e_{g_0} b \alpha_h(e_{g_0}) - e_{g_0} e_{h g_0} b\| \\
&\leq \|e_{g_0}\| \cdot \|b\| \cdot \|\alpha_h(e_{g_0}) - e_{h g_0}\| + \|e_{g_0}\| \cdot \|b e_{h g_0} - e_{h g_0} b\| \\
&< M\varepsilon_0 + \varepsilon_0 \leq \varepsilon.
\end{aligned}$$

This completes the proof of Kishimoto's condition. $\square$

The following two results are stated for discrete groups rather than merely for finite groups. The finite group case is all that is needed here. The proofs when G is finite are a bit simpler, because one can omit the step in which an element of $C^*_{\mathrm{r}}(G, A, \alpha)$ is approximated by an element of $C_{\mathrm{c}}(G, A)$, eliminating some of the estimates, but otherwise the proofs are the same.

The next theorem is contained in Theorem 3.1 of [142], and follows the proof of Theorem 3.2 of [67].

Theorem 14.22. Let A be a C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a minimal action (Definition 14.19) of a discrete group G on A. Assume that α satisfies Kishimoto's condition (Definition 14.20). Then $C^*_{\mathrm{r}}(G, A, \alpha)$ is simple.

Proof. Let $J \subset C_{\mathrm{c}}(G, A)$ be a proper ideal. For $g \in G$ let u_g be the standard unitary of Notation 8.7.

We first claim that $J \cap A$ is a G-invariant ideal in A. That it is an ideal is clear. Let $g \in G$. If A is unital, then $u_g \in C^*_{\mathrm{r}}(G, A, \alpha)$, and for $a \in J \cap A$ we have $\alpha_g(a) = u_g a u_g^* \in J \cap A$. In the general case, let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for A. Then for $\lambda \in \Lambda$, the elements $e_\lambda u_g$ and $u_g e_\lambda = \alpha_g(e_\lambda) u_g$ are in $C^*_{\mathrm{r}}(G, A, \alpha)$, so

$$\alpha_g(a) = \lim_{\lambda} \alpha_g(e_\lambda a e_\lambda) = \lim_{\lambda} u_g e_\lambda a e_\lambda u_g \in J \cap A.$$

The claim is proved.

We next claim that $J \cap A = \{0\}$. Since α is minimal, we need only rule out $A \subset J$. Suppose $A \subset J$. Let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for A. For $a \in A$ and $g \in G$, we have $a u_g = \lim_\lambda a e_\lambda u_g \in J$. Therefore $C_{\mathrm{c}}(G, A) \subset J$, whence $J = C^*_{\mathrm{r}}(G, A, \alpha)$. This contradiction proves the claim.

Let $E\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be the standard conditional expectation, as in Definition 9.18. We now claim that if $a \in J$ then $E(a^* a) = 0$. Given the claim, since E is faithful (Proposition 9.16(4)), this implies that $a^* a = 0$, whence $a = 0$. So $J = \{0\}$, proving the theorem.

We prove the claim. Let $a \in J$ and let $\varepsilon > 0$. We show that $\|E(a^* a)\| < \varepsilon$. Choose $y \in C_{\mathrm{c}}(G, A)$ with $\|y - a\|$ so small that $\|y^* y - a^* a\| < \frac{\varepsilon}{5}$. Then there are a finite set $F \subset G$ and elements $b_g \in A$ for $g \in F$ such that $y^* y = \sum_{g \in F} b_g u_g$. Without loss of generality $1 \in F$. We must have $b_1 = E(y^* y) \geq 0$. Also

$$\|b_1 - E(a^* a)\| \leq \|y^* y - a^* a\| < \tfrac{\varepsilon}{5}. \tag{14.4}$$

Suppose $b_1 = 0$. Then (14.4) implies $\|E(a^* a)\| < \varepsilon$, as desired. So we may assume $b_1 \neq 0$.

Set

$$x = \|b_1\|^{-1} b_1 \qquad \text{and} \qquad \varepsilon_0 = \min\left(\frac{1}{2}, \frac{\varepsilon}{5 \cdot \mathrm{card}(F)}\right).$$

Apply Kishimoto's condition with $F \setminus \{1\}$ in place of F, with

$$S = \{b_g \colon g \in F \setminus \{1\}\},$$

with x as given, and with ε_0 in place of ε. Let c be the resulting element.

We can now estimate

$$\begin{aligned}\|c y^* y c - c b_1 c\| &= \left\| \textstyle\sum_{g \in F \setminus \{1\}} c b_g u_g c \right\| = \left\| \textstyle\sum_{g \in F \setminus \{1\}} c b_g \alpha_g(c) u_g \right\| \\ &\leq \sum_{g \in F \setminus \{1\}} \|c b_g \alpha_g(c)\| < \mathrm{card}(F) \varepsilon_0 \leq \frac{\varepsilon}{5}.\end{aligned}$$

Therefore, using $\|c\| \leq 1$,

$$\|ca^*ac - cb_1c\| \leq \|a^*a - y^*y\| + \|cy^*yc - cb_1c\| < \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \frac{2\varepsilon}{5}.$$

Let $\pi\colon C^*_{\mathrm{r}}(G, A, \alpha) \to C^*_{\mathrm{r}}(G, A, \alpha)/J$ be the quotient map. Since $J \cap A = \{0\}$, the restriction $\pi|_A$ is injective, so $\|\pi(cb_1c)\| = \|cb_1c\|$. On the other hand, $ca^*ac \in J$, so $\pi(ca^*ac) = 0$. Thus

$$\|cb_1c\| = \|\pi(cb_1c)\| = \|\pi(cb_1c - ca^*ac)\| \leq \|cb_1c - ca^*ac\| < \frac{2\varepsilon}{5}.$$

The choice of c and the relation $\varepsilon_0 \leq \frac{1}{2}$ imply that

$$\|cb_1c\| > \|b_1\|\big(1 - \varepsilon_0\big) \geq \tfrac{1}{2}\|b_1\|.$$

Thus $\|b_1\| \leq 2\|cb_1c\| < \frac{4\varepsilon}{5}$. Combining this with (14.4), we get $\|E(a^*a)\| < \varepsilon$. This completes the proof. ☐

The next theorem is contained in Theorem 4.2 of [127].

Theorem 14.23. Let A be a C*-algebra which has property (SP), and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on A. Assume that α satisfies Kishimoto's condition (Definition 14.20). Then for every nonzero hereditary subalgebra $D \subset C^*_{\mathrm{r}}(G, A, \alpha)$, there is a nonzero projection $p \in D$ which is Murray-von Neumann equivalent to a projection in A.

The proof of Lemma 16.23 is very similar but done in an easier context, so one may want to read the proof of that lemma first.

Proof of Theorem 14.23. Let $E\colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be the standard conditional expectation (Definition 9.18). Choose $a \in D_+ \setminus \{0\}$. Since E is faithful (Proposition 9.16(4)), we have $E(a) \neq 0$. By scaling, we may assume $\|E(a)\| = 1$. Choose $y \in C_{\mathrm{c}}(G, A)$ with $\big\|y - a^{1/2}\big\|$ so small that $\|y^*y - a\| < \frac{1}{4}$. Then there are a finite set $F \subset G$ and elements $b_g \in A$ for $g \in F$ such that $y^*y = \sum_{g \in F} b_g u_g$. Without loss of generality $1 \in F$. Set

$$\delta = \frac{1}{2(\mathrm{card}(F) + 2)}.$$

Apply Kishimoto's condition with $F \setminus \{1\}$ in place of F, with $S = \{b_g \colon g \in F \setminus \{1\}\}$, with $x = E(a)$, and with δ in place of ε. Let c be the resulting element.

For $g \in G$ let u_g be the standard unitary of Notation 8.7. We can now estimate

$$\begin{aligned}\|cy^*yc - cE(y^*y)c\| &= \Big\|\sum\nolimits_{g \in F \setminus \{1\}} cb_g u_g c\Big\| = \Big\|\sum\nolimits_{g \in F \setminus \{1\}} cb_g \alpha_g(c) u_g\Big\| \\ &\leq \sum_{g \in F \setminus \{1\}} \|cb_g \alpha_g(c)\| < \mathrm{card}(F)\delta.\end{aligned}$$

Therefore, using $\|c\| \leq 1$,

$$\|cac - cE(a)c\| \leq 2\|a - yy^*\| + \|cy^*yc - cE(y^*y)c\| < \tfrac{1}{2} + \mathrm{card}(F)\delta. \tag{14.5}$$

Let $f, f_0\colon [0, 1] \to [0, 1]$ be the continuous functions which are linear on $[0,\, 1-2\delta]$ and $[1 - 2\delta,\, 1]$, and satisfy

$$f(0) = f_0(0) = 0, \quad f_0(1 - 2\delta) = 0, \quad f(1 - 2\delta) = 1, \quad \text{and} \quad f(1) = f_0(1) = 1.$$

Then $ff_0 = f_0$. Also $\|cE(a)c\| > 1 - \delta$, so $f_0(cE(a)c) \neq 0$. Use property (SP) to choose a nonzero projection e in the hereditary subalgebra of A generated by

$f_0(cE(a)c)$. Since $ff_0 = f_0$, we have $f(cE(a)c)e = e$. Since $|f(t) - t| \leq 2\delta$ for $t \in [0, 1]$, we get $\|cE(a)ce - e\| < 2\delta$. Combining this estimate with (14.5), we get

$$\|ecace - e\| \leq \|cac - cE(a)c\| + \|e\| \cdot \|cE(a)ce - e\| < \tfrac{1}{2} + \mathrm{card}(F)\delta + 2\delta = 1.$$

Set $z_0 = a^{1/2}ce$. Then $z_0^* z_0 = ecace \in eC^*_{\mathrm{r}}(G, A, \alpha)e$, and moreover satisfies $\|z_0^* z_0 - e\| < 1$. Evaluating functional calculus in $eC^*_{\mathrm{r}}(G, A, \alpha)e$, we may therefore set $r = (z_0^* z_0)^{-1/2}$. Then $z = z_0 r$ satisfies $z^* z = e$. Also $p = zz^*$ is a projection such that

$$p = a^{1/2}cer^2eca^{1/2} \in a^{1/2}C^*_{\mathrm{r}}(G, A, \alpha)a^{1/2} \subset D.$$

Since p is Murray-von Neumann equivalent to e, the proof is complete. □

We are now ready for the proof of Theorem 14.17.

Proof. We will use Lemma 11.36, taking T to be (following Notation 8.7 for the standard unitaries in the crossed product)

$$T = \{u_g : g \in G\} \cup \{a \in A : \|a\| \leq 1\}.$$

Accordingly, let $S \subset T$ be finite, let $\varepsilon > 0$, and let $c \in C^*(G, A, \alpha)_+ \setminus \{0\}$. We may take

$$S = \{u_g : g \in G\} \cup F$$

with $F \subset A$ finite and $\|a\| \leq 1$ for all $a \in F$. We further write

$$S = \{a_1, a_2, \ldots, a_N\}.$$

In Lemma 14.16, choose $\delta > 0$ for the number $\frac{\varepsilon}{4}$ in place of ε. Since A has property (SP) by Corollary 11.39 and α satisfies Kishimoto's condition by Lemma 14.21, we can apply Theorem 14.23 to find a nonzero projection $q \in A$ which is Murray-von Neumann equivalent to a projection in $\overline{cC^*(G, A, \alpha)c}$. Again using the fact that A has property (SP), use Lemma 11.21 to choose nonzero orthogonal projections $q_1, q_2 \in qAq$. Apply the strengthening of the tracial Rokhlin property in Lemma 14.15, with F as given, with δ in place of ε, and with q_1 in place of x, getting projections $e_g \in A$ for $g \in G$ as there. In particular, the projection $e = \sum_{g \in G} e_g$ is G-invariant and satisfies $1 - e \precsim q_1$. The choice of δ using Lemma 14.16 implies that there is a unital homomorphism $\varphi \colon M_n \otimes e_1Ae_1 \to eAe$ such that for $j = 1, 2, \ldots, N$ there are

$$x_j \in M_n \otimes e_1Ae_1 \qquad \text{and} \qquad y_j \in (1 - e)A(1 - e)$$

with $\|[\varphi(x_j) + y_j] - a_j\| < \frac{\varepsilon}{4}$. Moreover, we have $\varphi(e_{1,1} \otimes a) = a$ for all $a \in e_1Ae_1$.

Use Lemma 11.17 to choose a nonzero projection $f \in e_1Ae_1$ such that $f \precsim q_2$. Since A has tracial rank zero, so does e_1Ae_1 (by Lemma 11.40) and therefore also so does $M_n \otimes e_1Ae_1$ (by Lemma 11.41). Therefore there exist a projection $p_0 \in M_n \otimes e_1Ae_1$, a unital finite dimensional subalgebra $D_0 \subset p_0(M_n \otimes e_1Ae_1)p_0$, and $d_1, d_2, \ldots, d_N \in D_0$ such that:

(1) $\|[x_j, p_0]\| < \frac{\varepsilon}{2}$ for $j = 1, 2, \ldots, N$.
(2) $\|p_0x_jp_0 - d_j\| < \frac{\varepsilon}{2}$ for $j = 1, 2, \ldots, N$.
(3) $1 - p_0 \precsim e_{1,1} \otimes f$.

Set $p = \varphi(p_0)$ and $D = \varphi(D_0)$. Then

$$\begin{aligned} 1 - p &= (1 - e) + (e - p) \\ &= (1 - e) + \varphi(1 - p_0) \precsim (1 - e) + \varphi(e_{1,1} \otimes f) = (1 - e) + f \precsim q_1 + q_2 \leq q, \end{aligned}$$

so $1-p$ is Murray-von Neumann equivalent to a projection in $\overline{cC^*(G,A,\alpha)c}$. Next, for $j = 1, 2, \ldots, N$, we have $py_j = y_j p = 0$. So

$$\|[p, \varphi(x_j) + y_j]\| = \|[p, \varphi(x_j)]\| \leq \|[x_j, p_0]\| < \frac{\varepsilon}{2},$$

whence

$$\|[p, a_j]\| \leq 2\|a_j - [\varphi(x_j) + y_j]\| + \|[p, \varphi(x_j) + y_j)]\| < 2\left(\frac{\varepsilon}{4}\right) + \frac{\varepsilon}{2} = \varepsilon.$$

Moreover, $\varphi(d_j) \in D$ and

$$\begin{aligned}\|pa_jp - \varphi(d_j)\| &\leq \|a_j - [\varphi(x_j) + y_j]\| + \big\|p[\varphi(x_j) + y_j]p - \varphi(d_j)\big\| \\ &= \|a_j - [\varphi(x_j) + y_j]\| + \|p\varphi(x_j)p - \varphi(d_j)\| \\ &\leq \|a_j - [\varphi(x_j) + y_j]\| + \|p_0 x_j p_0 - d_j\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

This completes the proof. $\square$

Part 4. An Introduction to Crossed Products by Minimal Homeomorphisms

15. Minimal Actions and their Crossed Products

In this section, we discuss free and essentially free minimal actions of countable discrete groups on compact metric spaces, with emphasis on minimal homeomorphisms (actions of $\mathbb{Z}$). We give two simplicity proofs, using very different methods. One works for free minimal actions, and the method gives further information, as well as some information when the action is not minimal. See Theorem 15.20 and Theorem 15.22. The second proof is a special case of a more general simplicity theorem; the case we prove allows some simplification of the argument. Our theorem is Theorem 15.10, and its proof is given before Theorem 15.25. The full theorem is stated as Theorem 15.25. Both proofs end with an argument related to the proof that Kishimoto's condition (Definition 14.20) implies simplicity of the crossed product (Theorem 14.22), but the two proofs use quite different routes to get there.

We recall Definition 2.1, specialized to the case of locally compact groups and spaces. It is also the specialization of Definition 14.19 to the commutative case.

Definition 15.1. Let a locally compact group G act continuously on a locally compact space X. The action is called *minimal* if whenever $T \subset X$ is a closed subset such that $gT \subset T$ for all $g \in G$, then $T = \varnothing$ or $T = X$.

In short, there are no nontrivial invariant closed subsets. This is the topological analog of an ergodic action on a measure space (Definition 2.6). It is equivalent (Lemma 2.2) that every orbit be dense.

If the action of G on X is not minimal, then there is a nontrivial invariant closed subset $T \subset X$, and $C^*(G, X \setminus T)$ is a nontrivial ideal in $C^*(G, X)$. See Theorem 8.32. Thus $C^*(G, X)$ is not simple. In fact, $C^*_{\mathrm{r}}(G, X)$ is not simple, by Theorem 9.24(4).

For the case $G = \mathbb{Z}$, the conventional terminology is a bit different.

Definition 15.2. Let X be a locally compact Hausdorff space, and let $h\colon X \to X$ be a homeomorphism. Then h is called *minimal* if whenever $T \subset X$ is a closed subset such that $h(T) = T$, then $T = \varnothing$ or $T = X$.

Almost all work on minimal homeomorphisms has been on compact spaces. For these, we have the following equivalent conditions.

Lemma 15.3. Let X be a compact Hausdorff space, and let $h\colon X \to X$ be a homeomorphism. Then the following are equivalent:

(1) h is minimal.
(2) Whenever $T \subset X$ is a closed subset such that $h(T) \subset T$, then $T = \varnothing$ or $T = X$.
(3) Whenever $U \subset X$ is an open subset such that $h(U) = U$, then $U = \varnothing$ or $U = X$.
(4) Whenever $U \subset X$ is an open subset such that $h(U) \subset U$, then $U = \varnothing$ or $U = X$.
(5) For every $x \in X$, the orbit $\{h^n(x)\colon n \in \mathbb{Z}\}$ is dense in X.
(6) For every $x \in X$, the forward orbit $\{h^n(x)\colon n \geq 0\}$ is dense in X.

Conditions (1), (3), and (5) are equivalent even when X is only locally compact, and in fact there is an analog for actions of arbitrary groups. Minimality does not imply the other three conditions without compactness, as can be seen by considering the homeomorphism $n \mapsto n+1$ of $\mathbb{Z}$. (This is the case $G = \mathbb{Z}$ of Example 2.12.) Also, even for compact X, it isn't good enough to merely have the existence of some dense orbit, as can be seen by considering the homeomorphism $n \mapsto n+1$ on the two point compactification $\mathbb{Z} \cup \{\pm\infty\}$ of $\mathbb{Z}$. (This action is one of those described in Example 2.15.)

Exercise 15.4. Prove Lemma 15.3.

We recall a few examples.

Example 15.5. Let G be a locally compact group, let $H \subset G$ be a closed subgroup, and let G act on G/H be translation, as in Example 2.12. This action is minimal: there are no nontrivial invariant subsets, closed or not.

Example 15.5 is a "trivial" example of a minimal action. Here are several more interesting ones.

Example 15.6. The irrational rotations in Example 2.16 are minimal homeomorphisms.

Example 15.7. The homeomorphism $x \mapsto x+1$ on the p-adic integers (Example 2.21) is minimal. The orbit of 0 is $\mathbb{Z}$, which is dense, essentially by definition. Every other orbit is a translate of this one, so is also dense. (This is a special case of Proposition 2.18.)

Example 15.8. The shift homeomorphism of $\{0,1\}^{\mathbb{Z}}$ (Example 2.20) and the action of $\mathrm{SL}_2(\mathbb{Z})$ on $S^1 \times S^1$ (Example 2.30) are not minimal. In fact, they have fixed points.

Other examples of minimal homeomorphisms include Furstenberg transformations (Example 2.19) and generalizations (some of which are discussed after Example 2.19), odometers (Definition 2.22; see Exercise 2.23), restrictions of Denjoy homeomorphisms of the circle to their minimal sets ([234]), and certain irrational time maps of suspension flows, studied in [122]. There are many others, such as those discussed after Example 2.38 and those of Theorem 2.42 and Theorem 2.45.

The C*-algebras associated with many minimal homeomorphisms have been studied: Furstenberg transformations and generalizations in [189], [129], [146], Example 4.9 of [204], Sections 2 and 3 of [206], and [237], restricted Denjoy homeomorphisms in [234], irrational time maps of suspension flows in [122], certain classes of minimal homeomorphisms of $S^1 \times X$ in [154], [155], and [156], and certain classes of minimal homeomorphisms of $S^1 \times S^1 \times X$ in [268]. Again, there are others not mentioned here.

Minimal actions are plentiful: a Zorn's Lemma argument shows that every nonempty compact G-space X contains a nonempty invariant closed subset on which the restricted action is minimal.

The transformation group C*-algebra of a minimal action need not be simple. Consider, for example, the trivial action of a group G (particularly an abelian group) on a one point space, for which the transformation group C*-algebra is $C^*(G)$.

Let a locally compact group G act continuously on a locally compact space X. Recall from Definition 2.3 that the action is free if whenever $g \in G \setminus \{1\}$ and $x \in X$, then $gx \neq x$, and is essentially free if whenever $g \in G \setminus \{1\}$, the set $\{x \in X : gx = x\}$ has empty interior.

Remark 15.9. Let X be an infinite compact Hausdorff space, and let $h \colon X \to X$ be a minimal homeomorphism. Then the corresponding action of $\mathbb{Z}$ on X is free. Indeed, if for some $n \neq 0$ and $x \in X$, we have $h^n(x) = x$, then the orbit of x is finite, hence closed, and is clearly invariant. Now minimality contradicts infiniteness of X.

Of course, nothing like Remark 15.9 is true for general groups. For example, let G act freely and minimally on X, let H be some other group, and let $G \times H$ act on X via $(g, h)x = gx$.

Recall from Proposition 2.4 that an essentially free minimal action of an abelian group is free, and from the discussion after Definition 2.3 that essential freeness is not the right concept for nonminimal actions. Example 2.35 gives an action of a countable discrete group which is minimal and essentially free, but not free.

Let a locally compact group G act continuously on a locally compact space X. Recall from Definition 1.5 that the corresponding action $\alpha \colon G \to \mathrm{Aut}(C_0(X))$ is given by $\alpha_g(f)(x) = f(g^{-1}x)$ for $g \in G$, $f \in C_0(X)$, and $x \in X$. Also recall (Definition 8.20 and Definition 9.6) that we abbreviate $C^*(G, C_0(X), \alpha)$ to $C^*(G, X)$ and $C^*_{\mathrm{r}}(G, C_0(X), \alpha)$ to $C^*_{\mathrm{r}}(G, X)$.

The following result is essentially a special case of the corollary at the end of [5]; see the discussion before the corollary and the Remark before Lemma 1 of [5]. We state a much more general result from [5] below (Theorem 15.25).

Theorem 15.10. Let a discrete group G act minimally and essentially freely on a locally compact space X. Then $C^*_{\mathrm{r}}(G, X)$ is simple.

Essential freeness of the action is not necessary. The reduced transformation group C*-algebra for the trivial action of the free group on two generators on a one point space is simple, by Theorem 6.6. However, minimality is certainly necessary. This follows from Theorem 9.24(4).

Corollary 15.11. Let X be an infinite compact Hausdorff space, and let $h \colon X \to X$ be a minimal homeomorphism. Then $C^*(\mathbb{Z}, X, h)$ is simple.

Proof. This follows from Theorem 15.10 and the fact that $\mathbb{Z}$ is amenable, so that the full and reduced crossed products are equal by Theorem 9.7. □

Our proof of Theorem 15.10 will follow [5], and will be given at the end of this section. We first discuss some special cases and different proofs.

First, we point out that, when G is amenable and the action is free, and probably even when the action is only essentially free, Theorem 15.10 can be derived from the theorem of Gootman and Rosenberg described in Remark 10.17. See Corollary 8.22 of [292] for the free case.

Next, we give a simple proof for the special case of an irrational rotation on the circle. It introduces some important ideas which we, regretfully, will not develop further. (Also see Proposition 2.56 of [292].)

Theorem 15.12. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let $h_\theta \colon S^1 \to S^1$ be the homeomorphism $h_\theta(\zeta) = e^{2\pi i \theta}\zeta$. Then $C^*(\mathbb{Z}, S^1, h_\theta)$ is simple.

Proof. Following Example 10.25, we identify $C^*(\mathbb{Z}, S^1, h_\theta)$ with the universal C*-algebra A_θ in Example 3.10 generated by unitaries u and v satisfying $vu = e^{2\pi i \theta}uv$, by identifying v with the function $\zeta \mapsto \zeta$ on S^1 and identifying u with the standard unitary of the crossed product.

Following Example 3.15, let $\beta \colon S^1 \to \operatorname{Aut}(A_\theta)$ be the action such that $\beta_\zeta(u) = \zeta u$ and $\beta_\zeta(v) = v$ for $\zeta \in S^1$. Using normalized Haar measure in the integral, we define a linear map $E \colon A_\theta \to A_\theta$ by $E(a) = \int_{S^1} \beta_\zeta(a)\, d\zeta$. (The special case of Banach space valued integration theory needed here, essentially for continuous functions on a compact interval with respect to Lebesgue measure, is easily treated by elementary methods.) One checks that $E(v^n u^m) = v^n$ for $m, n \in \mathbb{Z}$. Since the elements $v^n u^m$ span a dense subset of A_θ, it follows that E is equal to the the standard conditional expectation coming from the crossed product structure (Definition 9.18).

Now let $I \subset A_\theta$ be a nonzero closed ideal. We claim that $E(I) \subset I$. First, check that, for $\zeta = e^{2\pi i k \theta}$ with $k \in \mathbb{Z}$, and for $m, n \in \mathbb{Z}$, we have

$$\beta_\zeta(v^n u^m) = e^{2\pi i k m \theta} v^n u^m = v^k (v^n u^m) v^{-k}.$$

Therefore $\beta_\zeta(a) = v^k a v^{-k}$ for all $a \in A_\theta$. In particular, $\beta_\zeta(I) \subset I$. Since $\{e^{2\pi i k \theta} \colon k \in \mathbb{Z}\}$ is dense in S^1 (by Lemma 2.17), it follows from continuity of the action that $\beta_\zeta(I) \subset I$ for all $\zeta \in S^1$. The claim now follows by integration.

We finish the proof by showing that $I = A_\theta$. Choose a nonzero positive element $a \in I$. Let $f = E(a)$, which is a nonzero nonnegative function in $I \cap C(S^1)$. Then $u^k f u^{-k}$, which is the function in $C(S^1)$ given by $\zeta \mapsto f\big(e^{-2\pi i k \theta}\zeta\big)$, is also in $I \cap C(S^1)$. Let $U = \{\zeta \in S^1 \colon f(\zeta) \neq 0\}$. Then $u^k f u^{-k}$ is strictly positive on $e^{2\pi i k \theta} U$. The set $\bigcup_{k \in \mathbb{Z}} e^{2\pi i k \theta} U$ is a nonempty invariant open subset of S^1, and it therefore equal to S^1. By compactness, there is a finite set $S \subset \mathbb{Z}$ such that $\bigcup_{k \in S} e^{2\pi i k \theta} U = S^1$. Then $\sum_{k \in S} u^k f u^{-k}$ is a strictly positive function on S^1, and is hence invertible. Since it is in I, we conclude that $I = A_\theta$. $\square$

Remark 15.13. The action $\beta \colon S^1 \to \operatorname{Aut}(A_\theta)$ used in the proof of Theorem 15.12 is a special case of the dual action on a crossed product by an abelian group, as described in Remark 9.25.

The proof of Theorem 15.10 for $G = \mathbb{Z}$ given in [52] (see Theorem VIII.3.9 of [52]) is similar to the proof given for Theorem 15.12 above. However, it is harder to prove that $E(I) \subset I$, since there is no analog of the automorphism $\operatorname{Ad}(v)$. The proof in [52] uses the Rokhlin Lemma. (See the proof of Lemma VIII.3.7 of [52].)

We have avoided Rokhlin type arguments in this section. To obtain more information about simple transformation group C*-algebras, such arguments are necessary, at least with the current state of knowledge. Examples show that, in the absence of some form of the Rokhlin property, stronger structural properties of crossed products of noncommutative C*-algebras need not hold, even when they are simple. However, the Rokhlin Lemma is not actually needed for the proof in [52], and, in fact, the proof works for reduced crossed products by arbitrary (not necessarily amenable) discrete groups. We give a version of this proof here. The method has the added advantage of providing information about the tracial states on the crossed product, and of being easily adaptable to at least some Banach algebra versions of crossed products. However, it requires that the space X be compact.

The following definition is intended only for use in the proof of Proposition 15.19 and the lemmas leading up to it. For the definition to make sense, and for some of the lemmas, we do not need to require that the subset F be finite.

Definition 15.14. Let G be a discrete group, let X be a compact G-space, let $U \subset X$ be open, and let $F \subset G \setminus \{1\}$ be finite. We say that (F, U) is *inessential* if there exist $n \in \mathbb{Z}_{>0}$ and $s_1, s_2, \ldots, s_n \in C(X)$ such that $|s_k(x)| = 1$ for $k = 1, 2, \ldots, n$ and all $x \in X$, and such that for all $x \in U$ and $g \in F$, we have

$$\frac{1}{n} \sum_{k=1}^{n} s_k(x)\overline{s_k(g^{-1}x)} = 0.$$

Lemma 15.15. Let G be a discrete group, let X be a compact G-space, let $g \in G \setminus \{1\}$, and let $x \in X$ be a point such that $gx \neq x$. Then there exists an open set $U \subset X$ with $x \in U$ such that $(\{g\}, U)$ is inessential in the sense of Definition 15.14.

Proof. Choose an open set $U \subset X$ with $x \in U$ such that $\overline{U} \cap g^{-1}\overline{U} = \varnothing$. Take $n = 2$, and take s_1 to be the constant function 1. Choose a continuous function $r \colon X \to \mathbb{R}$ such that $r(x) = 0$ for $x \in \overline{U}$ and $r(x) = \pi$ for $x \in g^{-1}\overline{U}$. Set $s_2(x) = \exp(ir(x))$ for $x \in X$. For $x \in U$, we have

$$\frac{1}{n} \sum_{k=1}^{n} s_k(x)\overline{s_k(g^{-1}x)} = \frac{1}{2}\big[1 \cdot 1 + 1 \cdot (-1)\big] = 0.$$

Thus $(\{g\}, U)$ is inessential. $\square$

The next two lemmas are based on the same calculation, namely (15.2) in the proof of Lemma 15.16.

Lemma 15.16. Let G be a discrete group, let X be a compact G-space, let $U, V \subset X$ be open, and let $F \subset G \setminus \{1\}$ be finite. If (F, U) and (F, V) are both inessential, then so is $(F,\, U \cup V)$.

Proof. By definition, there exist $m, n \in \mathbb{Z}_{>0}$ and continuous functions

$$r_1, r_2, \ldots, r_m, s_1, s_2, \ldots, s_n \colon X \to S^1$$

such that for every $g \in F$, we have

(15.1)

$$\frac{1}{m} \sum_{j=1}^{m} r_j(x)\overline{r_j(g^{-1}x)} = 0 \text{ for } x \in U \qquad \text{and} \qquad \frac{1}{n} \sum_{k=1}^{n} s_k(x)\overline{s_k(g^{-1}x)} = 0 \text{ for } x \in V.$$

The functions $r_j s_k$ are continuous functions from X to S^1, and we have

$$\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n}(r_j s_k)(x)\overline{(r_j s_k)(g^{-1}x)} = \left(\frac{1}{m}\sum_{j=1}^{m} r_j(x)\overline{r_j(g^{-1}x)}\right)\left(\frac{1}{n}\sum_{k=1}^{n} s_k(x)\overline{s_k(g^{-1}x)}\right). \tag{15.2}$$

By (15.1), this product vanishes for $x \in U$ and also for $x \in V$. □

Lemma 15.17. Let G be a discrete group, let X be a compact G-space, let $U \subset X$ be open, and let $E, F \subset G \setminus \{1\}$ be finite. If (E, U) and (F, U) are both inessential, then so is $(E \cup F,\, U)$.

Proof. By definition, there exist $m, n \in \mathbb{Z}_{>0}$ and continuous functions

$$r_1, r_2, \ldots, r_m, s_1, s_2, \ldots, s_n \colon X \to S^1$$

such that for every $x \in U$, we have

$$\frac{1}{m}\sum_{j=1}^{m} r_j(x)\overline{r_j(g^{-1}x)} = 0 \text{ for } g \in E \qquad \text{and} \qquad \frac{1}{n}\sum_{k=1}^{n} s_k(x)\overline{s_k(g^{-1}x)} = 0 \text{ for } g \in F.$$

The calculation in (15.2) in the proof of Lemma 15.16 shows that for all $x \in U$ and $g \in E \cup F$, we have

$$\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n}(r_j s_k)(x)\overline{(r_j s_k)(g^{-1}x)} = 0.$$

This completes the proof. □

Lemma 15.18. Let G be a discrete group, let X be a free compact G-space, and let $F \in G \setminus \{1\}$ be finite. Then (F, X) is inessential.

Proof. Let $g \in G \setminus \{1\}$. Use compactness of X and Lemma 15.15 to find n and open sets $U_1, U_2, \ldots, U_n \subset X$ such that $(\{g\}, U_k)$ is inessential for $k = 1, 2, \ldots, n$ and such that $\bigcup_{k=1}^{n} U_k = X$. Then $n-1$ applications of Lemma 15.16 show that $(\{g\}, X)$ is inessential. Since F is finite, repeated application of Lemma 15.17 implies that (F, X) is inessential. □

Proposition 15.19. Let G be a discrete group, let X be a free compact G-space, and let $E \colon C^*_{\mathrm{r}}(G, X) \to C(X)$ be the standard conditional expectation (Definition 9.18), viewed as a map $C^*_{\mathrm{r}}(G, X) \to C^*_{\mathrm{r}}(G, X)$. Then for every $a \in C^*_{\mathrm{r}}(G, X)$ and $\varepsilon > 0$, there exist $n \in \mathbb{Z}_{>0}$ and $s_1, s_2, \ldots, s_n \in C(X)$ such that $|s_k(x)| = 1$ for $k = 1, 2, \ldots, n$ and all $x \in X$, and such that

$$\left\| E(a) - \frac{1}{n}\sum_{k=1}^{n} s_k a s_k^* \right\| < \varepsilon.$$

Proof. Let $\alpha \colon G \to \mathrm{Aut}(C(X))$ be the induced action (Definition 1.5), that is, $\alpha_g(f)(x) = f(g^{-1}x)$ for $g \in G$, $f \in C(X)$, and $x \in X$. Also, for $g \in G$ let $u_g \in C^*_{\mathrm{r}}(G, X)$ be the standard unitary (Notation 8.7).

Choose a finite set $F \subset G$ and elements $b_g \in C(X)$ for $g \in G$ such that, with $b = \sum_{g \in F} b_g u_g$, we have

$$\|a - b\| < \frac{\varepsilon}{2}.$$

Without loss of generality $1 \in F$. By Lemma 15.18 and Definition 15.14, there exist $n \in \mathbb{Z}_{>0}$ and $s_1, s_2, \ldots, s_n \in C(X)$ such that $|s_k(x)| = 1$ for $k = 1, 2, \ldots, n$ and all $x \in X$, and such that for all $x \in U$ and $g \in F \setminus \{1\}$, we have

$$\frac{1}{n}\sum_{k=1}^{n} s_k(x)\overline{s_k(g^{-1}x)} = 0.$$

Define $P\colon C^*_{\mathrm{r}}(G,X) \to C^*_{\mathrm{r}}(G,X)$ by

$$P(c) = \frac{1}{n}\sum_{k=1}^{n} s_k c s_k^*$$

for $c \in C^*_{\mathrm{r}}(G,X)$. We have to show that $\|E(a) - P(a)\| < \varepsilon$. Since $\|s_k\| = 1$ for all k, we have $\|P\| \le 1$. Therefore

$$\begin{aligned}\|E(a) - P(a)\| &\le \|E(a) - E(b)\| + \|E(b) - P(b)\| + \|P(b) - P(a)\| \\ &< \frac{\varepsilon}{2} + \|E(b) - P(b)\| + \frac{\varepsilon}{2} = \|E(b) - P(b)\| + \varepsilon.\end{aligned}$$

So it suffices to prove that $P(b) = E(b)$.

Let $g \in F \setminus \{1\}$. Then

$$P(b_g u_g) = \frac{1}{n}\sum_{k=1}^{n} s_k b_g u_g s_k^* = b_g\left(\frac{1}{n}\sum_{k=1}^{n} s_k \alpha_g(s_k^*)\right)u_g.$$

Moreover, for $x \in X$, we have

$$\frac{1}{n}\sum_{k=1}^{n}[s_k\alpha_g(s_k^*)](x) = \frac{1}{n}\sum_{k=1}^{n} s_k(x)\overline{s_k(g^{-1}x)} = 0.$$

Thus $P(b_g u_g) = 0$. Also,

$$P(b_1 u_1) = b_1 \cdot \frac{1}{n}\sum_{k=1}^{n} s_k s_k^* = b_1 = E(b).$$

Thus, $P(b) = E(b)$, as desired. □

Theorem 15.20. Let G be a discrete group, and let X be a free minimal compact G-space. Then $C^*_{\mathrm{r}}(G,X)$ is simple.

Proof. Let $I \subset C^*_{\mathrm{r}}(G,X)$ be a proper closed ideal.

We first claim that $I \cap C(X) = \{0\}$. If not, let $f \in I \cap C(X)$ be nonzero. Choose a nonempty open set $U \subset X$ on which f does not vanish. By minimality, we have $\bigcup_{g\in G} gU = X$. Since X is compact, there is a finite set $S \subset G$ such that $\bigcup_{g\in S} gU = X$. Define $b \in C(X)$ by

$$b(x) = \sum_{g\in S} f(g^{-1}x)\overline{f(g^{-1}x)}$$

for $x \in X$. Then $b(x) > 0$ for all $x \in X$, so b is invertible. For $g \in G$ let $u_g \in C^*_{\mathrm{r}}(G,X)$ be the standard unitary (Notation 8.7). Then $b = \sum_{g\in S} u_g f f^* u_g^* \in I$. So I contains an invertible element, contradicting the assumption that I is proper. This proves the claim.

Let $E\colon C^*_{\mathrm{r}}(G,X) \to C(X)$ be the standard conditional expectation (Definition 9.18), viewed as a map $C^*_{\mathrm{r}}(G,X) \to C^*_{\mathrm{r}}(G,X)$. We claim that $E(a) = 0$

for all $a \in I$. It suffices to show that $E(a) \in I$. To prove this, let $\varepsilon > 0$. Use Proposition 15.19 to choose $n \in \mathbb{Z}_{>0}$ and $s_1, s_2, \ldots, s_n \in C(X)$ such that $|s_k(x)| = 1$ for $k = 1, 2, \ldots, n$ and all $x \in X$, and such that

$$\left\| E(a) - \frac{1}{n} \sum_{k=1}^{n} s_k a s_k^* \right\| < \varepsilon.$$

We have $\frac{1}{n} \sum_{k=1}^{n} s_k a s_k^* \in I$. Since $\varepsilon > 0$ is arbitrary, this implies that $E(a) \in \overline{I} = I$. The claim is proved.

Now let $a \in I$. For all $g \in G$, we have $a u_g \in I$, so $E(a u_g) = 0$. Proposition 9.16(1) now implies that $a = 0$. $\square$

We can use the same methods to identify all the tracial states on $C^*_{\mathrm{r}}(G, X)$. This result requires that the action be free, but not necessarily minimal. The main point is contained in the following proposition. The proof is taken from the proof of Corollary VIII.3.8 of [52].

Proposition 15.21. Let G be a discrete group, let X be a free compact G-space, and let $A \subset C^*_{\mathrm{r}}(G, X)$ be a subalgebra such that $C(X) \subset A$. Let $E \colon C^*_{\mathrm{r}}(G, X) \to C(X)$ be the standard conditional expectation (Definition 9.18). Then for every tracial state $\tau \colon A \to \mathbb{C}$, there exists a Borel probability measure μ on X such that for all $a \in A$ we have

$$\tau(a) = \int_X E(a)\, d\mu.$$

Proof. We prove that $\tau = (\tau|_{C(X)}) \circ E$. The statement then follows by applying the Riesz Representation Theorem to $\tau|_{C(X)}$.

Let $a \in A$ and let $\varepsilon > 0$. We prove that $|\tau(a) - \tau(E(a))| < \varepsilon$. Use Proposition 15.19 to choose $n \in \mathbb{Z}_{>0}$ and $s_1, s_2, \ldots, s_n \in C(X)$ such that $|s_k(x)| = 1$ for $k = 1, 2, \ldots, n$ and all $x \in X$, and such that

$$\left\| E(a) - \frac{1}{n} \sum_{k=1}^{n} s_k a s_k^* \right\| < \varepsilon.$$

Since $s_1, s_2, \ldots, s_n \in A$, we have $\tau(s_k a s_k^*) = \tau(a)$ for $k = 1, 2, \ldots, n$. Therefore

$$\left| \tau(a) - \tau(E(a)) \right| = \left| \tau\left(\frac{1}{n} \sum_{k=1}^{n} s_k a s_k^* \right) - \tau(E(a)) \right| \leq \left\| E(a) - \frac{1}{n} \sum_{k=1}^{n} s_k a s_k^* \right\| < \varepsilon.$$

This completes the proof. $\square$

Theorem 15.22. Let G be a discrete group, and let X be a free compact metrizable G-space. Let $E \colon C^*_{\mathrm{r}}(G, X) \to C(X)$ be the standard conditional expectation (Definition 9.18). For a G-invariant Borel probability measure μ on X, define a linear functional τ_μ on $C^*_{\mathrm{r}}(G, X)$ by

$$\tau_\mu(a) = \int_X E(a)\, d\mu.$$

Then $\mu \mapsto \tau_\mu$ is an affine bijection from the G-invariant Borel probability measures on X to the tracial states on $C^*_{\mathrm{r}}(G, X)$. Its inverse sends τ to the measure obtained from the functional $\tau|_{C(X)}$ via the Riesz Representation Theorem.

The only reason for restricting to metrizable spaces X is to avoid the technicalities surrounding regularity and the uniqueness part of the Riesz Representation Theorem on spaces which are not second countable.

Proof of Theorem 15.22. By Example 11.31, if μ is a G-invariant Borel probability measure on X, then τ_μ is a tracial state on $C^*_{\mathrm{r}}(G, X)$. Clearly $\tau_\mu(f) = \int_X f \, d\mu$ for $f \in C(X)$. This implies that $\mu \mapsto \tau_\mu$ is injective and that the description of its inverse is correct on the range of this map.

It remains only to prove that $\mu \mapsto \tau_\mu$ is surjective. Let τ be a tracial state on $C^*_{\mathrm{r}}(G, X)$. Proposition 15.21 provides a Borel probability measure μ on X such that $\tau(a) = \int_X E(a) \, d\mu$ for all $a \in C^*_{\mathrm{r}}(G, X)$. For $g \in G$ and $f \in C(X)$, using the fact that τ is a trace at the second step, we have

$$\int_X f(g^{-1}x) \, d\mu(x) = \tau(u_g f u_g^*) = \tau(f) = \int_X f \, d\mu.$$

Uniqueness in the Riesz Representation Theorem now implies that μ is G-invariant. This completes the proof. $\square$

We now turn to the direct proof of Theorem 15.10. We need several lemmas, which are special cases of the corresponding lemmas in [5].

Lemma 15.23. Let A be a C*-algebra, let $B \subset A$ be a subalgebra, and let ω be a state on A such that $\omega|_B$ is multiplicative. Then for all $a \in A$ and $b \in B$, we have $\omega(ab) = \omega(a)\omega(b)$ and $\omega(ba) = \omega(b)\omega(a)$.

This is a special case of Theorem 3.1 of [40]. (The corresponding lemma in [5] also follows from Theorem 3.1 of [40].)

Proof of Lemma 15.23. We prove $\omega(ab) = \omega(a)\omega(b)$. The other equation will follow by using adjoints and the relation $\omega(c^*) = \overline{\omega(c)}$.

If A is not unital, then ω extends to a state on the unitization A^+. Thus, we may assume that A is unital. Also, if ω is multiplicative on B, one easily checks that ω is multiplicative on $B + \mathbb{C} \cdot 1$. Thus, we may assume that $1 \in B$.

We recall from the Cauchy-Schwarz inequality that $|\omega(x^*y)|^2 \leq \omega(y^*y)\omega(x^*x)$. Replacing x by x^*, we get $|\omega(xy)|^2 \leq \omega(y^*y)\omega(xx^*)$. Now let $a \in A$ and $b \in B$. Then

$$\big|\omega(ab) - \omega(a)\omega(b)\big|^2 = \big|\omega\big(a[b - \omega(b) \cdot 1]\big)\big|^2 \leq \omega\big([b - \omega(b) \cdot 1]^*[b - \omega(b) \cdot 1]\big)\omega(aa^*).$$

Since ω is multiplicative on B, we have

$$\omega\big([b - \omega(b) \cdot 1]^*[b - \omega(b) \cdot 1]\big) = \omega\big([b - \omega(b) \cdot 1]^*\big)\omega\big(b - \omega(b) \cdot 1)\big) = 0.$$

So $\big|\omega(ab) - \omega(a)\omega(b)\big|^2 = 0$. $\square$

Lemma 15.24. Let G be a discrete group, and let X be a locally compact G-space. Let $x \in X$, let $g \in G$, and assume that $gx \neq x$. Let $\mathrm{ev}_x \colon C_0(X) \to \mathbb{C}$ be the evaluation map $\mathrm{ev}_x(f) = f(x)$ for all $f \in C_0(X)$, and let ω be a state on $C^*_{\mathrm{r}}(G, X)$ which extends ev_x. Then $\omega(f u_g) = 0$ for all $f \in C_0(X)$.

Proof. Let $\alpha \colon G \to \mathrm{Aut}(C_0(X))$ be $\alpha_g(f)(x) = f(g^{-1}x)$ for $f \in C_0(X)$, $g \in G$, and $x \in X$ (as in Definition 1.5; recalled before Theorem 15.10). Choose $f_0 \in C_0(X)$ such that $f_0(x) = 1$ and $f_0(gx) = 0$. Applying Lemma 15.23 to ω at the second and fourth steps, with $A = C^*_{\mathrm{r}}(G, X)$ and $B = C_0(X)$, and using $\omega(f_0) = 1$ at the first step, we have

$$\begin{aligned}\omega(fu_g) &= \omega(f_0)\omega(fu_g) = \omega(f_0 f u_g) = \omega\big(f u_g \alpha_g^{-1}(f_0)\big) \\ &= \omega(fu_g)\omega\big(\alpha_g^{-1}(f_0)\big) = \omega(fu_g) f_0(gx) = 0.\end{aligned}$$

This completes the proof. □

Proof of Theorem 15.10. Let $I \subset C^*_{\mathrm{r}}(G, X)$ be a nonzero closed ideal.

First suppose $I \cap C_0(X) = 0$. Choose $a \in I$ with $a \neq 0$. Let

$$E \colon C^*_{\mathrm{r}}(G, X) \to C_0(X)$$

be the standard conditional expectation (Definition 9.18). Then $E(a^*a) \neq 0$ by Proposition 9.16(4). Choose $b \in C_{\mathrm{c}}(G, C_0(X), \alpha)$ such that $\|b - a^*a\| < \frac{1}{4}\|E(a^*a)\|$. We can write $b = \sum_{g \in S} b_g u_g$ for some finite set $S \subset G$ and with $b_g \in C_0(X)$ for $g \in S$. Without loss of generality $1 \in S$. Since $E(a^*a)$ is a positive element of $C_0(X)$, there is $x_0 \in X$ such that $E(a^*a)(x_0) = \|E(a^*a)\|$. Essential freeness implies that

$$\big\{x \in X \colon gx \neq x \text{ for all } g \in S \setminus \{1\}\big\}$$

is the intersection of finitely many dense open subsets of X, and is therefore a dense open subset of X. In particular, there is $x \in X$ so close to x_0 that $E(a^*a)(x) > \frac{3}{4}\|E(a^*a)\|$, and also satisfying $gx \neq x$ for all $g \in S \setminus \{1\}$.

The set $C_0(X) + I$ is a C*-subalgebra of $C^*_{\mathrm{r}}(G, X)$. Let $\omega_0 \colon C_0(X) + I \to \mathbb{C}$ be the following composition:

$$C_0(X) + I \longrightarrow (C_0(X) + I)/I \xrightarrow{\cong} C_0(X)/(C_0(X) \cap I) = C_0(X) \xrightarrow{\mathrm{ev}_x} \mathbb{C}.$$

Then ω_0 is a homomorphism. Use the Hahn-Banach Theorem in the usual way to get a state $\omega \colon C^*_{\mathrm{r}}(G, X) \to \mathbb{C}$ which extends ω_0. Since $a^*a \in I$, we have $\omega(a^*a) = 0$.

We now have, using Lemma 15.24 at the fifth step,

$$\begin{aligned}\tfrac{1}{4}\|E(a^*a)\| &> \|b - a^*a\| \geq |\omega(b - a^*a)| = |\omega(b)| = \Big|\sum\nolimits_{g \in S} \omega(b_g u_g)\Big| \\ &= |\omega(b_1)| = |\omega_0(b_1)| = |b_1(x)| \geq E(a^*a)(x) - \|E(a^*a) - b_1\| \\ &\geq E(a^*a)(x) - \|a^*a - b\| > \tfrac{3}{4}\|E(a^*a)\| - \tfrac{1}{4}\|E(a^*a)\| = \tfrac{1}{2}\|E(a^*a)\|.\end{aligned}$$

This contradiction shows that $I \cap C_0(X) \neq 0$.

Since $I \cap C_0(X)$ is an ideal in $C_0(X)$, it has the form $C_0(U)$ for some nonempty open set $U \subset X$. We claim that U is G-invariant. Let $g \in G$ and let $f \in C_0(U)$. Let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for $C_0(X)$. Then the elements $e_\lambda u_g$ are in $C^*_{\mathrm{r}}(G, X)$, and we have $(e_\lambda u_g) f (e_\lambda u_g)^* = e_\lambda \alpha_g(f) e_\lambda$, which converges to $\alpha_g(f)$. We also have $(e_\lambda u_g) f (e_\lambda u_g)^* \in I \cap C_0(X)$, since I is an ideal. So $\alpha_g(C_0(U)) \subset C_0(U)$ for all $g \in G$, and the claim follows.

Since U is open, invariant, and nonempty, we have $U = X$. One easily checks that an approximate identity for $C_0(X)$ is also an approximate identity for $C^*_{\mathrm{r}}(G, X)$, so $I = C^*_{\mathrm{r}}(G, X)$, as desired. □

Theorem 15.10 generalizes, with essentially the same proof, to crossed products of actions of discrete groups on noncommutative C*-algebras A satisfying a kind of essential freeness condition for the action on the space of unitary equivalence classes of irreducible representations of A. Here is the general statement; it is the corollary after Theorem 1 in [5].

Theorem 15.25. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Suppose that α is minimal (Definition 14.19), that is, A has no nontrivial α-invariant ideals. Suppose further that α is topologically free, that is, with $\widehat{A}$ being the space of unitary equivalence classes of irreducible representations of A with the hull-kernel topology, the following property holds: for every finite set $F \subset G \setminus \{1\}$, the set

$$\bigl\{x \in \widehat{A}\colon gx \neq x \text{ for all } g \in F\bigr\}$$

is dense in $\widehat{A}$. Then $C^*_{\mathrm{r}}(G, A, \alpha)$ is simple.

The changes to the proof include using irreducible representations in place of the maps ev_x, and completely positive maps to $L(H)$ in place of states.

As discussed in [5] (see the remark after the corollary after Theorem 1), this result implies Theorem 3.1 of [142]. We state the following important special case:

Theorem 15.26. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a simple C*-algebra A. Suppose that α_g is outer for every $g \in G \setminus \{1\}$. Then $C^*_{\mathrm{r}}(G, A, \alpha)$ is simple.

The original proof of Theorem 3.1 of [142] proceeded via Kishimoto's condition (Definition 14.20) and a generalization of Theorem 14.22.

Theorem 15.26 fails for actions of $\mathbb{R}$ and S^1. We give examples based on calculations in [134] (originally Theorem 4.4 of [141], but the statement in [134] is more explicit). See the beginning of Section 2 and Definition 2.1 of [134] for the notation. Theorem 4 of [77] shows that the automorphisms which appear there are all outer unless they are trivial. We will use Theorem 4.8 of [134], verifying condition (iii) there. (See Definition 4.7 of [134] for the notation.) For S^1 take $n = 2$, take $G = S^1$ (so that $\Gamma = \mathbb{Z}$), and take $\omega_1 = 1$ and $\omega_2 = 0$. Then $\Omega_2 = \mathbb{Z}_{\geq 0} \neq \mathbb{Z}$, so the crossed product is not simple. The action one gets this way is the action $\alpha\colon S^1 \to \mathrm{Aut}(\mathcal{O}_2)$ determined by $\alpha_\zeta(s_1) = \zeta s_1$ and $\alpha_\zeta(s_2) = s_2$ for $\zeta \in S^1$. It is the restriction of an action from Example 3.20 to a subgroup. For $\mathbb{R}$, take $n = 3$, take $G = \mathbb{R}$ (so that $\Gamma = \mathbb{R}$), and take $\omega_1 = 1$, $\omega_2 = \sqrt{2}$, and $\omega_3 = 0$. Then $\Omega_3 \subset \{0\} \cup [1, \infty) \neq \mathbb{R}$, so again the crossed product is not simple. The action one gets this way is the action $\beta\colon \mathbb{R} \to \mathrm{Aut}(\mathcal{O}_3)$ determined by

$$\beta_t(s_1) = \exp(it)s_1, \qquad \beta_t(s_2) = \exp(i\sqrt{2}t)s_2, \qquad \text{and} \qquad \beta_t(s_3) = s_3$$

for $t \in \mathbb{R}$. It is the restriction of an action from Example 3.20 to a nonclosed subgroup.

For S^1, alternatively, consider of the usual gauge action of S^1 on $\mathcal{O}_\infty$ (the restriction of the action of Example 4.4 to the scalar multiples of the identity). Its strong Connes spectrum is $\mathbb{Z}_{>0}$ (Remark 5.2 of [141]), so its crossed product is not simple (Theorem 3.5 of [141]). This action is pointwise outer by Theorem 4 of [77].

16. Classifiability: Introduction and a Special Case

We discuss the structure and classification of transformation group C*-algebras of minimal homeomorphisms. We will later say a little about free minimal actions of more complicated groups, but less is known.

Our first main goal is the main result of [157] (Theorem 4.6 there), which gives conditions under which $C^*(\mathbb{Z}, X, h)$ has tracial rank zero. (See Definition 11.35.) Such transformation group C*-algebras are automatically nuclear and satisfy the

Universal Coefficient Theorem, so the conditions we give imply that $C^*(\mathbb{Z}, X, h)$ is in a class covered by a classification theorem. Here is the statement; the map ρ will be explained afterwards (Definition 16.14), along with a reformulation of the condition involving it which does not mention K-theory (Remark 16.15). For any compact metric space X, we let $\dim(X)$ be its covering dimension. (We sometimes just refer to dimension.) See the discussion starting after Corollary 16.2.

This theorem is not the best known result; by now, classifiability and related results are known under much more general conditions. For example, see Theorem 0.1 and Theorem 0.2 of [286], and also Theorem 19.19 (from [74]). Classifiability is known to fail for some minimal homeomorphisms, such as those of [93].

Theorem 16.1 (Theorem 4.6 of [157])**.** Let X be an infinite compact metric space with finite covering dimension, and let $h\colon X \to X$ be a minimal homeomorphism. Suppose that $\rho\big(K_0(C^*(\mathbb{Z}, X, h))\big)$ is dense in $\mathrm{Aff}\big(\mathrm{T}(C^*(\mathbb{Z}, X, h))\big)$. Then $C^*(\mathbb{Z}, X, h)$ is a simple unital C*-algebra with tracial rank zero which satisfies the Universal Coefficient Theorem.

There is machinery available to compute the range of ρ in the above theorem without computing $C^*(\mathbb{Z}, X, h)$. See, for example, [80].

Corollary 16.2 (Corollary 4.7 of [157])**.** Let X be an infinite compact metric space with finite covering dimension, and let $h\colon X \to X$ be a minimal homeomorphism. Suppose that $\rho\big(K_0(C^*(\mathbb{Z}, X, h))\big)$ is dense in $\mathrm{Aff}\big(\mathrm{T}(C^*(\mathbb{Z}, X, h))\big)$. Then $C^*(\mathbb{Z}, X, h)$ is a simple AH algebra with no dimension growth and with real rank zero.

We give a brief explanation of dimension for compact spaces, with definitions but without proofs, to put the finite dimensionality hypothesis of Theorem 16.1 in context. This material is also background for the discussion of the mean dimension of a homeomorphism (Definition 23.3).

Dimension theory attempts to assign a dimension to each topological space (usually in some restricted class) in such a way as to generalize the dimension of a manifold, in particular, the relation $\dim(\mathbb{R}^n) = n$, and to preserve expected properties of the dimension. There are a number of books on dimension theory; the one I have so far found most useful is [197]. (A warning on terminology there: "bicompact" is used for "compact Hausdorff". See Definition 1.5.4 of [197].) The mean dimension of a homeomorphism h of a space X should perhaps be thought of as saying how much more of the space X one sees with every iteration of h, with "how much one sees" being measured in some sense by dimension.

There are at least two quite different general approaches to the problem of assigning dimensions to spaces. One assumes the existence of a metric, and attempts to quantify how the "size" of a ball in the space shrinks with its radius. This approach leads to the Hausdorff dimension and its relatives. The result depends on the metric, need not be an integer, and can be strictly positive for the Cantor set (depending on the metric one uses). Such dimensions have so far played no role in the structure theory of C*-algebras, which is not surprising since $C(X)$ does not depend on the metric on X.

The approach more useful here relies entirely on topological properties of X, takes integer values, and is zero on the Cantor set, regardless of the metric. The three most well known dimension theories of this kind are the covering dimension $\dim(X)$ (Section 3.1 of [197]), the small inductive dimension $\mathrm{ind}(X)$ (Section 4.1

of [197]), and the large inductive dimension $\mathrm{Ind}(X)$ (Section 4.2 of [197]). There are three others that should be mentioned: for compact X, the topological stable rank $\mathrm{tsr}(C(X, \mathbb{R}))$ of the algebra $C(X, \mathbb{R})$ of continuous *real* valued functions on X (topological stable rank is discussed briefly after Definition 11.1 but for complex C*-algebras); for metrizable X the infimum, over all metrics ρ defining the topology, of the Hausdorff dimension of (X, ρ); and for compact metrizable X the cohomological dimension as described in [61] (with integer coefficients). For nonempty compact metrizable X, these all agree (except that one must use $\mathrm{tsr}(C(X, \mathbb{R})) - 1$, and in the case of cohomological dimension with integer coefficients require that $\dim(X) < \infty$), and for specific pairs of dimension theories, it is often known that they agree under much weaker conditions. For $\dim(X)$, $\mathrm{ind}(X)$, and $\mathrm{Ind}(X)$ see Corollary 4.5.10 of [197]. Agreement with $\mathrm{tsr}(C(X, \mathbb{R})) - 1$ is essentially Proposition 3.3.2 of [197] (not stated in that language). Agreement with the infimum of the Hausdorff dimensions of (X, ρ) is in Section 7.4 of [117]. When $\dim(X) < \infty$, agreement with cohomological dimension with integer coefficients is Theorem 1.4 of [61]; without the condition $\dim(X) < \infty$, Theorem 7.1 of [61] shows that agreement can fail.

We give a two warnings about dimension theories. First, they find the dimension of the highest dimensional part of the space. A space like $\mathbb{R}^n$ is homogeneous (in a very strong sense: the diffeomorphism group acts transitively), as is a connected compact manifold without boundary. Even for a connected compact manifold with boundary, it seems intuitively clear that the dimension as seen at any point should be the same. However, a finite complex or a disconnected compact manifold may well have parts which should be considered to have different dimensions. All dimension theories I know of assign to a finite simplicial complex the dimension given by the largest standard (combinatorial) dimension of any of its simplices, even if there are other simplices of much lower dimension which are not contained in any higher dimensional simplex. There are at least some notions of "local dimension" at a point, which attempt to account for this kind of behavior, but the theory seems to be much less well developed.

We will primarily be interested in spaces X which admit minimal homeomorphisms, or minimal actions of other countable groups. Such spaces clearly have at least a weak form of homogeneity, since each orbit is dense and the action is transitive on orbits. We know little about what one can really get from this, but Example 2.26 shows that it does not imply that the local dimension is the same at every point.

Second, they do not necessarily have the properties one expects, or the properties they have are weaker than what one expects. Some such examples are presented or at least mentioned [197]. We list just a few. The notes to Chapter 8 of [197] mention an example due to Filippov: if $1 \leq m \leq n \leq 2m - 1$, there is a compact Hausdorff space X such that

$$\dim(X) = 1, \qquad \mathrm{ind}(X) = m, \qquad \text{and} \qquad \mathrm{Ind}(X) = n.$$

The conventions usually take $\dim(\varnothing) = -1$, so that the standard inequality

$$\dim(X \times Y) \leq \dim(X) + \dim(Y) \tag{16.1}$$

fails if $X = \varnothing$ or $Y = \varnothing$. However, this inequality can fail even if $X \neq \varnothing$ and $Y \neq \varnothing$. See Example 9.3.7 of [197]. If X and Y are nonempty compact Hausdorff spaces, then (16.1) does hold (Proposition 3.2.6 of [197]; see Section 9.3 of [197] for an assortment of weaker conditions under which (16.1) holds). There are nonempty compact metric spaces X and Y such that $\dim(X \times Y) < \dim(X) + \dim(Y)$; in [61] combine Example 1.3(1), Example 1.9, and the example after Corollary 3.8. There is even a nonempty compact metric space X such that $\dim(X \times X) < 2\dim(X)$; for example, combine [170] and [147].

Two expected properties that are true are given in Proposition 16.10 and Proposition 16.11.

The dimension theory most useful so far for minimal homeomorphisms is the covering dimension, which is defined using open covers. We thus start by stating the basic concepts used to define the covering dimension. We make all our definitions for finite open covers of compact Hausdorff spaces, although the earlier ones make sense in much greater generality (for more general spaces, not requiring that the covers be open, and sometimes not even requiring that the covers be finite).

By a finite open cover $\mathcal{U}$ of a compact Hausdorff space X, we mean a finite collection $\mathcal{U}$ of open subsets of X such that $X = \bigcup_{U \in \mathcal{U}} U$. (This convention follows [164].) Possibly (following Section 3.1 of [197]) one should instead use indexed families $(U_i)_{i \in I}$ of open subsets, for a finite index set I; this formulation allows repetitions among the sets. We will not need this refinement. (It is easy to check that it makes no difference in the definition of covering dimension, since one can simply delete repeated sets.)

Notation 16.3. Let X be a compact Hausdorff space. We write $\mathrm{Cov}(X)$ for the set of all finite open covers of X.

Definition 16.4. Let X be a compact Hausdorff space, and let $\mathcal{U}$ be a finite open cover of X. The *order* $\mathrm{ord}(\mathcal{U})$ of $\mathcal{U}$ is the least number $n \in \mathbb{Z}_{>0}$ such that the intersection of any $n + 2$ distinct elements of $\mathcal{U}$ is empty.

That is, $\mathrm{ord}(\mathcal{U})$ is the largest $n \in \mathbb{Z}_{>0}$ such that there are $n+1$ distinct sets in $\mathcal{U}$ whose intersection is not empty. An alternative formulation is

$$\mathrm{ord}(\mathcal{U}) = -1 + \sup_{x \in X} \sum_{U \in \mathcal{U}} \chi_U(x).$$

The normalization is chosen so that if $\mathcal{U}$ is cover of X by disjoint open sets, and $X \neq \varnothing$, then $\mathrm{ord}(\mathcal{U}) = 0$: the intersection of any two distinct sets in $\mathcal{U}$ is empty, but the sets themselves need not be empty.

Definition 16.5. Let X be a compact Hausdorff space, and let $\mathcal{U}$ and $\mathcal{V}$ be finite open covers of X. Then $\mathcal{V}$ *refines* $\mathcal{U}$ (written $\mathcal{V} \prec \mathcal{U}$) if for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$.

That is, every set in $\mathcal{V}$ is contained in some set in $\mathcal{U}$.

Definition 16.6. Let X be a compact Hausdorff space, and let $\mathcal{U}$ be a finite open cover of X. We define the *dimension* $\mathcal{D}(\mathcal{U})$ of $\mathcal{U}$ by

$$\mathcal{D}(\mathcal{U}) = \inf\big(\big\{ \mathrm{ord}(\mathcal{V}) \colon \mathcal{V} \in \mathrm{Cov}(X) \text{ and } \mathcal{V} \prec \mathcal{U}\big\}\big).$$

That is, $\mathcal{D}(\mathcal{U})$ is the least possible order of a finite open cover which refines $\mathcal{U}$.

Definition 16.7 (Definition 3.1.1 of [197])**.** Let X be a nonempty compact Hausdorff space. The *covering dimension* $\dim(X)$ is

$$\dim(X) = \sup\big(\big\{\mathcal{D}(\mathcal{U}) \colon \mathcal{U} \in \mathrm{Cov}(X)\big\}\big).$$

By convention, $\dim(\varnothing) = -1$.

That is, $\dim(X)$ is the supremum of $\mathcal{D}(\mathcal{U})$ over all finite open covers $\mathcal{U}$ of X.

We will say that a compact Hausdorff space X is *totally disconnected* if there is a base for the topology of X consisting of compact open sets. (This seems to be the standard definition for this class of spaces. In [197], a different definition is used, but for compact Hausdorff spaces it is equivalent. See Proposition 3.1.3 of [197].)

Exercise 16.8 (Proposition 3.1.3 of [197])**.** Let X be a nonempty compact Hausdorff space. Prove that $\dim(X) = 0$ if and only if X is totally disconnected.

Exercise 16.9. Prove that $\dim([0,1]) = 1$.

We have $\dim([0,1]) \neq 0$ by Exercise 16.8. To show $\dim([0,1]) \leq 1$, consider open covers of $[0,1]$ consisting of intervals

$$[0, \beta_0),\ (\alpha_1, \beta_1),\ (\alpha_2, \beta_2),\ \ldots,\ (\alpha_{n-1}, \beta_{n-1}),\ (\alpha_n, 1]$$

such that $\alpha_j \leq \beta_{j-1}$ but $\beta_{j-1} < \alpha_{j+1}$, and $\beta_j - \alpha_j$ is small, for all j. The intervals this cover $[0,1]$, but $[0, \beta_0)$ is disjoint from (α_2, β_2), etc.

One sees that $\dim([0,1]^2) \leq 2$ by using open covers consisting of small neighborhoods of the tiles in a fine hexagonal tiling. In general, one has $\dim(\mathbb{R}^n) = n$ (Theorem 3.2.7 of [197]), but proving this is nontrivial. Most proofs rely on some version of the Brouwer Fixed Point Theorem, and thus, in effect, on algebraic topology.

Proposition 16.10 (Proposition 3.1.5 of [197])**.** Let X be a topological space and let $Y \subset X$ be closed. Them $\dim(Y) \leq \dim(X)$.

Proposition 16.11 (Special case of Theorem 3.2.5 of [197])**.** Let X be a compact Hausdorff space and let $Y_1, Y_2, \ldots, Y_n \subset X$ be closed subsets such that $X = \bigcup_{k=1}^n Y_k$. Them $\dim(X) \leq \max_{1 \leq k \leq n} \dim(Y_k)$.

We now give the definitions related to the map ρ which appears in the statement of Theorem 16.1. Recall from Definition 11.23 that a tracial state on A is a state τ on A such that $\tau(ba) = \tau(ab)$ for all $a, b \in A$, and that $\mathrm{T}(A)$ is the set of all tracial states on A, equipped with the relative weak* topology inherited from the Banach space dual of A.

Remark 16.12. Let A be a unital C*-algebra. Then $\mathrm{T}(A)$ is a compact convex subset of the Banach space dual of A (with the weak* topology). Convexity is immediate, and compactness follows from the fact that $\mathrm{T}(A)$ is closed in the set of all states on A.

If A is not unital, then compactness can fail.

Definition 16.13. Let E be a topological vector space, and let $\Delta \subset E$ be a compact convex set. We let $\mathrm{Aff}(\Delta)$ be the real Banach space of real valued continuous affine functions on Δ, with the supremum norm.

We will need a condition which is normally expressed using the following map involving the K_0-group of a C*-algebra. However, the condition can be stated without using K-theory, and we give the explanation afterwards.

Definition 16.14. Let A be a unital C*-algebra. We let $\rho_A \colon K_0(A) \to \mathrm{Aff}(\mathrm{T}(A))$ be the homomorphism determined by $\rho([p])(\tau) = \tau(p)$ for $\tau \in \mathrm{T}(A)$ and p a projection in some matrix algebra over A. The trace τ is taken to be defined on $M_n(A)$

via the unnormalized version of the tracial state in Example 11.25. That is, we define τ on $M_n(A)$ by

$$\tau\left(\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}\right) = \sum_{k=1}^{n} \tau(a_{k,k}).$$

The map ρ is well defined by Lemma 11.33(1) (extended with the same proof to traces which don't necessarily have norm 1).

When A is clear from the context, we often abbreviate ρ_A to ρ.

Remark 16.15. We will often use the hypothesis that the map $\rho_A \colon K_0(A) \to \mathrm{Aff}(\mathrm{T}(A))$ of Definition 16.14 have dense range. This hypothesis can be stated without using K-theory as follows. Let $R \subset \mathrm{Aff}(\mathrm{T}(A))$ be the set all functions $\tau \mapsto \tau(p)$, as p runs through all the projections in $M_n(A)$ for all n (using the notation of Definition 16.14). Then the condition is that the additive subgroup of $\mathrm{Aff}(\mathrm{T}(A))$ generated by R be dense in $\mathrm{Aff}(\mathrm{T}(A))$.

In the rest of this section, we give the proof of Theorem 16.1 in the special case in which $\dim(X) = 0$, following the method of [157]. Since X is assumed to be infinite and to admit a minimal homeomorphism, it can have no isolated points, and therefore must be the Cantor set. This restriction simplifies the argument greatly. In particular, one need not deal with recursive subhomogeneous C*-algebras, KK-theory, or subsets of X with "small boundary". We will give some parts of the proof of the general case in the next section, but we will have to cite several theorems without giving proofs.

It is implicit in Section 8 of [107], with the main step having been done in [230], that these transformation group C*-algebras are AT algebras (direct limits of circle algebras) with real rank zero. The result we prove is weaker and the proof is longer. General theory (Lin's classification theorem for C*-algebras with tracial rank zero, Theorem 5.2 of [153], a K-theory calculation using the Pimsner-Voiculescu exact sequence [221], and results on the range of the Elliott invariant) shows that the AT algebra result follows from the theorem we prove here. Our reason for giving this proof is to illustrate a technical method.

Lemma 16.16. Let A be a simple unital C*-algebra. Suppose that for every finite subset $F \subset A$, every $\varepsilon > 0$, and every nonzero positive element $c \in A$, there exists a nonzero projection $p \in A$ and a unital AF subalgebra $B \subset A$ with $p \in B$ such that:

(1) $\| [a, p] \| < \varepsilon$ for all $a \in F$.
(2) $\mathrm{dist}(pap,\, pBp) < \varepsilon$ for all $a \in F$.
(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{cAc}$.

Then A has tracial rank zero (Definition 11.35).

Proof. Let $F \subset A$ be a finite subset, let $\varepsilon > 0$, and let $c \in A$ be a nonzero positive element. Choose p and B as in the hypotheses, with F and c as given and with $\frac{1}{2}\varepsilon$ in place of ε. Let $F_0 \subset pBp$ be a finite set such that $\mathrm{dist}(pap,\, F_0) < \frac{1}{2}\varepsilon$ for all $a \in F$. Since $p \in B$, the algebra pBp is also AF. Choose a unital finite dimensional subalgebra $D \subset pBp$ such that $\mathrm{dist}(b, D) < \frac{1}{2}\varepsilon$ for all $b \in F_0$. Then $\mathrm{dist}(pap,\, D) < \varepsilon$ for all $a \in F$. $\square$

Exercise 16.17. Let A be a unital C*-algebra, and let $S \subset A$ be a subset which generates A as a C*-algebra. Assume that the condition of Lemma 16.16 holds for all finite subsets $F \subset S$. Prove that A has tracial rank zero.

Definition 16.18. Let X be a compact metric space, and let $h \colon X \to X$ be a homeomorphism. In the transformation group C*-algebra $C^*(\mathbb{Z}, X, h)$, we normally write u for the standard unitary representing the generator of $\mathbb{Z}$. (This unitary is called u_1 in Notation 8.7.) For a closed subset $Y \subset X$, we define the C*-subalgebra $C^*(\mathbb{Z}, X, h)_Y$ to be

$$C^*(\mathbb{Z}, X, h)_Y = C^*(C(X),\, C_0(X \setminus Y)u) \subset C^*(\mathbb{Z}, X, h).$$

We call it the *Y-orbit breaking subalgebra* of $C^*(\mathbb{Z}, X, h)$.

This subalgebra was introduced by Putnam in Section 3 of [229], specifically in the case that X is the Cantor set. There, and in all subsequent papers, the definition

$$C^*(\mathbb{Z}, X, h)_Y = C^*(C(X),\, uC_0(X \setminus Y)) \subset C^*(\mathbb{Z}, X, h).$$

was used. As will be seen in the course of the proof of Lemma 16.20, and later, the analysis of the structure of $C^*(\mathbb{Z}, X, h)_Y$ for $\mathrm{int}(Y) \neq \varnothing$ depends on Rokhlin towers constructed from Y. When Y is compact and open, the Rokhlin towers take a standard form, given in (16.2) below: there are positive integers $n(0) < n(1) < \cdots < n(l)$ and subsets $Y_0, Y_1, \ldots, Y_l \subset Y$ such that

$$Y = \coprod_{k=0}^{l} Y_k \qquad \text{and} \qquad X = \coprod_{k=0}^{l} \coprod_{j=0}^{n(k)-1} h^j(Y_k).$$

The sequences $Y_k,\, h(Y_k), \ldots, h^{n(k)-1}(Y_k)$ are called *Rokhlin towers*. The sets Y_k are the *bases* of the towers, and the numbers $n(k)$ are their *heights*. The reason for changing the convention in the definition of $C^*(\mathbb{Z}, X, h)_Y$ is that the old convention leads to Rokhlin towers with bases $h(Y_0), h(Y_1), \ldots, h(Y_l)$ instead of $Y_0, Y_1, \ldots, Y_l$, so that the useful partition of X becomes

$$X = \coprod_{k=0}^{l} \coprod_{j=1}^{n(k)} h^j(Y_k).$$

We do not need groupoids at this stage, but they do seem to be needed for useful analogs of $C^*(\mathbb{Z}, X, h)_Y$ in more general situations. They are also used in the usual computation of the K-theory of $C^*(\mathbb{Z}, X, h)_{\{y\}}$ for $y \in X$; see the discussion of the proof of Theorem 17.25 (after the proof of Lemma 17.22). We therefore describe briefly how to realize $C^*(\mathbb{Z}, X, h)_Y$ in terms of groupoids. Readers not familiar with groupoids should skip this description.

Remark 16.19. The algebra $C^*(\mathbb{Z}, X, h)_Y$ is the C*-algebra of a subgroupoid of the transformation group groupoid $\mathbb{Z} \ltimes X$ made from the action of $\mathbb{Z}$ on X generated by h. Informally, we "break" every orbit each time it goes through Y.

Here is a more formal description. The notation here differs slightly from the most common notation. We take $\mathbb{Z} \ltimes X$ to be the set

$$\big\{(h^n(x),\, n,\, x) \colon x \in X \text{ and } n \in \mathbb{Z}\big\} \subset X \times \mathbb{Z} \times X,$$

with the groupoid operation determined by

$$(h^{m+n}(x),\, m,\, h^n(x)) \cdot (h^n(x),\, n,\, x) = (h^{m+n},\, m+n,\, x)$$

and with other products undefined. This is the transformation group groupoid made from the action of $\mathbb{Z}$ on X generated by h. See Example 1.2a of [238], which, as well as having different notation, uses a right action instead of a left action. With this notation, $C^*(\mathbb{Z}, X, h)_Y$ is the C*-algebra of the open subgroupoid $G \subset \mathbb{Z} \ltimes X$ which contains $(x, 0, x)$ for all $x \in X$, and such that for $n \in \mathbb{Z}_{>0}$ and $x \in X$ we have $(h^n(x), n, x) \in G$ if and only if $h(x), h^2(x), \ldots, h^n(x) \in X \setminus Y$ and $(h^{-n}(x), -n, x) \in G$ if and only if $x, h^{-1}(x), \ldots, h^{-n+1}(x) \in X \setminus Y$.

If Y has nonempty interior, then all the orbits are finite, and the orbit of $x \in X$ is as follows. Let $j_0 \leq 0$ be the greatest nonpositive integer such that $h^{j_0}(x) \in Y$, and let $j_1 > 0$ be the least strictly positive integer such that $h^{j_1}(x) \in Y$. Then the orbit of x is

$$h^{j_0}(x), \, h^{j_0+1}(x), \, \ldots, \, h^{j_1-2}(x), \, h^{j_1-1}(x).$$

The following lemma is a special case of Theorem 3.3 of [229]. Note the standing assumption of minimality throughout [229], stated in Section 1 there. (Theorem 3.3 of [229] does not assume that Y is open. The requirement that Y be open is easily removed by choosing compact open sets $Y_1 \supset Y_2 \supset \cdots$ in X such that $\bigcap_{n=1}^{\infty} Y_n = Y$, and observing that $C^*(\mathbb{Z}, X, h)_Y$ is the closure of the increasing union of the subalgebras $C^*(\mathbb{Z}, X, h)_{Y_n}$. Compare with Remark 17.20 below.)

Lemma 16.20. Let X be the Cantor set, and let $h \colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a nonempty compact open subset. Then $C^*(\mathbb{Z}, X, h)_Y$ is an AF algebra.

Proof. The proof depends on the construction of Rokhlin towers, which is a crucial element of many structure results for crossed products. (For the construction in more general spaces X, see Definition 17.2, Lemma 17.3, Definition 17.4, and Lemma 17.5.)

We first claim that there is $N \in \mathbb{Z}_{>0}$ such that $\bigcup_{n=1}^{N} h^{-n}(Y) = X$. (This is Lemma 17.3 in the general case.) Set $U = \bigcup_{n=1}^{\infty} h^{-n}(Y)$, which is a nonempty open subset of X such that $U \subset h(U)$. Then $Z = X \setminus \bigcup_{n=1}^{\infty} h^{-n}(Y)$ is a closed subset of X such that $h(Z) \subset Z$, and $Z \neq X$. Therefore $Z = \varnothing$ by Lemma 15.3. So $U = X$, and the claim now follows from compactness of X.

It follows that for each fixed $y \in Y$, the sequence of iterates $h(y), h^2(y), \ldots$ of y under h must return to Y in at most N steps. Define the *first return time* $r(y)$ to be

$$r(y) = \min \bigl(\bigl\{ n \geq 1 \colon h^n(y) \in Y \bigr\} \bigr) \leq N.$$

(This is Definition 17.2 in the general case.) Let $n(0) < n(1) < \cdots < n(l) \leq N$ be the distinct values of r. Set

$$Y_k = \bigl\{ y \in Y \colon r(y) = n(k) \bigr\}.$$

Then the sets Y_k are compact, open, and partition Y, and the sets $h^j(Y_k)$, for $1 \leq j \leq n(k)$, partition X:

$$Y = \coprod_{k=0}^{l} Y_k \qquad \text{and} \qquad X = \coprod_{k=0}^{l} \coprod_{j=0}^{n(k)-1} h^j(Y_k). \tag{16.2}$$

Further set $X_k = \bigcup_{j=0}^{n(k)-1} h^j(Y_k)$. The sets X_k then also partition X. (This part is much messier in the general case; see Definition 17.4 and Lemma 17.5.)

Define $p_k \in C(X) \subset C^*(\mathbb{Z}, X, h)_Y$ by $p_k = \chi_{X_k}$. We claim that p_k commutes with all elements of $C^*(\mathbb{Z}, X, h)_Y$. It suffices to prove that p_k commutes with all elements of $C(X)$ and with all elements fu with $f \in C_0(X \setminus Y)$. Clearly p_k commutes with every element of $C(X)$. Next, for any compact open subset $Z \subset X$, we have $u\chi_Z u^* = \chi_{h(Z)}$. In particular,

$$up_k u^* = \sum_{j=0}^{n(k)-1} u\chi_{h^j(Y_k)}u^* = \sum_{j=1}^{n(k)} \chi_{h^j(Y_k)} = p_k - \chi_{Y_k} + \chi_{h^{n(k)}(Y_k)}.$$

So

$$up_k = \big(p_k + \chi_{Y_k} - \chi_{h^{n(k)}(Y_k)}\big)u. \tag{16.3}$$

Now let $f \in C(X)$ vanish on Y. Multiply (16.3) on the left by f. Since $Y_k \subset Y$ and $h^{n(k)}(Y_k) \subset Y$, we get $f\chi_{Y_k} = f\chi_{h^{n(k)}(Y_k)} = 0$, whence $fup_k = fp_k u = p_k fu$. The claim is proved.

We now have

$$C^*(\mathbb{Z}, X, h)_Y = \bigoplus_{k=0}^{l} p_k C^*(\mathbb{Z}, X, h)_Y p_k.$$

It therefore suffices to prove that $p_k C^*(\mathbb{Z}, X, h)_Y p_k$ is AF for each k.

It is easy to see that $C^*(\mathbb{Z}, X, h)_Y$ is the C*-algebra generated by $C(X)$ and $(\chi_{X\setminus Y})u$. Therefore $p_k C^*(\mathbb{Z}, X, h)_Y p_k$ is the C*-algebra generated by $C(X_k)$ and (using (16.3) at the first step of the calculation)

$$\begin{aligned} p_k(\chi_{X\setminus Y})up_k = (\chi_{X_k\setminus Y_k})(\chi_{X_k\setminus h^{n(k)}(Y_k)})u &= \sum_{j=1}^{n(k)-1} (\chi_{h^j(Y_k)})u \\ &= \sum_{j=1}^{n(k)-1} (\chi_{h^{j+1}(Y_k)})u(\chi_{h^j(Y_k)}). \end{aligned}$$

One can now check, although it is a bit tedious to write out the details (see Exercise 16.21), that there is an isomorphism

$$\psi_k \colon p_k C^*(\mathbb{Z}, X, h)_Y p_k \to M_{n(k)} \otimes C(Y_k) \tag{16.4}$$

such that for $f \in C(X_k)$ we have

$$\psi_k(f) = \operatorname{diag}\big(f|_{Y_k},\ f\circ h|_{Y_k},\ \ldots,\ f\circ h^{n(k)-1}|_{Y_k}\big)$$

and (using matrix units in $M_{n(k)}$ labelled as $(e_{i,j})_{i,j=0}^{n(k)-1}$) for $1 \leq j \leq n(k)-1$ we have

$$\psi_k\big(\chi_{h^j(Y_k)}u\chi_{h^{j-1}(Y_k)}\big) = e_{j,\,j-1} \otimes 1.$$

(Theorem 17.19 below gives the much messier statement needed when the space X is not totally disconnected, and its proof is given in full.) The algebra $M_{n(k)}\otimes C(Y_k)$ is AF because Y_k is totally disconnected. □

Exercise 16.21. Prove that ψ_k as in (16.4) in the proof above is in fact an isomorphism.

This exercise is preparation for reading the proof of Theorem 17.19.

The proof of Lemma 16.20 shows, in the notation used in it, that

$$C^*(\mathbb{Z}, X, h)_Y \cong \bigoplus_{k=0}^{l} C(Y_k, M_{n(k)}), \tag{16.5}$$

via an isomorphism constructed from the system of Rokhlin towers associated with Y. See Theorem 17.19 for what happens for more general spaces X.

Lemma 16.22. Let X be the Cantor set, and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a nonempty compact open subset. Let $N \in \mathbb{Z}_{>0}$, and suppose that $Y, h(Y), \ldots, h^{N-1}(Y)$ are disjoint. Then the projections $\chi_{h^{-1}(Y)}$ and $\chi_{h^{N-1}(Y)}$ are Murray-von Neumann equivalent in $C^*(\mathbb{Z}, X, h)_Y$.

The proof is short, but we explain in terms of the Rokhlin towers and the decomposition

$$C^*(\mathbb{Z}, X, h)_Y \cong \bigoplus_{k=0}^{l} M_{n(k)} \otimes C(Y_k)$$

why one should expect it to be true. First, all the towers have height at least N. So passing from χ_Y to $\chi_{h^{N-1}(Y)}$ amounts to replacing, in each summand $M_{n(k)} \otimes C(Y_k)$ and using the indexing in the proof of Lemma 16.20, the projection $e_{0,0} \otimes 1$ with $e_{N-1,\, N-1} \otimes 1$. These are certainly Murray-von Neumann equivalent. Passing from χ_Y to $\chi_{h^{-1}(Y)}$ corresponds to going off the bottoms of the Rokhlin towers. This need not send Y_k to $h^{n(k)-1}(Y_k)$, so need not send $e_{0,0} \otimes 1$ to $e_{n(k)-1,\, n(k)-1} \otimes 1$. But Y is also equal to $\coprod_{k=0}^{l} h^{n(k)}(Y_k)$, so it *does* send $Y = \coprod_{k=0}^{l} Y_k$ to $\coprod_{k=0}^{l} h^{n(k)-1}(Y_k) = h^{-1}(Y)$. The projection corresponding to χ_Y is

$$\big(e_{0,0} \otimes 1,\, e_{0,0} \otimes 1,\, \ldots,\, e_{0,0} \otimes 1\big) \in \bigoplus_{k=0}^{l} M_{n(k)} \otimes C(Y_k)$$

and the identification of $h^{-1}(Y)$ shows that projection corresponding to $\chi_{h^{-1}(Y)}$ is

$$\big(e_{n(0)-1,\, n(0)-1} \otimes 1,\, e_{n(1)-1,\, n(1)-1} \otimes 1,\, \ldots,\, e_{n(l)-1,\, n(l)-1} \otimes 1\big).$$

These clearly *are* Murray-von Neumann equivalent.

Proof of Lemma 16.22. We use the notation for Murray-von Neumann equivalence in Notation 11.5. First, observe that if $Z \subset X$ is a compact open subset such that $Y \cap Z = \varnothing$, then $v = \chi_Z u \in C^*(\mathbb{Z}, X, h)_Y$ and satisfies $vv^* = \chi_Z$ and $v^* v = \chi_{h^{-1}(Z)}$. Thus $\chi_Z \sim \chi_{h^{-1}(Z)}$.

An induction argument, taking successively

$$Z = h(Y), \quad Z = h^2(Y), \quad Z = h^N(Y),$$

now shows that $\chi_Y \sim \chi_{h^{N-1}(Y)}$. Also, taking $Z = X \setminus Y$ gives $\chi_{X \setminus Y} \sim \chi_{X \setminus h^{-1}(Y)}$. Since $C^*(\mathbb{Z}, X, h)_Y$ is an AF algebra, it follows that $\chi_Y \sim \chi_{h^{-1}(Y)}$. The result follows by transitivity. $\square$

The proof of the next lemma is closely related to the first part of the proof of Theorem 15.10 (which is at the end of Section 15) and especially to the proof of Theorem 14.23. Indeed, we could get the result from Theorem 14.23 by proving that α satisfies Kishimoto's condition (Definition 14.20).

Lemma 16.23. Let X be the Cantor set, and let $h\colon X \to X$ be a minimal homeomorphism. Let $c \in C^*(\mathbb{Z}, X, h)$ be a nonzero positive element. Then there exists a nonzero projection $p \in C(X)$ such that p is Murray-von Neumann equivalent in $C^*(\mathbb{Z}, X, h)$ to a projection in $\overline{cC^*(\mathbb{Z}, X, h)c}$.

Proof. Let $E\colon C^*(\mathbb{Z}, X, h) \to C(X)$ be the standard conditional expectation (Definition 9.18). It follows from Proposition 9.16(4) and Exercise 9.17(3) that $E(c)$ is a nonzero positive element of $C(X)$. Choose a nonempty compact open subset $K_0 \subset X$ and $\delta > 0$ such that the function $E(c)$ satisfies $E(c)(x) > 4\delta$ for all $x \in K_0$. Choose a finite sum $b = \sum_{n=-N}^{N} b_n u^n \in C^*(\mathbb{Z}, X, h)$ such that $\|b - c\| < \delta$. Since the action of $\mathbb{Z}$ induced by h is free, there is a nonempty compact open subset $K \subset K_0$ such that the sets $h^{-N}(K)$, $h^{-N+1}(K)$, $\ldots$, $h^{N}(K)$ are disjoint. Set $p = \chi_K \in C(X)$. For $n \in \{-N, -N+1, \ldots, N\} \setminus \{0\}$, the disjointness condition implies that $pu^n p = 0$. Therefore $pbp = pb_0 p = pE(b)p$. Using this equation at the first step and Exercise 9.17(4) at the second step, we get

$$\|pcp - pE(c)p\| \leq \|pcp - pbp\| + \|pE(b)p - pE(c)p\| \leq 2\|c - b\| < 2\delta. \tag{16.6}$$

Since $K \subset K_0$, the function $pE(c)p$ is invertible in $pC(X)p$. In the following calculation, inverses are taken in $pC^*(\mathbb{Z}, X, h)p$. With this convention, $[pE(c)p]^{-1}$ exists and satisfies $\|[pE(c)p]^{-1}\| < \frac{1}{4}\delta^{-1}$. The estimate (16.6) now implies that pcp is invertible in $pC^*(\mathbb{Z}, X, h)p$. Let $a = (pcp)^{-1/2}$, calculated in $pC^*(\mathbb{Z}, X, h)p$. Set $v = apc^{1/2}$. Then

$$vv^* = apcpa = (pcp)^{-1/2}(pcp)(pcp)^{-1/2} = p$$

and

$$v^*v = c^{1/2}pa^2pc^{1/2} \in \overline{cC^*(\mathbb{Z}, X, h)c}.$$

This completes the proof. $\square$

Most of the proof of the following lemma is taken from [157]. The definition of $C^*(\mathbb{Z}, X, h)_Y$ is different, as explained after Definition 16.18, and the notation in the proof has been changed accordingly.

Lemma 16.24. Let X be the Cantor set, and let $h\colon X \to X$ be a minimal homeomorphism. Let $y \in X$. Then for any $\varepsilon > 0$, any nonempty open set $U \subset X$, and any finite subset $F \subset C(X)$, there is a compact open set $Y \subset X$ containing y and a projection $p \in C^*(\mathbb{Z}, X, h)_Y$ such that:

(1) $\|pa - ap\| < \varepsilon$ for all $a \in F \cup \{u\}$.
(2) $pap \in pC^*(\mathbb{Z}, X, h)_Y p$ for all $a \in F \cup \{u\}$.
(3) There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.

The key point in the proof of Lemma 16.24 is the estimate (16.14) below. We outline the method. For a suitable small compact open set $Y \subset X$ which contains y, set $q_n = \chi_{h^n(Y)}$. There will be a large number $N \in \mathbb{Z}_{>0}$ such that $Y, h(Y), \ldots, h^{N-1}(Y)$ are disjoint, and such that $h^N(y)$ is close to y. The projection $1 - p$ will be called e in the proof. The naive choice for e will turn out to be $f = \sum_{n=0}^{N-1} q_n$. This projection commutes exactly with all elements of $C(X)$. Also, it is easy to check that $fC^*(\mathbb{Z}, X, h)f \subset C^*(\mathbb{Z}, X, h)_Y$. If we had $h^N(Y) = Y$, it would also commute with u. It is of course not possible to have $h^N(Y) = Y$, since then $\bigcup_{n=0}^{N-1} h^n(Y)$ would be a nontrivial h-invariant closed set. The main idea of the proof is to modify this naive choice to get a projection e which approximately commutes with u but which still has approximate versions of the other good properties of f.

There is a big difference between what we need to do and what would happen if we were working in von Neumann algebras. In the von Neumann algebra setting, we would be given an ergodic probability measure μ on X, and it would be enough to ask that $q_N - q_0$ be small in trace derived from μ. Thus it would be sufficient to have $\mu(Y)$ small, which is extremely easy to arrange. The analog of Lemma 16.24 would be essentially trivial: just take Y small enough, pay no attention to how close $h^N(y)$ is to y, and take $e = \sum_{n=0}^{N-1} q_n$. In the C* setting, unless $h^N(Y)$ is exactly equal to Y, we get $\|q_N - q_0\| = 1$. Therefore we must work much harder.

Proof of Lemma 16.24. We abbreviate $A = C^*(\mathbb{Z}, X, h)$ and $A_Y = C^*(\mathbb{Z}, X, h)_Y$. Let d be the metric on X. Choose $N_0 \in \mathbb{Z}_{>0}$ so large that $4\pi/N_0 < \varepsilon$. Choose $\delta_0 > 0$ with $\delta_0 < \frac{1}{2}\varepsilon$ and so small that $d(x_1, x_2) < 4\delta_0$ implies $|f(x_1) - f(x_2)| < \frac{1}{4}\varepsilon$ for all $f \in F$. Choose $\delta > 0$ with $\delta \leq \delta_0$ and such that whenever $d(x_1, x_2) < \delta$ and $0 \leq k \leq N_0$, then $d(h^{-k}(x_1), h^{-k}(x_2)) < \delta_0$.

Since h is minimal, there is $N > N_0 + 1$ such that $d(h^N(y), y) < \delta$. Choose $N + N_0 + 1$ disjoint nonempty open subsets $U_{-N_0}, U_{-N_0+1}, \ldots, U_N \subset U$. Using minimality again, choose $r_{-N_0}, r_{-N_0+1}, \ldots, r_N \in \mathbb{Z}$ such that $h^{r_l}(y) \in U_l$ for $l = -N_0, -N_0 + 1, \ldots, N$. Since h is has no periodic points, there is a compact open set $Y \subset X$ such that:

(1) $y \in Y$.
(2) The sets
$$h^{-N_0}(Y),\, h^{-N_0+1}(Y),\, \ldots,\, Y,\, h(Y),\, \ldots,\, h^N(Y)$$
are disjoint.
(3) The sets
$$h^{-N_0}(Y),\, h^{-N_0+1}(Y),\, \ldots,\, Y,\, h(Y),\, \ldots,\, h^N(Y)$$
all have diameter less than δ.
(4) $h^{r_l}(Y) \subset U_l$ for $l = -N_0, -N_0 + 1, \ldots, N$.

Set $q_0 = \chi_Y$. For $n = -N_0, -N_0 + 1, \ldots, N$ set
$$T_n = h^n(Y) \qquad \text{and} \qquad q_n = u^n q_0 u^{-n} = \chi_{h^n(Y)} = \chi_{T_n}.$$

We now have a sequence of projections, in principle going to infinity in both directions:
$$\ldots,\, q_{-N_0},\, \ldots,\, q_{-1},\, q_0,\, q_1,\, \ldots,\, q_{N-N_0},\, \ldots,\, q_{N-1},\, q_N,\, \ldots.$$
The ones shown are orthogonal, and conjugation by u is the shift on this sequence. The projections q_0 and q_N are the characteristic functions of compact open sets which are disjoint but close to each other, and similarly for the projections q_{-1} and q_{N-1}, for the projections q_{-2} and q_{N-2}, down to the projections q_{-N_0} and q_{N-N_0}. We are now going to use Berg's technique [19] to splice this sequence along the pairs of indices $(-N_0, N - N_0)$ through $(0, N)$, obtaining a loop of length N on which conjugation by u is approximately the cyclic shift.

Lemma 16.22 provides a partial isometry $w \in A_Y$ such that $w^*w = q_{-1}$ and $ww^* = q_{N-1}$.

For $t \in [0,1]$ define

$$v(t) = \cos(\pi t/2)(q_{-1} + q_{N-1}) + \sin(\pi t/2)(w - w^*) \in A_Y. \tag{16.7}$$

Then $v(t)$ is a unitary in the corner

$$(q_{-1} + q_{N-1})A_Y(q_{-1} + q_{N-1}).$$

To see what is happening, we write elements of this corner in 2×2 matrix form, with the $(1,1)$ entry corresponding to $q_{-1}A_Y q_{-1}$. That is, there is a homomorphism

$$\varphi \colon M_2 \to (q_{-1} + q_{N-1})A_Y(q_{-1} + q_{N-1})$$

such that

$$\varphi(e_{1,1}) = q_{-1}, \qquad \varphi(e_{1,2}) = w^*, \qquad \varphi(e_{2,1}) = w, \qquad \text{and} \qquad \varphi(e_{2,2}) = q_{N-1}.$$

If we identify M_2 with its image under φ, we get

$$q_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad w = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad q_{N-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(these are just the definitions), and

$$w - w^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad v(t) = \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

For $k = 0, 1, \ldots, N_0$ define

$$z_k = u^{-k+1} v(k/N_0) u^{k-1}, \tag{16.8}$$

which is in

$$(q_{-k} + q_{N-k})A(q_{-k} + q_{N-k})$$

and is a unitary in this corner.

We claim that $z_k \in A_Y$ for $k = 0, 1, \ldots, N_0$. We have $z_0 = q_0 + q_N \in C(X) \subset A_Y$. Also, $z_1 \in A_Y$ by construction. For $k = 2, 3 \ldots, N_0$, set $a_k = q_{-1}u^{k-1}$ and $b_k = q_{N-1}u^{k-1}$. Since $uq_n u^* = q_{n+1}$ for all n, we can write these as

$$\begin{aligned} a_k &= q_{-1}(uq_{-2}u^{-1})(u^2 q_{-3}u^{-2}) \cdots (u^{k-2}q_{-k+1}u^{-k+2})u^{k-1} \\ &= (q_{-1}u)(q_{-2}u) \cdots (q_{-k+1}u) \end{aligned} \tag{16.9}$$

and

$$\begin{aligned} b_k &= q_{N-1}(uq_{N-2}u^{-1})(u^2 q_{N-3}u^{-2}) \cdots (u^{k-2}q_{N-k+1}u^{-k+2})u^{k-1} \\ &= (q_{N-1}u)(q_{N-2}u) \cdots (q_{N-k+1}u). \end{aligned} \tag{16.10}$$

Since $N_0 < N$, the projections

$$q_{-1}, q_{-2}, \ldots, q_{-N_0+1}, q_{N-1}, q_{N-2}, \ldots, q_{N-N_0+1}$$

are all characteristic functions of sets disjoint from Y. The factorizations in (16.9) and (16.10) therefore show that $a_k, b_k \in A_Y$. Now one checks that

$$z_k = (a_k + b_k)^* v(k/N_0)(a_k + b_k),$$

which is in A_Y. This is the claim. Thus

$$z_k \in (q_{-k} + q_{N-k})A_Y(q_{-k} + q_{N-k})$$

and is a unitary in this corner.

From (16.7), it is easy to get $\|v(t_1) - v(t_2)\| \leq 2\pi|t_1 - t_2|$ for $t_1, t_2 \in [0, 1]$. Using (16.8), for $k = 0, 1, \ldots, N_0$ we therefore get

$$\|uz_{k+1}u^* - z_k\| = \big\|v\big((k+1)/N_0\big) - v(k/N_0)\big\| \leq \frac{2\pi}{N_0} < \frac{\varepsilon}{2}. \tag{16.11}$$

Now define $e_n = q_n$ for $n = 0, 1, \ldots, N - N_0$. For $n = N - N_0,\ N - N_0 + 1,\ \ldots, N$, define k by $n = N - k$, and set $e_n = z_k q_{-k} z_k^*$. These are clearly all elements of A_Y. The two definitions for $n = N - N_0$ agree because, in the obvious block decomposition (similar to that used above) of

$$(q_{-N_0} + q_{N-N_0}) A_Y (q_{-N_0} + q_{N-N_0}),$$

we get

$$z_{N_0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so that $z_{N_0} q_{-N_0} z_{N_0}^* = q_{N-N_0}$. (One can check this formula by a direct calculation.) Moreover, $z_0 = q_0 + q_N$, so $e_N = e_0$.

Putting things together, we have

$$ue_{n-1}u^* = e_n \tag{16.12}$$

for $n = 1, 2, \ldots, N - N_0$, and also $ue_N u^* = e_1$. For $N - N_0 < n \leq N$ we define k by $n = N - k$ and use (16.11) and $uq_{-k-1}u^* = q_{-k}$ to get

$$\begin{aligned} \|ue_{n-1}u^* - e_n\| &= \big\|uz_{k+1}q_{-k-1}z_{k+1}u^* - z_k q_{-k} z_k\big\| \\ &= \big\|(uz_{k+1}u^*)q_{-k}(uz_{k+1}u^*) - q_{-k}\big\| \\ &\leq 2\big\|uz_{k+1}u^* - z_k\big\| < \varepsilon. \end{aligned} \tag{16.13}$$

Set

$$e = \sum_{n=1}^{N} e_n \qquad \text{and} \qquad p = 1 - e,$$

both of which are in A_Y. We verify that p satisfies (1), (2), and (3).

We verify (1) and (2). Consider u first. Since $e_N = e_0$, we have

$$ueu^* - e = \sum_{n=1}^{N} (ue_{n-1}u^* - e_n).$$

For $n = 1, 2, \ldots, N - N_0$, equation (16.12) applies, so that in fact

$$ueu^* - e = \sum_{n=N-N_0+1}^{N} (ue_{n-1}u^* - e_n).$$

For the indices used in this sum, the inequality (16.13) applies, so the terms in the sum have norm less than ε. They are orthogonal since, with k determined by $n = N - k$,

$$ue_{n-1}u^* - e_n \in (q_{-k} + q_{N-k}) A_Y (q_{-k} + q_{N-k}).$$

Therefore

$$\|ueu^* - e\| < \varepsilon. \tag{16.14}$$

So $\|upu^* - p\| = \|-ueu^* + e\| < \varepsilon$. Furthermore, since $p \in A_Y$ and $p \leq 1 - q_0 = 1 - \chi_Y$, we get

$$pup = p(1 - \chi_Y)up \in A_Y.$$

This is (1) and (2) for the element $u \in F \cup \{u\}$.

Next, let $f \in F$. Define sets S_n for $n = 1, 2, \ldots, N$ by

$$S_1 = T_1, \;\; S_2 = T_2, \;\; \ldots, \;\; S_{N-N_0-1} = T_{N-N_0-1}$$

and

$$S_{N-N_0} = T_{N-N_0} \cup T_{-N_0}, \;\; S_{N-N_0+1} = T_{N-N_0+1} \cup T_{-N_0+1}, \;\; \ldots, \;\; S_N = T_N \cup T_0.$$

These sets are disjoint. The sets $T_0, T_1, \ldots, T_N$ all have diameter less than δ. We have $d(h^N(y), y) < \delta$, so the choice of δ implies that $d(h^n(y), h^{n-N}(y)) < \delta_0$ for $n = -N_0, -N_0+1, \ldots, N$. Also, $T_{n-N} = h^{n-N}(T_0)$ has diameter less than δ. Therefore $T_{n-N} \cup T_n$ has diameter less than $2\delta + \delta_0 \leq 3\delta_0$. It follows that S_n has diameter less than $3\delta_0$ for $n = 1, 2, \ldots, N$. Since f varies by at most $\frac{1}{4}\varepsilon$ on any set with diameter less than $4\delta_0$, and since the sets $S_1, S_2, \ldots, S_N$ are disjoint, there is $g \in C(X)$ which is constant on each of these sets and satisfies $\|f - g\| < \frac{1}{2}\varepsilon$.

Let the values of g on these sets be λ_1 on S_1 through λ_N on S_N. Then $ge_n = e_n g = \lambda_n e_n$ for $0 \leq n \leq N - N_0$. For $N - N_0 < n \leq N$ we use

$$e_n \in (q_{n-N} + q_n) A_Y (q_{n-N} + q_n)$$

to get, using the same calculations as above at the third and fourth steps,

$$ge_n = g(q_{n-N} + q_n)e_n = \lambda_n (q_{n-N} + q_n) e_n = e_n (q_{n-N} + q_n) g = e_n g.$$

Since $\|f - g\| < \frac{1}{2}\varepsilon$ and $ge = eg$, it follows that

$$\|pf - fp\| = \|fe - ef\| < \varepsilon.$$

This is (1) for f. That $pfp \in A_Y$ follows from the fact that f and p are in this subalgebra. So we also have (2) for f.

It remains only to verify (3). Using $h^{r_l}(Y) \subset U_l$ for $l = -N_0, -N_0+1, \ldots, N$ and disjointness of the sets $U_{-N_0}, U_{-N_0+1}, \ldots, U_N$ at the third step, and defining $Z = \bigcup_{l=-N_0}^{N} h^{r_l}(Y) \subset U$, we get (with Murray-von Neumann equivalence in A)

$$1 - p = e \leq \sum_{l=-N_0}^{N} q_l \sim \sum_{l=-N_0}^{N} \chi_{h^{r_l}(Y)} = \chi_Z.$$

This completes the proof. ∎

Proof of Theorem 16.1 when X is the Cantor set. We use Exercise 16.17. Let $u \in C^*(\mathbb{Z}, X, h)$ be the standard unitary (called u_1 in Notation 8.7). Take the set S in Exercise 16.17 to be $S = C(X) \cup \{u\}$. We verify the conditions (1), (2), and (3) in Lemma 16.16 for finite sets $F \subset S$. We may clearly assume that u is in our finite subset, so let $F_0 \subset C(X)$ be finite, let $c \in A_+ \setminus \{0\}$, let $\varepsilon > 0$, and take $F = F_0 \cup \{u\}$. Use Lemma 16.23 to find a nonempty compact open set $U \subset X$ such that χ_U is Murray-von Neumann equivalent to a projection $q \in \overline{cC^*(\mathbb{Z}, X, h)c}$. Choose any $y \in X$. Apply Lemma 16.24 with U, ε, and y as given, and with F_0 in place of F. Let p and Y be the resulting projection and compact open set. Let Z be as in part (3) of Lemma 16.24. Then $\|ap - pa\| < \varepsilon$ for all $a \in F$, which is (1) of Lemma 16.16. Also, $C^*(\mathbb{Z}, X, h)_Y$ is an AF algebra by Lemma 16.20, so $pC^*(\mathbb{Z}, X, h)_Y p$ is a corner of an AF algebra, hence AF. For $a \in F$, part (2) of Lemma 16.24 gives $\text{dist}(a, pC^*(\mathbb{Z}, X, h)_Y p) = 0 < \varepsilon$, which is (1) of Lemma 16.16. Finally, using part (3) of Lemma 16.24 at the first step, we get

$$1 - p \precsim \chi_Z \leq \chi_U \sim q \in \overline{cC^*(\mathbb{Z}, X, h)c}.$$

This is (3) of Lemma 16.16. ∎

17. Minimal Homeomorphisms of Finite Dimensional Spaces

In this section, we describe what needs to be done to prove the general case of Theorem 16.1. We do not give full details, since to do so would require substantial excursions into parts of C*-algebras which have little to do with dynamics, namely the structure of direct limits of subhomogeneous C*-algebras and K-theory. We do give proofs of some the parts which are related to dynamics.

In this section, we need to assume some familiarity with K-theory. We don't discuss K-theory here. Instead, we refer to [290] for a gentle introduction and [23] for a more extensive treatment.

For applications to classification, the following generalization of Theorem 16.1 is useful. See [286]; the statement is essentially Proposition 4.6 of [286]. Since some parts of the proof involve essentially no extra work, we describe parts of the proof of the generalization.

Theorem 17.1 ([286])**.** Let X be an infinite compact metric space with finite covering dimension, and let $h\colon X \to X$ be a minimal homeomorphism. Let D be $\mathbb{C}$ or a UHF algebra of the form $\bigotimes_{n=1}^{\infty} M_l$ for some prime l. Suppose that, following the notation of Definition 16.14, $\rho\big(K_0(D \otimes C^*(\mathbb{Z}, X, h))\big)$ is dense in $\mathrm{Aff}\big(\mathrm{T}(D \otimes C^*(\mathbb{Z}, X, h))\big)$. Then $D \otimes C^*(\mathbb{Z}, X, h)$ is a simple unital C*-algebra with tracial rank zero which satisfies the Universal Coefficient Theorem.

In the description we give of the proof, presumably D can be any UHF algebra.

If X is an odd sphere of dimension at least 3 and h is uniquely ergodic (see Theorem 2.42), then h satisfies the hypotheses of Theorem 17.1 when D is any UHF algebra but not when $D = \mathbb{C}$. More generally, it follows from Proposition 3.12(b) of [27] that the hypotheses of Theorem 17.1 are satisfied whenever D is a UHF algebra and the projections in $\bigcup_{n=1}^{\infty} M_n(C^*(\mathbb{Z}, X, h))$ distinguish the traces on $C^*(\mathbb{Z}, X, h)$.

The first complication involves the construction of Rokhlin towers, as in the proof of Lemma 16.20 and the discussion after Definition 16.18. The sets Y and Y_k used there can't be chosen to be compact and open (indeed, if X is connected, there will be no nontrivial compact open sets), so that the projections χ_{Y_k} are not in $C(X)$ (and not in $C^*(\mathbb{Z}, X, h)$ either). It turns out that one must take Y to be closed with nonempty interior, and replace the sets Y_k by their closures. Then they are no longer disjoint. The algebra $C^*(\mathbb{Z}, X, h)_Y$ is now a very complicated subalgebra of $\bigoplus_{k=0}^{l} M_{n(k)} \otimes C(Y_k)$. It is what is known as a recursive subhomogeneous algebra. See Definition 17.10 and Theorem 17.14 below. We give a complete proof of Theorem 17.14 since, as far as we know, no complete proof has yet been published. It is taken with little change from the unpublished paper [162].

This modification will lead to further difficulties. Such algebras are generally not AF, and may have few or no nontrivial projections. The hypothesis on the range of ρ (which was not used in Section 16, although it is automatic when X is the Cantor set) must be used to produce sufficiently many nonzero projections and approximating finite dimensional subalgebras.

The following definition and lemma formalize the first return time used in the proof of Lemma 16.20.

Definition 17.2. Let X be an infinite compact metric space and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$, and let $x \in Y$. The *first return time* $r_Y(x)$

of x to Y is the smallest integer $n \geq 1$ such that $h^n(x) \in Y$. We set $r_Y(x) = \infty$ if no such n exists. If Y is understood, we may simply write $r(x)$.

Lemma 17.3. Let X be an infinite compact metric space, let $h\colon X \to X$ be a minimal homeomorphism, and let $Y \subset X$. If $\operatorname{int}(Y) \neq \varnothing$, then $\sup_{x \in Y} r_Y(x) < \infty$.

Proof. Set $U = \bigcup_{n=1}^{\infty} h^{-n}(\operatorname{int}(Y))$. Clearly $h^{-1}(U) \subset U$. Applying Lemma 15.3(4) to h^{-1}, and using $U \neq \varnothing$, we get $U = X$. Therefore the sets $h^{-n}(\operatorname{int}(Y))$, for $n \geq 1$, form an open cover of the compact set $\overline{Y}$. Choose a finite subcover. The largest value of n used is an upper bound for $\{r_Y(x)\colon x \in Y\}$. $\square$

Definition 17.4. Let $Y \subset X$ be closed with $\operatorname{int}(Y) \neq \emptyset$. The *modified Rokhlin tower* associated with Y consists of the subsets and numbers

$$Y_0, Y_1, \ldots, Y_l \subset Y, \quad Y_0^{\bullet}, Y_1^{\bullet}, \ldots, Y_l^{\bullet} \subset Y, \quad \text{and} \quad 1 \leq n(0) < n(1) < \cdots < n(l),$$

defined as follows. We let $n(0) < n(1) < \cdots < n(l)$ be the distinct values of the first return time to Y (there are only finitely many, by Lemma 17.3), and we define

$$Y_k = \overline{\{x \in Y\colon r(x) = n(k)\}} \qquad \text{and} \qquad Y_k^{\bullet} = \operatorname{int}\big(\{x \in Y\colon r(x) = n(k)\}\big)$$

for $k = 0, 1, \ldots, l$.

We warn that there is no reason to expect $Y_k^{\bullet}$ to be dense in Y_k, or even that $Y_k^{\bullet} = \operatorname{int}(Y_k)$.

Lemma 17.5. Let $Y \subset X$ be closed with $\operatorname{int}(Y) \neq \varnothing$. Then (following the notation of Definition 17.4):

(1) Suppose $0 \leq k,\, k' \leq l$ and $0 \leq j,\, j' \leq n(k) - 1$, with $(k, j) \neq (k', j')$. Then $h^{-j}(Y_k^{\bullet}) \cap h^{-j'}(Y_{k'}) = \varnothing$.
(2) $X = \bigcup_{k=0}^{l} \bigcup_{j=0}^{n(k)-1} h^j(Y_k)$.
(3) $X = \bigcup_{j=1}^{n(l)} h^{-j}(Y)$.
(4) $Y = \bigcup_{k=0}^{l} Y_k$.
(5) For $k = 0, 1, \ldots, l$ and $y \in Y_k$, if $r(y) < n(k)$ then $y \in Y_k \setminus Y_k^{\bullet}$.

Proof. We begin with an argument from the proof of Lemma 17.3. Set $U = \bigcup_{j=1}^{\infty} h^j(\operatorname{int}(Y))$. Clearly $h(U) \subset U$. Since $U \neq \varnothing$, Lemma 15.3(4) implies $U = X$. In particular, $\bigcup_{j=1}^{\infty} h^j(Y) = X$.

It is now essentially immediate from the construction that

$$X = \coprod_{k=0}^{l} \coprod_{j=0}^{n(k)-1} h^j\left(\{x \in Y\colon r(x) = n(k)\}\right).$$

Part (2) follows since

$$\{x \in Y\colon r(x) = n(k)\} \subset Y_k.$$

Part (3) follows by applying $h^{-n(l)}$ to part (2), since $n(l)$ is the largest of the $n(k)$.

For part (1), apply the disjointness part of the above together with the observation that $R \cap S = \varnothing$ implies $\operatorname{int}(R) \cap \overline{S} = \varnothing$.

In part (4), the inclusion $Y \subset \bigcup_{k=0}^{l} Y_k$ is immediate from

$$\{x \in Y\colon r(x) = n(k)\} \subset Y_k$$

for $k = 0, 1, \ldots, l$. The reverse inclusion follows from continuity of $h^{n(k)}$, the fact that Y is closed, and the relation

$$\{x \in Y : r(x) = n(k)\} \subset Y.$$

Part (5) follows from the relations

$$y \notin \{x \in Y : r(x) = n(k)\} \qquad \text{and} \qquad Y_k^{\bullet} \subset \{x \in Y : r(x) = n(k)\}.$$

This completes the proof. □

There is a homomorphism from $C^*(\mathbb{Z}, X, h)_Y$ to $\bigoplus_{k=0}^{l} C(Y_k, M_{n(k)})$, like the isomorphism of (16.5), but in general it is not surjective. To describe it, we need a description of $C^*(\mathbb{Z}, X, h)_Y$, which we provide in the following proposition. It is valid whether or not $\mathrm{int}(Y) = \varnothing$.

Proposition 17.6 (Proposition 7.5 of [213])**.** Let X be a compact Hausdorff space and let $h \colon X \to X$ be a homeomorphism. Let $u \in C^*(\mathbb{Z}, X, h)$ be the standard unitary generator (u_1 in Notation 8.7), and let $E \colon C^*(\mathbb{Z}, X, h) \to C(X)$ be the standard conditional expectation (E_1 in Definition 9.18). Let $Y \subset X$ be a nonempty closed subset. For $n \in \mathbb{Z}$, set

$$Z_n = \begin{cases} \bigcup_{j=0}^{n-1} h^j(Y) & n > 0 \\ \varnothing & n = 0 \\ \bigcup_{j=1}^{-n} h^{-j}(Y) & n < 0. \end{cases}$$

Then

$$C^*(\mathbb{Z}, X, h)_Y = \big\{ a \in C^*(\mathbb{Z}, X, h) \colon E(a u^{-n}) \in C_0(X \setminus Z_n) \text{ for all } n \in \mathbb{Z} \big\} \tag{17.1}$$

and

$$\overline{C^*(\mathbb{Z}, X, h)_Y \cap C(X)[\mathbb{Z}]} = C^*(\mathbb{Z}, X, h)_Y. \tag{17.2}$$

Proof. Define

$$B = \big\{ a \in C^*(\mathbb{Z}, X, h) \colon E(a u^{-n}) \in C_0(X \setminus Z_n) \text{ for all } n \in \mathbb{Z} \big\}$$

and

$$B_0 = B \cap C(X)[\mathbb{Z}].$$

We claim that B_0 is dense in B. We would like to write an element of B as $\sum_{k=-\infty}^{\infty} b_k u^k$ with $b_k \in C_0(X \setminus Z_k)$ for $k \in \mathbb{Z}$. Unfortunately, in general, such series need not converge. (See Remark 9.19. If $\mathrm{int}(T) \neq \varnothing$, then the series is necessarily finite and therefore does converge.) Instead, we use the Cesàro means. So let $b \in B$ and for $k \in \mathbb{Z}$ define $b_k = E(b u^{-k}) \in C_0(X \setminus Z_k)$. Then for $n \in \mathbb{Z}_{>0}$, the element

$$a_n = \sum_{k=-n+1}^{n-1} \left(1 - \frac{|k|}{n} \right) b_k u^k.$$

is clearly in B_0, and Theorem VIII.2.2 of [52] implies that $\lim_{n \to \infty} a_n = b$. The claim follows. In particular, (17.2) will now follow from (17.1), so we need only prove (17.1).

For $0 \leq m \leq n$ and $0 \geq m \geq n$, we clearly have $Z_m \subset Z_n$.

We claim that for all $n \in \mathbb{Z}$, we have

$$h^{-n}(Z_n) = Z_{-n}. \tag{17.3}$$

The case $n = 0$ is trivial, the case $n > 0$ is easy, and the case $n < 0$ follows from the case $n > 0$.

We next claim that for all $m, n \in \mathbb{Z}$, we have

$$Z_{m+n} \subset Z_m \cup h^m(Z_n).$$

The case $m = 0$ or $n = 0$ is trivial. For $m, n > 0$ and also for $m, n < 0$, it is easy to check that $Z_{m+n} \subset Z_m = h^m(Z_n)$.

Now suppose $m > 0$ and $-m \leq n < 0$. Then $0 \leq m + n \leq m$, so

$$Z_{m+n} \subset Z_m \subset Z_m \cup h^m(Z_n).$$

If $m > 0$ and $n < -m$, then

$$Z_{m+n} = \bigcup_{j=m+n}^{-1} h^j(Y) \subset \bigcup_{j=m+n}^{m-1} h^j(Y) = \bigcup_{j=0}^{m-1} h^j(Y) \cup \bigcup_{j=m+n}^{m-1} h^j(Y) = Z_m \cup h^m(Z_n).$$

Finally, suppose $m < 0$ and $n > 0$. Then, using (17.3) at the first and third steps, and the already done case $m > 0$ and $n < 0$ at the second step, we get

$$Z_{m+n} = h^{m+n}(Z_{-m-n}) \subset h^{m+n}\big(Z_{-m} \cup h^{-m}(Z_{-n})\big) = h^n(Z_m) \cup Z_n.$$

This completes the proof of the claim.

We now claim that B_0 is a *-algebra. It is enough to prove that if $f \in C_0(X \setminus Z_m)$ and $g \in C_0(X \setminus Z_n)$, then $(fu^m)(gu^n) \in B_0$ and $(fu^m)^* \in B_0$. For the first, we have $(fu^m)(gu^n) = f \cdot (g \circ h^{-m}) \cdot u^{m+n}$. Now $f \cdot (g \circ h^{-m})$ vanishes on $Z_m \cup h^m(Z_n)$, so the previous claim implies that $f \cdot (g \circ h^{-m}) \in C_0(X \setminus Z_{m+n})$. Also, $(fu^m)^* = u^{-m}\overline{f} = \big(\overline{f \circ h^m}\big)u$, and, using (17.3), the function $f \circ h^m$ vanishes on $h^{-m}(Z_m) = Z_{-m}$, so $(fu^m)^* \in B_0$. This proves the claim.

Since $C(X) \subset B_0$ and $C_0(X \setminus Y)u \subset B_0$, it follows that $C^*(\mathbb{Z}, X, h)_Y \subset \overline{B_0} = B$.

We next claim that for all $n \in \mathbb{Z}$, we have $C_0(X \setminus Z_n) \subset C^*(\mathbb{Z}, X, h)_Y$. For $n = 0$ this is trivial. Let $n > 0$, and let $f \in C_0(X \setminus Z_n)$. Define $f_0 = (\operatorname{sgn} \circ f)|f|^{1/n}$ and for $j = 1, 2, \ldots, n-1$ define $f_j = |f \circ h^j|^{1/n}$. Then $f_0, f_1, \ldots, f_{n-1} \in C_0(X \setminus Y)$. Therefore the element

$$a = (f_0 u)(f_1 u) \cdots (f_{n-1} u)$$

is in $C^*(\mathbb{Z}, X, h)_Y$. Moreover, we can write

$$\begin{aligned} a &= f_0(uf_1u^{-1})(u^2 f_2 u^{-2}) \cdots (u^{n-1} f_{n-1} u^{-(n-1)})u^n \\ &= f_0(f_1 \circ h^{-1})(f_2 \circ h^{-2}) \cdots (f_{n-1} \circ h^{-(n-1)})u^n = (\operatorname{sgn} \circ f)\big(|f|^{1/n}\big)^n u^n = fu^n. \end{aligned}$$

Finally, suppose $n < 0$, and let $f \in C_0(X \setminus Z_n)$. It follows from (17.3) that $f \circ h^n \in C_0(X \setminus Z_{-n})$, whence also $\overline{f \circ h^n} \in C_0(X \setminus Z_{-n})$. Since $-n > 0$, we therefore get

$$fu^n = \big(u^{-n}\overline{f}\big)^* = \big(\big(\overline{f \circ h^n}\big)u^{-n}\big)^* \in C^*(\mathbb{Z}, X, h)_Y.$$

The claim is proved.

It now follows that $B_0 \subset C^*(\mathbb{Z}, X, h)_Y$. Combining this result with $\overline{B_0} = B$ and $C^*(\mathbb{Z}, X, h)_Y \subset B$, we get $C^*(\mathbb{Z}, X, h)_Y = B$. ∎

Corollary 17.7. Let X be an infinite compact Hausdorff space and let $h\colon X \to X$ be a minimal homeomorphism. As in Proposition 17.6, let $u \in C^*(\mathbb{Z}, X, h)$ be the standard unitary generator. Let $Y \subset X$ be a closed subset such that $\operatorname{int}(Y) \neq \varnothing$.

Let Z_n be as in Proposition 17.6. Then there exists $N \in \mathbb{Z}_{\geq 0}$ such that $C^*(\mathbb{Z}, X, h)_Y$ has the Banach space direct sum decomposition

$$C^*(\mathbb{Z}, X, h)_Y = \bigoplus_{n=-N}^{N} C_0(X \setminus Z_n) u^n.$$

Proof. Define $N = \sup_{x \in Y} r_Y(x)$. Then N is finite by Lemma 17.3. Proposition 17.6 implies that

$$C^*(\mathbb{Z}, X, h)_Y = \sum_{n=-N}^{N} C_0(X \setminus Z_n) u^n.$$

The sum on the right is algebraically a direct sum, the subspaces are closed, and there are finitely many of them, so it is a Banach space direct sum by the Open Mapping Theorem. □

Notation 17.8. Assume $n(0)$, $n(1)$, $\ldots$, $n(l)$ are positive integers. (They will be the first return times associated with a minimal homeomorphism $h \colon X \to X$ and a closed subset $Y \subset X$ with nonempty interior.) Define $s_k^{(0)}$ and s_k in $M_{n(k)}$, or in $C(Z, M_{n(k)})$ for any Z, by

$$s_k^{(0)} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad s_k = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

The only difference is in the upper right corner, where s_k has the entry 1.

The formula for γ in the following proposition is based on a formula for the Cantor set case in [229]. Recall that our definition of $C^*(\mathbb{Z}, X, h)_Y$ differs from that in [229]. If we took

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\, u C_0(X \setminus Y)\big) \subset C^*(\mathbb{Z}, X, h),$$

as in [229], the correct formulas would use $Z_m = \bigcup_{j=0}^{m-1} h^{-j}(Y)$ for $m \in \mathbb{Z}_{\geq 0}$, and would be

$$\gamma_k(u^m f) = s_k^m \operatorname{diag}\big(f \circ h|_{Y_k},\, f \circ h^2|_{Y_k},\, \ldots,\, f \circ h^{n(k)}|_{Y_k}\big)$$

and

$$\gamma_k(f u^{-m}) = \operatorname{diag}\big(f \circ h|_{Y_k},\, f \circ h^2|_{Y_k},\, \ldots,\, f \circ h^{n(k)}|_{Y_k}\big) \cdot s_k^{-m}$$

for $f \in C_0(X \setminus Z_m)$.

This proposition and its proof are based on [162].

Proposition 17.9. Let X be an infinite compact Hausdorff space and let $h \colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be closed with $\operatorname{int}(Y) \neq \varnothing$. Adopt the notation of Definition 17.4 and Notation 17.8, and let Z_m be as in Proposition 17.6. For $k = 0, 1, \ldots, l$ there a unique linear map $\gamma_k \colon C^*(\mathbb{Z}, X, h)_Y \to C(Y_k, M_{n(k)})$ such that

$$\gamma_k(f u^m) = \operatorname{diag}\big(f|_{Y_k},\, f \circ h|_{Y_k},\, \ldots,\, f \circ h^{n(k)-1}|_{Y_k}\big) \cdot s_k^m$$

and

$$\gamma_k(u^{-m}f) = s_k^{-m}\cdot \operatorname{diag}\big(f|_{Y_k},\, f\circ h|_{Y_k},\, \ldots,\, f\circ h^{n(k)-1}|_{Y_k}\big)$$

for $f \in C_0(X \setminus Z_m)$. Moreover, the map

$$\gamma\colon C^*(\mathbb{Z}, X, h)_Y \to \bigoplus_{k=0}^{l} C(Y_k, M_{n(k)})$$

given by

$$\gamma(a) = (\gamma_0(a),\, \gamma_1(a),\, \ldots,\, \gamma_l(a)).$$

is a homomorphism of C*-algebras.

Proof. We first claim that if $f \in C(X)$ and $m \in \mathbb{Z}_{\geq 0}$, then $u^{-m}f \in C^*(\mathbb{Z}, X, h)_Y$ if and only if $f \in C_0(X \setminus Z_m)$. Since $u^{-m}f = (f \circ h^m)u^m$, the claim follows from Proposition 17.6 and the fact that $Z_{-m} = h^{-m}(Z_m)$ (so that f vanishes on Z_m if and only if $f \circ h^m$ vanishes on Z_{-m}).

Existence and uniqueness of the linear map γ_k now follows from the Banach space direct sum decomposition of Corollary 17.7.

It remains to check that γ is a homomorphism. We check that γ_k is a homomorphism for $k = 0, 1, \ldots, l$.

It is obvious that $\gamma_k(a^*) = \gamma_k(a)^*$ for $a \in C^*(\mathbb{Z}, X, h)_Y$. So we only need to prove multiplicativity.

Define $\sigma_k\colon C(X) \to C(Y_k, M_{n(k)})$ by

$$\sigma_k(f) = \operatorname{diag}\big(f|_{Y_k},\, f\circ h|_{Y_k},\, \ldots,\, f\circ h^{n(k)-1}|_{Y_k}\big).$$

Let Z_m be as in Proposition 17.6. We claim that if $f \in C(X \setminus Z_m)$ and $g \in C(X)$, then

$$s_k^m\sigma_k(f\circ h^m)\sigma_k(g) = \sigma(f)s_k^m\sigma_k(g) = \sigma(f)\sigma_k(g\circ h^{-m})s_k^m. \tag{17.4}$$

Define

$$b_1 = \operatorname{diag}\big((f\circ h^{n(k)}|_{Y_k})(g\circ h^{n(k)-m}|_{Y_k}),\, (f\circ h^{n(k)+1}|_{Y_k})(g\circ h^{n(k)-m+1}|_{Y_k}),\, \ldots,\\ (f\circ h^{n(k)+m-1}|_{Y_k})(g\circ h^{n(k)-1}|_{Y_k})\big),$$

$$b_2 = \operatorname{diag}\big((f|_{Y_k})(g\circ h^{n(k)-m}|_{Y_k}),\, (f\circ h|_{Y_k})(g\circ h^{n(k)-m+1}|_{Y_k}),\, \ldots,\\ (f\circ h^{m-1}|_{Y_k})(g\circ h^{n(k)-1}|_{Y_k})\big),$$

$$b_3 = \operatorname{diag}\big((f|_{Y_k})(g\circ h^{-m}|_{Y_k}),\, (f\circ h|_{Y_k})(g\circ h^{-m+1}|_{Y_k}),\, \ldots,\\ (f\circ h^{m-1}|_{Y_k})g\circ h^{-1}|_{Y_k}\big),$$

and

$$c = \operatorname{diag}\big((f\circ h^{m}|_{Y_k})(g|_{Y_k}),\, (f\circ h^{m+1}|_{Y_k})(g\circ h|_{Y_k}),\, \ldots,\\ (f\circ h^{n(k)-1}|_{Y_k})(g\circ h^{n(k)-m-1}|_{Y_k})\big).$$

Carrying out the matrix multiplications gives, for any $f, g \in C(X)$, the block matrix forms (in which the off diagonal blocks are square, $m \times m$ in the upper right and $(n(k)-m)\times(n(k)-m)$ in the lower left):

$$s_k^m\sigma_k(f\circ h^m)\sigma_k(g) = \begin{pmatrix} 0 & b_1 \\ c & 0\end{pmatrix}, \qquad \sigma(f)s_k^m\sigma_k(g) = \begin{pmatrix} 0 & b_2 \\ c & 0\end{pmatrix},$$

and

$$\sigma(f)\sigma_k(g \circ h^{-m})s_k^m = \begin{pmatrix} 0 & b_3 \\ c & 0 \end{pmatrix}.$$

Now we recall that f is required to vanish on

$$Z_m = Y \cup h(Y) \cup \cdots \cup h^{m-1}(Y).$$

Since $h^{n(k)}(Y_k) \subset Y$, it follows that $b_1 = b_2 = b_3 = 0$. Thus, all three products agree. This proves the claim.

Now let $p,\ q \in \mathbb{Z}_{\geq 0}$, let $f \in C(X \setminus Z_p)$, and let $g \in C(X \setminus Z_q)$. We claim that

$$\gamma_k\big((fu^p)(gu^q)\big) = \gamma_k(fu^p)\gamma_k(gu^q) \tag{17.5}$$

for any such p and q, that

$$\gamma_k\big((u^{-p}f)(gu^q)\big) = \gamma_k(u^{-p}f)\gamma_k(gu^q) \tag{17.6}$$

whenever $p \leq q$, and that

$$\gamma_k\big((fu^p)(u^{-q}g)\big) = \gamma_k(fu^p)\gamma_k(u^{-q}g) \tag{17.7}$$

whenever $q \leq p$. Given the claim, to prove multiplicativity it suffices to prove (17.6) when $p \geq q$, (17.7) when $q \geq p$, and

$$\gamma_k\big((u^{-p}f)(u^{-q}g)\big) = \gamma_k(u^{-p}f)\gamma_k(u^{-q}g)$$

for arbitrary $p,\ q \in \mathbb{Z}_{\geq 0}$. The first can be deduced by taking adjoints in (17.6), the second can be deduced by taking adjoints in (17.7), and the third can be deduced by taking adjoints in (17.5).

Using the first part of (17.4) with $m = p$ at the second step, we get

$$\begin{aligned} \gamma_k(fu^p)\gamma_k(gu^q) &= \sigma_k(f)s_k^p\sigma_k(g)s_k^q = \sigma_k(f)\sigma_k(g \circ h^{-p})s_k^{p+q} \\ &= \gamma_k\big(f(g \circ h^{-p})u^{p+q}\big) = \gamma_k\big((fu^p)(gu^q)\big), \end{aligned}$$

which is (17.5).

Assume that $p \leq q$. Then $\overline{g}$ vanishes on Z_p since $Z_p \subset Z_q$. Applying (17.4) with $m = p$ at the third step, we get

$$\begin{aligned} \gamma_k(u^{-p}f)\gamma_k(gu^q) &= s_k^{-p}\sigma_k(f)\sigma_k(g)s_k^q = \big[\sigma_k(\overline{g})\sigma_k(\overline{f})s_k^p\big]^* s_k^q \\ &= \big[s_k^p\sigma_k(\overline{g} \circ h^p)\sigma_k(\overline{f} \circ h^p)\big]^* s_k^q = \sigma_k(f \circ h^p)\sigma_k(g \circ h^p)s_k^{q-p} \\ &= \gamma_k\big((f \circ h^p)(g \circ h^p)u^{q-p}\big) = \gamma_k\big((u^{-p}f)(gu^q)\big), \end{aligned}$$

which is (17.6).

Now assume that $q \leq p$. Since $p - q \leq p$, we have $Z_{p-q} \subset Z_p$. So f vanishes on Z_{p-q}, and we can apply (17.4) with $m = p - q$ at the second step to get

$$\begin{aligned} \gamma_k(fu^p)\gamma_k(u^{-q}g) &= \sigma_k(f)s_k^{p-q}\sigma_k(g) = \sigma_k(f)\sigma_k(g \circ h^{q-p})s_k^{p-q} \\ &= \gamma_k\big(f(g \circ h^{q-p})u^{p-q}\big) = \gamma_k\big((fu^p)(u^{-q}g)\big). \end{aligned}$$

This is (17.7). The proof of the claim, and therefore of the proposition, is complete. □

What does the range of the homomorphism γ of Proposition 17.9 look like? To give a good answer, we start with the definition of a recursive subhomogeneous algebra. Essentially, it is a generalization of an algebra of the form $\bigoplus_{k=0}^{l} C(X_k,\ M_{n(k)})$, in which one is allowed to glue the summands together along the "boundaries" of the spaces X_k. As a very simple example, let $M_3 \oplus M_4 \subset M_7$ be the subalgebra consisting of all block diagonal matrices $a \oplus b$ with $a \in M_3$ and $b \in M_4$, and consider the C*-algebra

$$\big\{ f \in C([-1, 1], M_7) \colon f(t) \in M_3 \oplus M_4 \text{ for } t \in [-1, 0] \big\},$$

which is made by gluing together the algebras

$$C([-1, 0], M_3), \qquad C([-1, 0], M_4), \qquad \text{and} \qquad C([0, 1], M_7)$$

at the point 0. Here is an example with no nontrivial projections, using the same direct sum notation:

$$\big\{ f \in C([0,1], M_7) \colon f(0) \in M_3 \oplus M_4 \text{ and } f(1) \in M_2 \oplus M_5 \big\}.$$

Definition 17.10 (Definition 1.1 of [203])**.** The class of *recursive subhomogeneous algebras* is the smallest class $\mathcal{R}$ of C*-algebras such that:

(1) If X is a compact Hausdorff space and $n \in \mathbb{Z}_{>0}$, then $C(X, M_n) \in \mathcal{R}$.
(2) $\mathcal{R}$ is closed under the following pullback construction. Let $A \in \mathcal{R}$, let X be a compact Hausdorff space, let $X^{(0)} \subset X$ be closed, let $\varphi \colon A \to C\big(X^{(0)}, M_n\big)$ be any unital homomorphism, and let $\rho \colon C(X, M_n) \to C\big(X^{(0)}, M_n\big)$ be the restriction homomorphism. Then the pullback

$$A \oplus_{C(X^{(0)}, M_n)} C(X, M_n) = \{ (a, f) \in A \oplus C(X, M_n) \colon \varphi(a) = \rho(f) \}$$

(compare with Definition 2.1 of [199]) is in $\mathcal{R}$.

In (2) the choice $X^{(0)} = \varnothing$ is allowed (in which case $\varphi = 0$ is allowed). Thus the pullback could be an ordinary direct sum.

Remark 17.11. From the definition, it is clear that any recursive subhomogeneous algebra can be written in the form

$$R = \Big[\cdots \big[\big[C_0 \oplus_{C_1^{(0)}} C_1 \big] \oplus_{C_2^{(0)}} C_2 \big] \cdots \Big] \oplus_{C_l^{(0)}} C_l,$$

with $C_k = C(X_k, M_{n(k)})$ for compact Hausdorff spaces X_k and positive integers $n(k)$, with $C_k^{(0)} = C\big(X_k^{(0)}, M_{n(k)}\big)$ for compact subsets $X_k^{(0)} \subset X_k$ (possibly empty), and where the maps $C_k \to C_k^{(0)}$ are always the restriction maps. An expression of this type will be referred to as a *recursive subhomogeneous decomposition* of R. (The decomposition is very far from unique.)

We give parts of Definition 1.2 of [203].

Definition 17.12. Let R be a recursive subhomogeneous algebra, with a decomposition as in Remark 17.11. We associate with this decomposition:

(1) For $k = 0, 1, \ldots, l$, the *k-th stage algebra*

$$R^{(k)} = \Big[\cdots \big[\big[C_0 \oplus_{C_1^{(0)}} C_1 \big] \oplus_{C_2^{(0)}} C_2 \big] \cdots \Big] \oplus_{C_k^{(0)}} C_k,$$

obtained by using only the first $k+1$ algebras $C_0, C_1, \ldots, C_k$.
(2) Its *base spaces* $X_0, X_1, \ldots, X_l$ and *total space* $X = \coprod_{k=0}^{l} X_k$.
(3) Its *topological dimension* $\dim(X)$ (following Definition 16.7; here equal to $\max_k \dim(X_k)$).
(4) Its *matrix sizes* $n(0), \ldots, n(l)$.
(5) Its *minimum matrix size* $\min_k n(k)$.

(6) Its *standard representation* $\sigma = \sigma_R \colon R \to \bigoplus_{k=0}^{l} C(X_k, \, M_{n(k)})$, defined by forgetting the restriction to a subalgebra in each of the pullbacks in the decomposition.

By abuse of language, we will often refer to the base spaces, topological dimension, etc. of a recursive subhomogeneous algebra A, when they in fact apply to a particular recursive subhomogeneous decomposition. The minimum matrix size actually does not depend on the decomposition, since it is the smallest dimension of an irreducible representation of A. The base spaces certainly do, and even their dimensions do, as can be seen by considering the following example.

Example 17.13. Let X be an arbitrary compact Hausdorff space, set $X^{(0)} = X$, and define $\varphi \colon \mathbb{C} \to C(X)$ by $\varphi(\lambda) = \lambda \cdot 1$ for $\lambda \in \mathbb{C}$. Let $\rho \colon C(X) \to C(X)$ be $\rho = \mathrm{id}_{C(X)}$. Then $\mathbb{C} \oplus_{C(X)} C(X) \cong \mathbb{C}$, so we have a recursive subhomogeneous decomposition for $\mathbb{C}$ whose topological dimension is $\dim(X)$.

Theorem 17.14. Let X be an infinite compact metric space, let $h \colon X \to X$ be a minimal homeomorphism, and let $Y \subset X$ be a closed subset with $\mathrm{int}(Y) \neq \varnothing$. Then the algebra $C^*(\mathbb{Z}, X, h)_Y$ of Definition 16.18 is a recursive subhomogeneous algebra with topological dimension equal to $\dim(X)$, and whose base spaces are closed subsets of X.

The proof of this theorem is based on [162]; also see Section 2 of the survey [161]. It proceeds via several further lemmas.

Lemma 17.15. Let X be an infinite compact Hausdorff space and let $h \colon X \to X$ be a minimal homeomorphism. Let $Y \subset M$ be closed with $\mathrm{int}(Y) \neq \varnothing$. Then the homomorphism γ of Proposition 17.9 is unchanged if, for $k = 0, 1, \ldots, l$, we replace s_k by $s_k^{(0)}$ and s_k^{-1} by $\big(s_k^{(0)}\big)^*$ in the definition. That is, in the notation there, for $k = 0, 1, \ldots, l$, $m \in \mathbb{Z}_{\geq 0}$, and $f \in C_0(M \setminus Z_m)$, we have

$$\gamma_k(f u^m) = \mathrm{diag}\big(f|_{Y_k}, \, f \circ h|_{Y_k}, \, \ldots, \, f \circ h^{n(k)-1}|_{Y_k}\big)\big(s_k^{(0)}\big)^m$$

and

$$\gamma_k(u^{-m} f) = \big(\big(s_k^{(0)}\big)^*\big)^m \mathrm{diag}\big(f|_{Y_k}, \, f \circ h|_{Y_k}, \, \ldots, \, f \circ h^{n(k)-1}|_{Y_k}\big).$$

Proof. This follows by matrix multiplication from the fact that f vanishes on the sets $Y_k, \, h(Y_k), \, \ldots, \, h^{m-1}(Y_k)$. □

Corollary 17.16. Let X be an infinite compact Hausdorff space and let $h \colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be closed with $\mathrm{int}(Y) \neq \varnothing$. Let $E_k^{(m)} \colon C(Y_k, M_{n(k)}) \to C(Y_k, M_{n(k)})$ be the projection on the m-th subdiagonal, that is, identifying $C(Y_k, M_{n(k)})$ with $M_{n(k)}(C(Y_k))$, we have $E_k^{(m)}(b)_{m+j,j} = b_{m+j,j}$ for $j = 1, 2, \ldots, n(k) - m$ (if $m \geq 0$) and for $j = -m+1, \, -m+2, \, \ldots, \, n(k)$ (if $m \leq 0$), while $E_k^{(m)}(b)_{i,j} = 0$ for all other pairs (i, j). (In particular, if $m > n(k)$, then $E_k^{(m)} = 0$.) Set

$$D_m = \bigoplus_{k=0}^{l} E_k^{(m)}\big(C(Y_k, M_{n(k)})\big).$$

Let

$$\gamma_k \colon C^*(\mathbb{Z}, X, h)_Y \to C(Y_k, M_{n(k)}) \quad \text{and} \quad \gamma \colon C^*(\mathbb{Z}, X, h)_Y \to \bigoplus_{k=0}^{l} C(Y_k, M_{n(k)})$$

be as in Proposition 17.9. Then:

(1) There is a Banach space direct sum decomposition
$$\bigoplus_{k=0}^{l} C(Y_k, M_{n(k)}) = \bigoplus_{m=-n(l)}^{n(l)} D_m.$$

(2) For $k = 0, 1, \ldots, l$, $m \in \mathbb{Z}_{\geq 0}$, $f \in C_0(M \setminus Z_m)$, and $x \in Y_k$, the expression $\gamma_k(fu^m)(x)$ is given by the following matrix, in which the first nonzero entry is in row $m+1$.
$$\gamma_k(fu^m)(x) = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & & & & & \vdots \\ f \circ h^m(x) & 0 & & & & & \vdots \\ 0 & f \circ h^{m+1}(x) & & & & & \vdots \\ \vdots & & \ddots & & & & \vdots \\ 0 & \cdots & \cdots & f \circ h^{n(k)-1}(x) & 0 & \cdots & 0 \end{pmatrix}.$$

(3) For $m \geq 0$ and $f \in C_0(X \setminus Z_m)$,
$$\gamma_k(fu^m) \in E_k^{(m)}\big(C(Y_k, M_{n(k)})\big) \qquad \text{and} \qquad \gamma(fu^m) \in D_m$$
and
$$\gamma_k(u^{-m}f) \in E_k^{(-m)}\big(C(Y_k, M_{n(k)})\big) \qquad \text{and} \qquad \gamma(u^{-m}f) \in D_{-m}.$$

(4) The homomorphism γ is compatible with the vector space direct sum decomposition of Proposition 17.9 on its domain and the vector space direct sum decomposition of part (1) on its codomain.

Proof. The direct sum decomposition of part (1) is easy. The rest is all essentially immediate from Proposition 17.9 and Lemma 17.15. $\square$

Lemma 17.17. Let X be an infinite compact Hausdorff space and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be closed with $\mathrm{int}(Y) \neq \varnothing$. Then the homomorphism γ of Proposition 17.9 is injective.

Proof. By Corollary 17.16 and Corollary 17.7, it suffices to show that if $\gamma(fu^m) = 0$, with $m \geq 0$ and $f \in C_0(X \setminus Z_m)$, then $f = 0$. By the definition of γ, if $\gamma(fu^m) = 0$ then f vanishes on all sets $h^j(Y_k)$ for $k = 0, 1, \ldots, l$ and $j = 0, 1, \ldots, n(k) - 1$. These sets cover X by Lemma 17.5(2), so $f = 0$. $\square$

Lemma 17.18. Let X be an infinite compact Hausdorff space and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be closed with $\mathrm{int}(Y) \neq \varnothing$. Adopt the notation of Definition 17.4, and let γ be as in Proposition 17.9. An element
$$b = (b_0, b_1, \ldots, b_l) \in \bigoplus_{k=0}^{l} C(Y_k, M_{n(k)})$$
is in $\gamma(C^*(\mathbb{Z}, X, h)_Y)$ if and only if, whenever
$$r \in \mathbb{Z}_{>0}, \qquad k, t_1, t_2, \ldots, t_r \in \{0, 1, \ldots, l\}, \qquad n(t_1) + n(t_2) + \cdots + n(t_r) = n(k),$$

and

$$x \in (Y_k \setminus Y_k^{\bullet}) \cap Y_{t_1} \cap h^{-n(t_1)}(Y_{t_2}) \cap \cdots \cap h^{-[n(t_1)+n(t_2)+\cdots+n(t_{r-1})]}(Y_{t_r}),$$

then $b_k(x)$ is given by the block diagonal matrix

$$\begin{aligned} b_k(x) = \operatorname{diag}\big(&b_{t_1}(x),\ b_{t_2}(h^{n(t_1)}(x)), \\ &b_{t_3}(h^{n(t_1)+n(t_2)}(x)),\ \ldots,\ b_{t_r}(h^{n(t_1)+\cdots+n(t_{r-1})}(x))\big). \end{aligned} \tag{17.8}$$

Proof. For b, r, k, $t_1, t_2, \ldots, t_r$, and x as in the statement, let

$$\alpha_{x,r,k,t_1,t_2,\ldots,t_r}(b_0, b_1, \ldots, b_{k-1})$$

denote the block diagonal matrix on the right hand side of (17.8). We write

$$b^{(k-1)} = (b_0, b_1, \ldots, b_{k-1}).$$

Let $E_k^{(m)}$ be as in Corollary 17.16. By Corollary 17.16, it suffices to verify, for each fixed m, the statement of the lemma for elements $(b_0, b_1, \ldots, b_l)$ such that b_k is in the range of $E_k^{(m)}$ for $k = 0, 1, \ldots, l$. Using the adjoint, we may in fact restrict to the case $m \geq 0$.

We verify that elements of the range of γ satisfy the required relations. Let

$$r \in \mathbb{Z}_{>0}, \qquad t_1, t_2, \ldots, t_r \in \{0, 1, \ldots, l\}, \qquad n(t_1) + n(t_2) + \cdots + n(t_r) = n(k),$$

and

$$x \in (Y_k \setminus Y_k^{\bullet}) \cap Y_{t_1} \cap h^{-n(t_1)}(Y_{t_2}) \cap \cdots \cap h^{-[n(t_1)+n(t_2)+\cdots+n(t_{r-1})]}(Y_{t_r}).$$

Let $f \in C_0(X \setminus Z_m)$ and define

$$(b_0, b_1, \ldots, b_l) \in \bigoplus_{k=0}^{l} C(Y_k, M_{n(k)})$$

by $(b_0, b_1, \ldots, b_l) = \gamma(fu^m)$. By Corollary 17.16(2), the m-th subdiagonal of $b_k(x)$ is

$$\big(f \circ h^m(x),\ f \circ h^{m+1}(x),\ \ldots,\ f \circ h^{n(k)-1}(x)\big). \tag{17.9}$$

Similarly, the m-th subdiagonal of $\alpha_{x,r,k,t_1,t_2,\ldots,t_r}(b^{(k-1)})$ is given by the following formula (explanations afterwards):

(17.10)

$$\begin{aligned} \big(&f \circ h^m(x),\ f \circ h^{m+1}(x),\ \ldots,\ f \circ h^{n(t_1)-1}(x),\ 0,\ 0,\ \ldots,\ 0, \\ &f \circ h^{n(t_1)+m}(x),\ f \circ h^{n(t_1)+m+1}(x),\ \ldots,\ f \circ h^{n(t_1)+n(t_2)-1}(x),\ 0,\ 0,\ \ldots,\ 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \\ &f \circ h^{n(t_1)+\cdots+n(t_{r-1})+m}(x),\ f \circ h^{n(t_1)+\cdots+n(t_{r-1})+m+1}(x),\ \ldots, \\ &\qquad f \circ h^{n(t_1)+\cdots+n(t_{r-1})+n(t_r)-1}(x)\big). \end{aligned}$$

The sequences of zeros all have length m, except that if $n(t_i) \leq m$ then the subsequence

$$\begin{aligned} &f \circ h^{n(t_1)+\cdots+n(t_{i-1})+m}(x),\ f \circ h^{n(t_1)+\cdots+n(t_{i-1})+m+1}(x), \\ &\qquad \ldots,\ f \circ h^{n(t_1)+\cdots+n(t_i)-1}(x),\ 0,\ 0,\ \ldots,\ 0 \end{aligned} \tag{17.11}$$

should be read as a sequence of $n(t_i)$ zeros. Thus, for $i = 1, 2, \ldots, r-1$, the subsequence of the form (17.11) has total length $n(t_i)$, while for $i = r$ the corresponding subsequence (without zeros at the end) has total length $n(t_r) - m$.

The terms in (17.9) which have been replaced by zero in (17.10) are exactly those containing values of f at points of the form $h^{n(t_1)+\cdots+n(t_i)+j}(x)$ for $i = 1, 2, \ldots r-1$ and $j = 0, 1, \ldots m-1$. Since $h^{n(t_1)+\cdots+n(t_i)}(x) \in Y$, all these points are in Z_m, so that f is zero on them anyway. Therefore the sequences (17.9) and (17.10) are equal. We have show that elements of the range of γ satisfy the required relations.

For the converse, let $m \geq 0$, let

$$(b_0, b_1, \ldots, b_l) \in \bigoplus_{k=0}^{l} C(Y_k, M_{n(k)})$$

satisfy the relations in the statement of the lemma, and assume that b_k is in the range of $E_k^{(m)}$ for $k = 0, 1, \ldots, l$. Define continuous functions $f_k^{(j)} \colon h^j(Y_k) \to \mathbb{C}$, for $k = 0, 1, \ldots, l$ and $j = 0, 1, \ldots, n(k) - 1$, as follows. When $n(k) > m$, we specify that the m-th subdiagonal of b_k (starting at $(b_k)_{m+1,1}$) be given by

$$\big(f_k^{(m)} \circ h^m, \, f_k^{(m+1)} \circ h^{m+1}, \, \ldots, \, f_k^{(n(k)-1)} \circ h^{n(k)-1}\big).$$

That is,

$$f_k^{(j)} = (b_{j+1,\, j-m+1} \circ h^{-j})|_{Y_k}$$

for $j = m, m+1, \ldots, n(k) - 1$ We further set $f_k^{(j)} = 0$ for $k = 0, 1, \ldots, l$ and $j = 0, 1, \ldots, \min(n(k) - 1, \, m-1)$. We claim that there is a continuous function $f \colon X \to \mathbb{C}$ such that $f|_{h^j(Y_k)} = f_k^{(j)}$ for all j and k, and that f vanishes on Z_m. Given this, it is clear that $fu^m \in C^*(\mathbb{Z}, X, h)_Y$ and $\gamma(fu^m) = (b_0, b_1, \ldots, b_l)$.

Assume that f exists. We claim that f vanishes on Z_m. Recall that (Proposition 17.6 for the first equality and Lemma 17.5(4) for the second)

$$Z_m = \bigcup_{j=0}^{m-1} h^j(Y) = \bigcup_{j=0}^{m-1} \bigcup_{k=0}^{l} h^j(Y_k).$$

Let $j \in \{0, 1, \ldots, m\}$, let $k \in \{0, 1, \ldots, l\}$, and let $x \in h^j(Y_k)$. We need to show that $f(x) = 0$. If $j \leq n(k) - 1$, then $f_k^{(j)} = 0$ is immediate from the definition, so $f(x) = 0$. So assume $j > n(k) - 1$. Let $s \in \mathbb{Z}_{\geq 0}$ be the least nonnegative integer such that $h^{-s}(x) \in Y$. Then $s \leq j$. Set $x_0 = h^{-s}(x)$. Choose $i \in \{0, 1, \ldots, l\}$ such that $r(x_0) = n(i)$. Then $h(x_0), \, h^2(x_0), \, \ldots, h^s(x_0) \notin Y$, so

$$s \leq \min(n(i) - 1, \, j) \leq \min(n(i) - 1, \, m-1) \qquad \text{and} \qquad x = h^s(x_0) \in h^s(Y_{n(i)}).$$

So $f(x) = 0$ by the case considered first. This proves the claim.

It remains only to prove that f exists and is continuous. Since the sets $h^j(Y_k)$ are closed, it suffices to prove that if

$$0 \leq k_1, k_2 \leq l, \qquad 0 \leq j_1 \leq n(k_1) - 1, \qquad 0 \leq j_2 \leq n(k_2) - 1,$$

and

$$x \in h^{j_1}(Y_{k_1}) \cap h^{j_2}(Y_{k_2}),$$

then $f_{k_1}^{(j_1)}(x) = f_{k_2}^{(j_2)}(x)$. Without loss of generality $j_1 \leq j_2$.

First assume $j_1 = j_2$. Call this number j. Then without loss of generality $k_1 \leq k_2$. If $k_1 = k_2$, there is nothing to prove, so we may assume that $k_1 < k_2$. So $n(k_1) < n(k_2)$. Let $x_0 = h^{-j}(x)$. Then $x_0 \in Y_{k_1} \cap Y_{k_2}$. Since $n(k_1) < n(k_2)$ is a return time for x_0, we have $x_0 \in Y_{k_2} \setminus Y_{k_2}^{\bullet}$ by Lemma 17.5(5). Choose $t_1 \in \{0, 1, \ldots, l\}$ such that $n(t_1)$ is the first return time of $h^{n(k_1)}(x_0)$ to Y. So $h^{n(k_1)}(x_0) \in Y_{t_1}$. If $n(k_1) + n(t_1) < n(k_2)$, we can choose $t_2 \in \{0, 1, \ldots, l\}$ such that $n(t_2)$ is the first return time of $h^{n(k_1)+n(t_1)}(x_0)$ to Y. So $h^{n(k_1)+n(t_1)}(x_0) \in Y_{t_2}$. Proceed inductively. Since the numbers $n(t_1)$, $n(t_1) + n(t_2)$, $\ldots$ are successive return times of $h^{n(k_1)}(x_0)$ to Y, and since $h^{n(k_2)}(x_0) \in Y$, there is r such that

$$n(k_1) + n(t_1) + n(t_2) + \cdots + n(t_r) = n(k_2).$$

Then

$$\begin{aligned} x_0 \in & (Y_{k_2} \setminus Y_{k_2}^{\bullet}) \cap Y_{k_1} \cap h^{-n(k_1)}(Y_{t_1}) \\ & \cap h^{-[n(k_1)+n(t_1)]}(Y_{t_2}) \cap \cdots \cap h^{-[n(k_1)+n(t_1)+n(t_2)+\cdots+n(t_{r-1})]}(Y_{t_r}), \end{aligned}$$

so $\alpha_{x_0, r+1, k_2, k_1, t_1, t_2, \ldots, t_r}(b^{(k_2-1)}) = b_{k_2}(x_0)$.

If $0 \leq j \leq m-1$, then $f_{k_1}^{(j)}(x)$ and $f_{k_2}^{(j)}(x)$ are both zero. Otherwise, $f_{k_1}^{(j)}(x) = (f_{k_1}^{(j)} \circ h^j)(x_0)$ is the $(j+1,\, j-m+1)$ entry of $b_{k_1}(x_0)$ and $f_{k_2}^{(j)}(x) = (f_{k_2}^{(j)} \circ h^j)(x_0)$ is the $(j+1,\, j-m+1)$ entry of $b_{k_2}(x_0)$. The relations in the statement of the lemma, with $k_1, t_1, t_2, \ldots, t_r$ in place of $t_1, t_2, \ldots, t_r$, therefore imply that $f_{k_1}^{(j)}(x) = f_{k_2}^{(j)}(x)$, as desired.

Now suppose $j_1 < j_2$. We split this case in two subcases, the first of which is

$$n(k_1) - j_1 \leq n(k_2) - j_2. \tag{17.12}$$

Suppose $j_2 < m$. Then also $j_1 < m$, so $f_{k_1}^{(j_1)}(x) = f_{k_2}^{(j_2)}(x) = 0$, as desired.

So we can assume $m \leq j_2 \leq n(k_2) - 1$. Define $x_0 = h^{-j_2}(x)$, giving

$$x_0 \in Y_{k_2} \cap h^{-(j_2 - j_1)}(Y_{k_1}).$$

Now $j_2 - j_1 < n(k_2)$ and is a return time of x_0 to Y, so $x_0 \in Y_{k_2} \setminus Y_{k_2}^{\bullet}$ by Lemma 17.5(5). Using the same argument as in the previous case, choose

$$t_1, t_2, \ldots, t_\mu \in \{0, 1, \ldots, l\}$$

such that $n(t_1)$, $n(t_1) + n(t_2)$, $\ldots$ are successive return times of x_0 to Y, and such that

$$n(t_1) + n(t_2) + \cdots + n(t_\mu) = j_2 - j_1.$$

Similarly, using (17.12) to get $n(k_1) + j_2 - j_1 \leq n(k_2)$, choose

$$t'_1, t'_2, \ldots, t'_\nu \in \{0, 1, \ldots, l\}$$

such that $n(t'_1)$, $n(t'_1) + n(t'_2)$, $\ldots$ are successive return times of $h^{n(k_1)+j_2-j_1}(x_0)$ to Y, and such that

$$n(t'_1) + n(t'_2) + \cdots + n(t'_\nu) = n(k_1) - (j_2 - j_1).$$

Then

$$\begin{aligned} x_0 \in & (Y_{k_2} \setminus Y_{k_2}^{\bullet}) \cap Y_{t_1} \cap h^{-n(t_1)}(Y_{t_2}) \cap \cdots \cap h^{-[n(t_1)+\cdots+n(t_{\mu-1})]}(Y_{t_\mu}) \\ & \cap h^{-[n(t_1)+\cdots+n(t_\mu)]}(Y_{k_1}) \cap h^{-[n(t_1)+\cdots+n(t_\mu)+n(k_1)]}(Y_{t'_1}) \cap \cdots \\ & \cap h^{-[n(t_1)+\cdots+n(t_\mu)+n(k_1)+n(t'_1)+\cdots+n(t'_{\nu-1})]}(Y_{t'_\nu}) \end{aligned}$$

and

$$n(t_1) + \cdots + n(t_\mu) + n(k_1) + n(t'_1) + \cdots + n(t'_\nu) = n(k_2).$$

so the hypothesized relations give

$$\alpha_{x_0,\mu+1+\nu,k_2,t_1,t_2,\ldots,t_\mu,k_1,t'_1,t'_2,\ldots,t'_\nu}(b^{(k_2-1)}) = b_{k_2}(x_0). \tag{17.13}$$

We compare the $(j_2+1,\, j_2-m+1)$ entries of the two sides of (17.13). The $(j_2+1,\, j_2-m+1)$ entry of $b_{k_2}(x_0)$ is $\big(f_{k_2}^{(j_2)} \circ h^{j_2}\big)(x_0) = f_{k_2}^{(j_2)}(x)$. We examine the $(j_2+1,\, j_2-m+1)$ entry of the left hand side. By considering the row number j_2+1, and using the relations

$$j_2 = n(t_1) + n(t_2) + \cdots + n(t_\mu) + j_1 \qquad \text{and} \qquad 0 \leq j_1 \leq n(k_1) - 1,$$

we see that the $(j_2+1,\, j_2-m+1)$ entry of the left hand side of (17.13) must be either in the diagonal block

$$(b_{k_1} \circ h^{j_2-j_1})(x_0) = (b_{k_1} \circ h^{n(t_1)+n(t_2)+\cdots+n(t_\mu)})(x_0)$$

in the formula for $\alpha_{x_0,\mu+1+\nu,k_2,t_1,t_2,\ldots,t_\mu,k_1,t'_1,t'_2,\ldots,t'_\nu}(b^{(k_2-1)})$, or in none of the diagonal blocks. If $j_1 < m$, then

$$j_2 - m + 1 < j_2 - j_1 + 1 = n(t_1) + n(t_2) + \cdots + n(t_\mu) + 1,$$

so the $(j_2+1,\, j_2-m+1)$ entry of the left hand side of (17.13) is in none of the diagonal blocks. Thus (17.13) implies that $f_{k_2}^{(j_2)}(x) = 0$, while $f_{k_1}^{(j_1)}(x) = 0$ by the definition of $f_{k_1}^{(j_1)}$. Thus $f_{k_1}^{(j_1)}(x) = f_{k_2}^{(j_2)}(x)$, as desired.

If instead $m \leq j_1 \leq n(k_1) - 1$, then

$$n(t_1)+n(t_2)+\cdots+n(t_\mu)+1 \leq j_2-m+1 \leq j_2 < n(t_1)+n(t_2)+\cdots+n(t_\mu)+n(t_k),$$

so the $(j_2+1,\, j_2-m+1)$ entry of the of the left hand side of (17.13) is in the diagonal block $(b_{k_1} \circ h^{j_2-j_1})(x_0)$. In fact, it is the $(j_1+1,\, j_1-m+1)$ entry of $(b_{k_1} \circ h^{j_2-j_1})(x_0)$. So (17.13) implies that

$$f_{k_2}^{(j_2)}(x) = \big(f_{k_1}^{(j_1)} \circ h^{j_1} \circ h^{j_2-j_1}\big)(x_0) = f_{k_1}^{(j_1)}(x),$$

as desired.

Now suppose that $n(k_1) - j_1 > n(k_2) - j_2$, the opposite of (17.12). We reduce this case to a strictly smaller value of $n(k_1) + n(k_2)$ together with instances of the cases already done, so that the desired equality $f_{k_1}^{(j_1)}(x) = f_{k_2}^{(j_2)}(x)$ follows by a finite descent argument. Set $x_0 = h^{-j_1}(x) \in Y_{k_1}$. Using the same argument as before, choose

$$t_1, t_2, \ldots, t_r \in \{0, 1, \ldots, l\}$$

such that $n(t_1)$, $n(t_1) + n(t_2)$, ... are successive return times of x_0 to Y, and such that

$$n(t_1) + n(t_2) + \cdots + n(t_r) = n(k_1).$$

Then

$$h^{n(k_2)-(j_2-j_1)}(x_0) = h^{n(k_2)-j_2}(x) \in h^{n(k_2)}(Y_{k_2}) \subset Y$$

and $n(k_2) - (j_2 - j_1) < n(k_1)$, so $r \geq 2$. Choose $i \in \{0, 1, \ldots, l\}$ such that

$$n(t_1) + n(t_2) + \cdots + n(t_{i-1}) \leq j_1 < n(t_1) + n(t_2) + \cdots + n(t_i),$$

and let

$$k_3 = t_i \qquad \text{and} \qquad j_3 = j_1 - [n(t_1) + n(t_2) + \cdots + n(t_{i-1})].$$

Then $0 \leq j_3 \leq n(k_3) - 1$. Define

$$y = h^{n(t_1)+n(t_2)+\cdots+n(t_{i-1})}(x_0) \in Y_{k_3}.$$

Then $h^{j_3}(y) = x$, so

$$x \in h^{j_3}(Y_{k_3}) \cap h^{j_1}(Y_{k_1}) \qquad \text{and} \qquad x \in h^{j_3}(Y_{k_3}) \cap h^{j_2}(Y_{k_2}).$$

We have

$$j_3 \leq j_1 \qquad \text{and} \qquad n(k_3) - j_3 \leq n(k_1) - j_1,$$

so the cases we have already done give $f_{k_1}^{(j_1)}(x) = f_{k_3}^{(j_3)}(x)$. Therefore it suffices to replace k_1 and j_1 by k_3 and j_3 in the statement to be proved. We have $n(k_3) < n(k_1)$ because $n(t_1) + n(t_2) + \cdots + n(t_r) = n(k_1)$ and $r \geq 2$. Meanwhile, $n(k_2)$ is the same as before. This is the required reduction.

The proof that f is well defined and continuous is now complete. $\square$

We now give a more precise statement of Theorem 17.14. It is the generalization of the isomorphism (16.5) gotten from the proof of Lemma 16.20.

Theorem 17.19. Let $Y \subset X$ be closed with $\operatorname{int}(Y) \neq \varnothing$. Let

$$\gamma \colon C^*(\mathbb{Z}, X, h)_Y \to \bigoplus_{k=0}^{l} C(Y_k, M_{n(k)})$$

be the homomorphism of Proposition 17.9. Then γ induces an isomorphism of $C^*(\mathbb{Z}, X, h)_Y$ with the recursive subhomogeneous algebra defined, in the notation of Remark 17.11 and Definition 17.12, as follows.

(1) l and $n(0)$, $n(1)$, $\ldots$, $n(l)$ are as in Definition 17.4.
(2) $X_k = Y_k$ for $k = 1, 2, \ldots, l$.
(3) $X_k^{(0)} = Y_k \cap \bigcup_{j=0}^{k-1} Y_j$ for $k = 1, 2, \ldots, l$.
(4) For $k = 0, 1, \ldots, l$, $x \in Y_k \cap \bigcup_{j=0}^{k-1} Y_j$ and $(b_0, b_1, \ldots, b_{k-1})$ in the image in $\bigoplus_{j=0}^{k-1} C(Y_j, M_{n(j)})$ of the $k-1$ stage algebra $R^{(k-1)}$, whenever

$$x \in (Y_k \setminus Y_k^{\bullet}) \cap Y_{t_1} \cap h^{-n(t_1)}(Y_{t_2}) \cap \cdots \cap h^{-[n(t_1)+n(t_2)+\cdots+n(t_{r-1})]}(Y_{t_r}),$$

with

$$n(t_1) + n(t_2) + \cdots + n(t_r) = n(k),$$

then

$$\varphi_k(b_0, b_1, \ldots, b_{k-1})(x) = \begin{pmatrix} b_{t_1}(x) & & & & \\ & b_{t_2}(h^{n(t_1)}(x)) & & & \\ & & b_{t_3}(h^{n(t_1)+n(t_2)}(x)) & & \\ & & & \ddots & \\ & & & & b_{t_r}(h^{n(t_1)+\cdots+n(t_{r-1})}(x)) \end{pmatrix}.$$

(5) For $k = 0, 1, \ldots, l$, ψ_k is the restriction map.

Moreover, the standard representation of $\gamma(C^*(\mathbb{Z}, X, h)_Y)$ is the inclusion map in $\bigoplus_{k=0}^{l} C(Y_k, M_{n(k)})$.

Proof. The main point is to show that the formula in (4) actually gives a well defined homomorphism

$$\varphi_k \colon R^{(k-1)} \to C\big(Y_k^{(0)}, M_{n(k)}\big).$$

We do this by induction on k. Once it is known that $\varphi_1, \varphi_2, \ldots, \varphi_{k-1}$ are well defined, it follows that $R^{(k-1)}$ as described is a recursive subhomogeneous algebra, and that its elements are exactly those sequences $(b_0, b_1, \ldots, b_{k-1})$ which satisfy the conditions of Lemma 17.18 up to $l = k-1$ (that is, the number k there is at most the number $k-1$ here). Having $R^{(k-1)}$, it makes sense to consider a homomorphism $\varphi_k \colon R^{(k-1)} \to C\big(Y_k^{(0)}, M_{n(k)}\big)$. If the one described in (4) is well defined, it will follow that $R^{(k)}$ is a recursive subhomogeneous algebra, and that its elements are exactly those sequences $(b_0, b_1, \ldots, b_{k-1})$ which satisfy the conditions of Lemma 17.18 up to $l = k$.

We start by showing that φ_1 is well defined. We have $Y_1^{(0)} = Y_1 \cap Y_0$. For $x \in Y_1^{(0)}$, let $0 = \nu_0, \nu_1, \ldots, \nu_r = n(1)$ be the successive return times of x to Y. We have $\nu_1 = n(0) < n(1)$, so $r \geq 2$. For $i = 0, 1, \ldots, r-1$, the first return time of $h^{\nu_i}(x)$ is strictly less than $n(1)$, so can only be $n(0)$. Therefore $n(1) = rn(0)$. Thus, if $Y_1^{(0)} \neq \varnothing$, then $n(1) = rn(0)$ and

$$Y_1^{(0)} = Y_1 \cap Y_0 \cap h^{-n(0)}(Y_0) \cap h^{-2n(0)}(Y_0) \cap \cdots \cap h^{-[n(1)-n(0)]}(Y_0).$$

If $Y_1^{(0)} = \varnothing$ then φ_1 is trivially well defined, and if $Y_1^{(0)} \neq \varnothing$, then φ_1 is well defined by the formula

$$\varphi_1(b) = \operatorname{diag}\big(b|_{Y_1^{(0)}},\, b \circ h^{n(0)}|_{Y_1^{(0)}},\, \ldots,\, b \circ h^{n(1)-n(0)}|_{Y_1^{(0)}}\big).$$

Now assume we have $R^{(k-1)}$. Let S be the set of all sequences $(t_1, t_2, \ldots, t_r)$ such that $n(t_1) + n(t_2) + \cdots + n(t_r) = n(k)$, with $r \geq 2$. In such a sequence, we have $t_i < k$ for $i = 1, 2, \ldots, r$. For $\sigma = (t_1, t_2, \ldots, t_r) \in S$, define

$$Y_k^{(\sigma)} = (Y_k \setminus Y_k^{\bullet}) \cap Y_{t_1} \cap h^{-n(t_1)}(Y_{t_2}) \cap \cdots \cap h^{-[n(t_1)+n(t_2)+\cdots+n(t_{r-1})]}(Y_{t_r}).$$

(Note that the intersection is the same if one uses Y_k in place of $Y_k \setminus Y_k^{\bullet}$.) By considering successive return times as in the initial step of the induction, one checks that

$$Y_k^{(0)} = \bigcup_{\sigma \in S} Y_k^{(\sigma)}.$$

Showing that φ_k is well defined is therefore equivalent to showing that if $\sigma\tau \in S$ and $x \in Y_k^{(\sigma)} \cap Y_k^{(\tau)}$, then the corresponding two formulas in (4) agree at x. For $b \in R^{(k-1)}$, call these expressions $\varphi_k^{(\sigma)}(b)(x)$ and $\varphi_k^{(\tau)}(b)(x)$.

Given $\tau = (t_1, t_2, \ldots, t_\nu) \in S$, define

$$R(\tau) = \{0,\, n(t_1),\, n(t_1) + n(t_2),\, \ldots, n(t_1) + \cdots + n(t_{\nu-1}),\, n(k)\},$$

the set of return times associated with τ. Let $\sigma, \tau \in S$, and let $x \in Y_k^{(\sigma)} \cap Y_k^{(\tau)}$. Let $\rho = (r_1, r_2, \ldots, r_\nu) \in S$ be the sequence using all return times of x. That is, $n(r_1)$ is the first return time of x, $n(r_2)$ is the first return time of $h^{n(r_1)}(x)$, etc. Then $x \in Y_0^{(\rho)}$ and $R(\rho)$ contains both $R(\sigma)$ and $R(\tau)$. It therefore suffices to prove agreement of the two formulas when $x \in Y_k^{(\sigma)} \cap Y_k^{(\tau)}$ and $R(\sigma) \subset R(\tau)$.

Assuming this, write $\tau = (t_1, t_2, \ldots, t_\nu)$ and

$$R(\sigma) = \{0,\, n(t_1) + \cdots + n(t_{j(1)}),\, n(t_1) + \cdots + n(t_{j(2)}),\, \ldots,\, n(t_1) + \cdots + n(t_{j(\mu)})\},$$

with

$$j(1) < j(2) < \cdots < j(\mu) \qquad \text{and} \qquad n(t_1) + n(t_2) + \cdots + n(t_{j(\mu)}) = n(k).$$

Then $\sigma = (s_1, s_2, \ldots, s_\mu)$, with

$$n(s_i) = n(t_{j(i-1)+1}) + n(t_{j(i-1)+2}) + \cdots + n(t_{j(i)})$$

for $i = 1, 2, \ldots, \mu$. Now $\varphi_k^{(\sigma)}(b)(x)$ is a block diagonal matrix, with blocks

$$b_{s_1}(x),\ b_{s_2}(h^{n(s_1)}(x)),\ b_{s_3}(h^{n(s_1)+n(s_2)}(x)),\ \ldots,\ b_{s_\mu}(h^{n(s_1)+n(s_2)|\cdots+n(s_{\mu-1})}(x)).$$

The induction hypothesis implies that

$$b_{s_1}(y) = \begin{pmatrix} b_{t_1}(y) & & & \\ & b_{t_2}(h^{n(t_1)}(y)) & & \\ & & \ddots & \\ & & & b_{t_{j(1)}}(h^{n(t_1)+\cdots+n(t_{j(1)-1})}(y)) \end{pmatrix}$$

for

$$y \in (Y_{s_1} \setminus Y_{s_1}^{\bullet}) \cap Y_{t_1} \cap h^{-n(t_1)}(Y_{t_2}) \cap \cdots \cap h^{-[n(t_1)+n(t_2)+\cdots+n(t_{j(1)-1})]}(Y_{t_{j(1)}}),$$

that

$$b_{s_2}(y) = \begin{pmatrix} b_{t_{j(1)+1}}(y) & & & \\ & b_{t_{j(1)+2}}(h^{n(t_{j(1)+1})}(y)) & & \\ & & \ddots & \\ & & & b_{t_{j(2)}}(h^{n(t_{j(1)+1})+\cdots+n(t_{j(2)-1})}(y)) \end{pmatrix}$$

for

$$\begin{aligned} y \in (Y_{s_2} \setminus Y_{s_2}^{\bullet}) \cap Y_{t_{j(1)+1}} \cap h^{-n(t_{j(1)+1})}(Y_{t_{j(1)+2}}) & \\ \cap \cdots \cap h^{-[n(t_{j(1)+1})+n(t_{j(1)+2})+\cdots+n(t_{j(2)-1})]}(Y_{t_{j(2)}}), & \end{aligned}$$

etc. Taking $y = x$ in the first,

$$y = h^{n(s_1)}(x) = h^{n(t_1)+\cdots+n(t_{j(1)})}(x)$$

in the second of these,

$$y = h^{n(s_1)+n(s_2)}(x) = h^{n(t_1)+\cdots+n(t_{j(2)})}(x)$$

in the third, etc., we get $\varphi_k^{(\sigma)}(b)(x) = \varphi_k^{(\tau)}(b)(x)$, as desired. This completes the induction, and the proof. $\square$

Proof of Theorem 17.14. The only part of the statement of Theorem 17.14 which is not in Theorem 17.19 is the statement that $C^*(\mathbb{Z}, X, h)_Y$ has topological dimension equal to $\dim(X)$. That the topological dimension is at most $\dim(X)$ follows from Theorem 17.19 and Proposition 16.10. That it is at least $\dim(X)$ follows from Theorem 17.19, Proposition 16.11, and Lemma 17.5(2). $\square$

The subalgebras we really want are of the form $C^*(\mathbb{Z}, X, h)_{\{y\}}$ for suitable $y \in X$, not $C^*(\mathbb{Z}, X, h)_Y$ with $\mathrm{int}(Y) \neq \varnothing$.

Remark 17.20. Let X be an infinite compact Hausdorff space and let $h \colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be closed. The case of immediate interest is $Y = \{y\}$ for some $y \in X$, but other choices are important, as for example in the discussion after Proposition 23.15. We can choose a decreasing sequence $Y_1 \supset Y_2 \supset \cdots$ of closed subsets of X with nonempty interiors such that

$$\bigcap_{n=1}^{\infty} Y_n = Y. \tag{17.14}$$

Then

$$C^*(\mathbb{Z}, X, h)_{Y_1} \subset C^*(\mathbb{Z}, X, h)_{Y_2} \subset \cdots$$

and

$$\overline{\bigcup_{n=1}^{\infty} C^*(\mathbb{Z}, X, h)_{Y_n}} = C^*(\mathbb{Z}, X, h)_Y.$$

That is,

$$C^*(\mathbb{Z}, X, h)_Y = \varinjlim_n C^*(\mathbb{Z}, X, h)_{Y_n}. \tag{17.15}$$

When $\dim(X) < \infty$, by Theorem 17.14 we have expressed $C^*(\mathbb{Z}, X, h)_Y$ as the direct limit of a direct system of recursive subhomogeneous algebras which has no dimension growth, in the sense of Corollary 1.9 of [204].

Moreover, we have the following result.

Proposition 17.21 (Proposition 2.5 of [157])**.** *Let X be an infinite compact Hausdorff space and let $h\colon X \to X$ be a minimal homeomorphism. Let $y \in X$. Then $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is infinite dimensional and simple.*

Infinite dimensionality is obvious. We refer to [157] for the proof of simplicity. We point out that a generalization of this result follows from Proposition 20.7, Proposition 20.10, and Theorem 19.5, whose proofs we give sketches of below.

Direct limits of direct systems of recursive subhomogeneous algebras with no dimension growth have a number of good properties, originally developed in [204] and [205]. By now, it is known that all such algebras are classifiable in the sense of the Elliott classification program. Here, we want to use the density of the range of the map

$$K_0\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big) \to \operatorname{Aff}\big(\mathrm{T}\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big)\big) \tag{17.16}$$

to conclude that $C^*(\mathbb{Z}, X, h)_{\{y\}}$ has tracial rank zero (Definition 11.35). More generally, if D is a UHF algebra, we want the map

$$K_0\big(D \otimes C^*(\mathbb{Z}, X, h)_{\{y\}}\big) \to \operatorname{Aff}\big(\mathrm{T}\big(D \otimes C^*(\mathbb{Z}, X, h)_{\{y\}}\big)\big) \tag{17.17}$$

to have dense range. Our hypotheses state that

$$K_0(C^*(\mathbb{Z}, X, h)) \to \operatorname{Aff}\big(\mathrm{T}(C^*(\mathbb{Z}, X, h))\big) \tag{17.18}$$

has dense range, or that

$$K_0\big(D \otimes C^*(\mathbb{Z}, X, h)\big) \to \operatorname{Aff}\big(\mathrm{T}\big(D \otimes C^*(\mathbb{Z}, X, h)\big)\big) \tag{17.19}$$

has dense range.

The following results take care of the differences. We need to deal with both tracial states and K-theory. We state the results, and discuss the proofs of the main ingredients afterwards.

Lemma 17.22 (Proposition 2.5 of [157])**.** *Let X be an infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism. Let $y \in X$. Then the restriction map $\mathrm{T}(C^*(\mathbb{Z}, X, h)) \to \mathrm{T}\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big)$ is a bijection and an affine homeomorphism.*

Corollary 17.23. *Let X be an infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism. Let $y \in X$. Let D is a UHF algebra. Then the restriction map*

$$\mathrm{T}(D \otimes C^*(\mathbb{Z}, X, h)) \to \mathrm{T}\big(D \otimes C^*(\mathbb{Z}, X, h)_{\{y\}}\big)$$

is a bijection and an affine homeomorphism.

The proof of Corollary 17.23 needs the following well known result.

Exercise 17.24. Let A be a unital C*-algebra, and let D be a unital C*-algebra with a unique tracial state. Then there is an affine homeomorphism

$$R\colon \mathrm{T}(D \otimes_{\mathrm{min}} A) \to \mathrm{T}(A)$$

such that for $\tau \in \mathrm{T}(D \otimes_{\mathrm{min}} A)$, the tracial state $R(\tau)$ is determined by $R(\tau)(a) = \tau(1 \otimes a)$ for $a \in A$.

Proof of Corollary 17.23. The result is immediate from Lemma 17.22 and Exercise 17.24. □

Theorem 17.25 (Theorem 4.1(3) of [204])**.** Let X be an infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism. Let $y \in X$. Then the inclusion map $\iota\colon C^*(\mathbb{Z}, X, h)_{\{y\}} \to C^*(\mathbb{Z}, X, h)$ induces an isomorphism

$$\iota_*\colon K_0\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big) \to K_0\big(C^*(\mathbb{Z}, X, h)\big).$$

We won't use this fact, but it is also true that

$$\iota_*\colon K_1\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big) \to K_1\big(C^*(\mathbb{Z}, X, h)\big)$$

is injective, with cokernel isomorphic to $\mathbb{Z}$, generated by the image in the cokernel of the K_1-class of the standard unitary u in $C^*(\mathbb{Z}, X, h)$. See Theorem 4.1(4) of [204].

Corollary 17.26. Let X be an infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism. Let $y \in X$. Let D is a UHF algebra. Then the inclusion map

$$\mathrm{id}_D \otimes \iota\colon D \otimes C^*(\mathbb{Z}, X, h)_{\{y\}} \to D \otimes C^*(\mathbb{Z}, X, h)$$

induces an isomorphism

$$(\mathrm{id}_D \otimes \iota)_*\colon K_0\big(D \otimes C^*(\mathbb{Z}, X, h)_{\{y\}}\big) \to K_0\big(D \otimes C^*(\mathbb{Z}, X, h)\big).$$

Proof. This result follows from the Künneth formula [251] and Theorem 17.25. □

One doesn't actually need the Künneth formula. If $D = \varinjlim_n M_{d(n)}$ with $d(1)|d(2)|\cdots$, then $\mathrm{id}_D \otimes \iota$ is the direct limit of the maps

$$\mathrm{id}_{M_{d(n)}} \otimes \iota\colon M_{d(n)} \otimes C^*(\mathbb{Z}, X, h)_{\{y\}} \to M_{d(n)} \otimes C^*(\mathbb{Z}, X, h),$$

which are all isomorphisms on K_0.

Corollary 17.27. Let X be an infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism. Let $y \in X$. Let D be $\mathbb{C}$ or a UHF algebra. Suppose that $\rho\big(K_0(D \otimes C^*(\mathbb{Z}, X, h))\big)$ is dense in $\mathrm{Aff}\big(\mathrm{T}(D \otimes C^*(\mathbb{Z}, X, h))\big)$. Then $\rho\big(K_0(D \otimes C^*(\mathbb{Z}, X, h)_{\{y\}})\big)$ is dense in $\mathrm{Aff}\big(\mathrm{T}(D \otimes C^*(\mathbb{Z}, X, h)_{\{y\}})\big)$.

Proof. If $D = \mathbb{C}$, combine Lemma 17.22 and Theorem 17.25. If D is a UHF algebra, combine Corollary 17.23 and Corollary 17.26. □

Lemma 17.22 is originally due to Qing Lin (via the closely related Proposition 16 of [159]), and its proof is sketched in the proof of Theorem 1.2 of [160]. We give the full proof here. We also point out that a more general result follows from Theorem 20.12 and Theorem 19.5, whose proofs we give sketches of below. That route directly uses the properties of $C^*(\mathbb{Z}, X, h)_{\{y\}}$ as a subalgebra of $C^*(\mathbb{Z}, X, h)$, but the proof we give here instead compares traces on both algebras to the set of invariant Borel probability measures on X.

The following lemma is a special case of Theorem 15.22.

Lemma 17.28. Let X be an infinite compact metric space, and let $h \colon X \to X$ be a minimal homeomorphism. Then the restriction map $\mathrm{T}(C^*(\mathbb{Z}, X, h)) \to \mathrm{T}(C(X))$ is a bijection from $\mathrm{T}(C^*(\mathbb{Z}, X, h))$ to the set of h-invariant Borel probability measures on X.

We need the analogous result for $C^*(\mathbb{Z}, X, h)_{\{y\}}$.

Lemma 17.29 (Proposition 16 of [159])**.** Let X be an infinite compact metric space, and let $h \colon X \to X$ be a minimal homeomorphism. Let $y \in X$. Then the restriction map $\mathrm{T}\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big) \to \mathrm{T}(C(X))$ is a bijection from $\mathrm{T}\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big)$ to the set of h-invariant Borel probability measures on X.

The proof is simpler than the original because we use Proposition 17.6.

Proof of Lemma 17.29. Applying Proposition 17.28 and restricting from $C^*(\mathbb{Z}, X, h)$ to $C^*(\mathbb{Z}, X, h)_{\{y\}}$, we see that every h-invariant Borel probability measure on X gives a tracial state on $C^*(\mathbb{Z}, X, h)_{\{y\}}$.

Now let τ be any tracial state on $C^*(\mathbb{Z}, X, h)_{\{y\}}$. Let μ be the Borel probability measure on X determined by $\tau(f) = \int_X f \, d\mu$ for $f \in C(X)$. The rest of the proof has two steps. The first step is to show that μ is h-invariant. Then Lemma 17.29 provides a tracial state τ_μ on $C^*(\mathbb{Z}, X, h)$. The formula, from Example 11.31, is

$$\tau_\mu\left(\sum_{n=-N}^{N} f_n u^n\right) = \int_X f_0 \, d\mu$$

for $N \in \mathbb{Z}_{>0}$ and $f_{_N}, f_{-N+1}, \ldots, f_N \in C(X)$. The second step of the proof is to show that $\tau_\mu|_{C^*(\mathbb{Z},X,h)_{\{y\}}} = \tau$.

For the first step, we show that $\int_X (f \circ h) \, d\mu = \int_X f \, d\mu$ for every $f \in C(X)$. This is clearly true for constant functions f. Therefore, it suffices to consider functions f such that $f(y) = 0$. For such a function f, write $f = f_1^* f_2$ with $f_1, f_2 \in C(X)$ such that $f_1(y) = f_2(y) = 0$. (For example, take $f_1 = |f|^{1/2}$ and $f_2 = (\mathrm{sgn} \circ f)|f|^{1/2}$.) Then $f_1 u, \, f_2 u \in C^*(\mathbb{Z}, X, h)_{\{y\}}$. So

$$f \circ h = u^* f u = (f_1 u)^* (f_2 u) \in C^*(\mathbb{Z}, X, h)_{\{y\}}.$$

We now use the trace property at the second step to get

$$\int_X (f \circ h) \, d\mu = \tau\big((f_1 u)^* (f_2 u)\big) = \tau\big((f_2 u)(f_1 u)^*\big) = \tau(f) = \int_X f \, d\mu.$$

Thus μ is h-invariant.

For the second step, it follows from Proposition 17.6 that $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is the closed linear span of all elements of the form $f u^n$, with $f \in C(X)$ and $n \in \mathbb{Z}$, which actually happen to be in $C^*(\mathbb{Z}, X, h)_{\{y\}}$. So it suffices to prove that if $f u^n \in C^*(\mathbb{Z}, X, h)_{\{y\}}$ and $n \neq 0$, then $\tau(f u^n) = 0$. Since h^n has no fixed points, there is an open cover of X consisting of sets U such that $h^n(U) \cap U = \varnothing$. Choose

$$g_1, g_2, \ldots, g_m \in C(X) \subset C^*(\mathbb{Z}, X, h)_{\{y\}}$$

which form a partition of unity subordinate to this cover. In particular, the supports of g_j and $g_j \circ h^{-n}$ are disjoint for all j. For $j = 1, 2, \ldots, m$ we have, using the trace property at the first step and the relation $u^n g u^{-n} = g \circ h^{-n}$ for any $g \in C(X)$ at the second step,

$$\tau(g_j f u^n) = \tau\big(g_j^{1/2} f u^n g_j^{1/2}\big) = \tau\big(g_j^{1/2} f \big(g_j^{1/2} \circ h^{-n}\big) u^n\big) = \tau(0) = 0.$$

Summing over j gives $\tau(fu^n) = 0$. □

Proof of Lemma 17.22. Let M be the set of h-invariant Borel probability measures on X. Lemma 17.28 shows that the restriction map $\mathrm{T}(C^*(\mathbb{Z}, X, h)) \to M$ is a bijection. Lemma 17.29 shows that the restriction map $\mathrm{T}\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big) \to M$ is a bijection. So the restriction map $\mathrm{T}(C^*(\mathbb{Z}, X, h)) \to \mathrm{T}\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big)$ is a bijection. The restriction map is clearly affine and continuous. Since its domain and codomain are compact Hausdorff, it is a homeomorphism. □

We now say something about the proof of Theorem 17.25. The usual proof uses Theorem 2.4 of [232], which relates the K-theory of the C*-algebra of a groupoid to the K-theory of the C*-algebra of a particular kind of subgroupoid, and is based on KK-theory computations in [231]. Example 2.6 of [232] contains the application to the K-theory of $C^*(\mathbb{Z}, X, h)_{\{y\}}$. Groupoids enter because of the interpretation of $C^*(\mathbb{Z}, X, h)$ as the C*-algebra of a transformation group groupoid (called $\mathbb{Z} \ltimes X$) and of $C^*(\mathbb{Z}, X, h)_Y$ (for an arbitrary closed subset $Y \subset X$) as the C*-algebra of an open subgroupoid of $\mathbb{Z} \ltimes X$. This interpretation is briefly outlined in Remark 16.19. The philosophy is that $\mathbb{Z} \ltimes X$ has many more (open) subgroupoids than $(\mathbb{Z}, X, h)$ has subobjects in the category of dynamical systems. The groupoid picture is not needed for the rest of what we do here, because of the concrete description of $C^*(\mathbb{Z}, X, h)_Y$, but it is needed for the generalization of the construction to actions of $\mathbb{Z}^d$. Unfortunately, we will not be able to discuss the relevant construction in these notes. See [202] for the special case in which X is the Cantor set.

We outline (without proofs) an alternate approach to the proof of Theorem 17.25, using partial actions. It is based on discussions with Ruy Exel. The partial action approach to this problem seems closely related to the subgroupoid approach. It is known, but, as far as we know, has not appeared in the literature. We presume it also generalizes to actions by groups other than $\mathbb{Z}$. For $\mathbb{Z}$, this approach avoids [231] and puts the K-theory computations in a somewhat more familiar context, namely a generalization (Theorem 17.31 below) of the Pimsner-Voiculescu exact sequence for crossed products by $\mathbb{Z}$ [221] to crossed products by partial actions.

We will follow [82] until we get to the point where K-theory appears, but we do not reproduce the definitions and statements of most of the theorems.

Let X be a compact Hausdorff space, let $h\colon X \to X$ be a homeomorphism, and let $Y \subset X$ be closed. We start with the topological partial action of $\mathbb{Z}$ on X obtained from the restriction and corestriction of h to a homeomorphism from $X \setminus Y$ to $X \setminus h(Y)$. Topological partial actions are defined in Definition 5.1 of [82], referring back to Definition 2.1 of [82] for partial actions on sets. Following the notation of [82], we define open subsets $D_n \subset X$ by, for $n \in \mathbb{Z}_{\geq 0}$,

$$D_n = X \setminus \big[Y \cup h(Y) \cup \cdots \cup h^{n-1}(Y)\big]$$

and

$$D_{-n} = X \setminus \big[h^{-n}(Y) \cup h^{-n+1}(Y) \cup \cdots \cup h^{-1}(Y)\big].$$

As a sign of what is to come, we point out that, with Z_n as in Proposition 17.6, we have $D_n = X \setminus Z_n$ for all $n \in \mathbb{Z}$. We further take $\theta_n\colon D_{-n} \to D_n$ to be the restriction and corestriction of h^n to D_{-n} and D_n for $n \in \mathbb{Z}$. One easily checks that $\big((D_n)_{n\in\mathbb{Z}},\, (\theta_n)_{n\in\mathbb{Z}}\big)$ is in fact a topological partial action of $\mathbb{Z}$ on X.

This partial action gives a C* partial action of $\mathbb{Z}$ on $C(X)$. Definition 6.4 of [82] gives the conditions for a partial action on an algebra, Definition 11.4 of [82] gives the additional conditions for a C* partial action, and the fact that a topological partial action on a locally compact Hausdorff space gives a C* partial action is Corollary 11.6 of [82].

Now form the algebraic crossed product by this partial action, as in Definition 8.3 of [82], and complete it to a C*-algebra as in Definition 11.11 of [82]. Call this C*-algebra $C^*(\mathbb{Z}, X, \theta)$. (In [82], the notation $C(X) \rtimes_{\theta} \mathbb{Z}$ is used.)

Lemma 17.30. Let X be a compact Hausdorff space, let $h \colon X \to X$ be a homeomorphism, and let $Y \subset X$ be closed. Let $\big((D_n)_{n \in \mathbb{Z}}, \, (\theta_n)_{n \in \mathbb{Z}} \big)$ and $C^*(\mathbb{Z}, X, \theta)$ be as in the discussion above. Let $\pi \colon C(X) \to C^*(\mathbb{Z}, X, h)$ be the standard inclusion of $C(X)$ in the ordinary C* crossed product (see the discussion after Remark 8.12), and let $n \mapsto u_n$ be the map from $\mathbb{Z}$ to the unitary group of $C^*(\mathbb{Z}, X, h)$ (Notation 8.7). Then (π, u) is a covariant representation of $\big((D_n)_{n \in \mathbb{Z}}, \, (\theta_n)_{n \in \mathbb{Z}} \big)$ in $C^*(\mathbb{Z}, X, h)$ in the sense of Definition 9.10 of [82], and the associated homomorphism $\gamma \colon C^*(\mathbb{Z}, X, \theta) \to C^*(\mathbb{Z}, X, h)$ of Proposition 13.1 of [82] is injective and has range $C^*(\mathbb{Z}, X, h)_Y$.

Proof. The proof that (π, u) is a covariant representation is immediate.

Let B be the algebraic partial crossed product of $C(X)$ by the partial action $\big((D_n)_{n \in \mathbb{Z}}, \, (\theta_n)_{n \in \mathbb{Z}} \big)$ (see Definition 8.3 of [82]). Following the notation there, write its elements as formal sums $\sum_{n \in \mathbb{Z}} a_n \delta_n$, with $a_n \in D_n$ for all $n \in \mathbb{Z}$ and $a_n = 0$ for all but finitely many $n \in \mathbb{Z}$. By construction (see Definition 11.11 of [82]), B is dense in $C^*(\mathbb{Z}, X, \theta)$. One checks, again directly from its definition and the definition of the homomorphism determined by a covariant representation, that the image in $C^*(\mathbb{Z}, X, h)$ under γ of B is, in the notation in the statement of Proposition 17.6, exactly

$$\big\{ a \in C(X)[\mathbb{Z}] \colon E(a u^{-n}) \in C_0(X \setminus Z_n) \text{ for all } n \in \mathbb{Z} \big\}.$$

So Proposition 17.6 implies that the range of γ is $C^*(\mathbb{Z}, X, h)_Y$.

It remain to prove that γ is injective. We construct a dual action of S^1 on $C^*(\mathbb{Z}, X, \theta)$. This is known, and works for any partial crossed product by $\mathbb{Z}$. For $\zeta \in S^1$, one checks that there is a covariant representation (σ, v) of $\big((D_n)_{n \in \mathbb{Z}}, \, (\theta_n)_{n \in \mathbb{Z}} \big)$ in $C^*(\mathbb{Z}, X, \theta)$ such that $\sigma(a) = a \delta_0$ for $a \in C(X)$ and $v_n = \zeta^{-n} \delta_n$. (This is essentially the same formula as that of Remark 9.25 for the dual action of S^1 on a crossed product by $\mathbb{Z}$.) Let $\beta_{\zeta} \colon C^*(\mathbb{Z}, X, \theta) \to C^*(\mathbb{Z}, X, \theta)$ be the corresponding homomorphism (Proposition 13.1 of [82]). Then $\beta_1 = \mathrm{id}_{C^*(\mathbb{Z}, X, \theta)}$ (this is clear), and $\beta_{\zeta_1} \circ \beta_{\zeta_2} = \beta_{\zeta_1 \zeta_2}$ for $\zeta, \zeta_2 \in S^1$ (this is easily checked by looking at what they do to elements of B). Therefore β_{ζ} is an automorphism for $\zeta \in S^1$ and $\zeta \mapsto \beta_{\zeta}$ is an action of S^1 on $C^*(\mathbb{Z}, X, \theta)$. Using Lemma 3.14, in the same way as in Example 3.15, one checks that this action is continuous.

It is clear that $\gamma \colon C^*(\mathbb{Z}, X, \theta) \to C^*(\mathbb{Z}, X, h)$ is equivariant when $C^*(\mathbb{Z}, X, \theta)$ is equipped with the action β and $C^*(\mathbb{Z}, X, h)$ is equipped with the dual action as in Remark 9.25. The fixed point algebras of both actions are easily checked to be the standard copies of $C(X)$, and thus the restriction of γ to the fixed point algebra $C^*(\mathbb{Z}, X, \theta)^{\beta}$ is injective. So γ is injective by Proposition 2.9 of [81]. □

The dual action argument in the proof of Lemma 17.30 can be replaced, with appropriate preparation, by Theorem 19.1(c) of [82], which applies to much more general situations.

The following result is a generalization of the Pimsner-Voiculescu exact sequence for the K-theory of crossed products by $\mathbb{Z}$ [221].

Theorem 17.31 (Theorem 7.1 of [81])**.** Let $\big((D_n)_{n \in \mathbb{Z}}, \, (\theta_n)_{n \in \mathbb{Z}} \big)$ be a partial action of $\mathbb{Z}$ on a C*-algebra A. Then there is a natural six term exact sequence

$$\begin{array}{ccccc} K_0(D_{-1}) & \longrightarrow & K_0(A) & \xrightarrow{\iota_*} & K_0(C^*(\mathbb{Z}, A, \alpha)) \\ \uparrow & & & & \downarrow \\ K_1(C^*(\mathbb{Z}, A, \alpha)) & \xleftarrow[\iota_*]{} & K_1(A) & \longleftarrow & K_1(D_{-1}). \end{array}$$

By naturality one gets a commutative diagram with exact rows, in which the bottom row is the usual Pimsner-Voiculescu exact sequence [221]:

$$\begin{array}{ccccccccc} \longrightarrow & K^0(X \setminus Y) & \longrightarrow & K^0(X) & \longrightarrow & K_0\big(C^*(\mathbb{Z}, X, h)_Y\big) & \longrightarrow & K^1(X \setminus Y) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & K^0(X) & \longrightarrow & K^0(X) & \longrightarrow & K_0\big(C^*(\mathbb{Z}, X, h)\big) & \longrightarrow & K^1(X) & \longrightarrow . \end{array}$$

Taking Y to be a one point set, it is now not hard to derive Theorem 17.25 and the corresponding result for $K_1\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big)$. We omit the details, but point out that the Five Lemma is not quite enough. One needs to show that the horizontal maps $K^0(X \setminus Y) \to K^0(X)$ and $K^0(X) \to K^0(X)$ shown have the same range, and that the vertical map $K^1(X \setminus Y) \to K^1(X)$ is an isomorphism.

We return to the description of steps in the proof of Theorem 16.1. The projections used in the proof when X is the Cantor set (the main part of the proof being Lemma 16.24) are in $C(X)$, and were gotten from Lemma 16.23. When X is connected, there are no nontrivial projections in $C(X)$, and a different approach is required. The following is a generalization of Lemma 16.23. It is both more elementary and more general than the corresponding argument in [157] (the main part of the proof of Theorem 4.5 there).

Lemma 17.32. Let X be an infinite compact metric space, and let $h \colon X \to X$ be a minimal homeomorphism. Let $B \subset C^*(\mathbb{Z}, X, h)$ be a unital subalgebra which contains $C(X)$ and has property (SP). Let $c \in C^*(\mathbb{Z}, X, h)$ be a nonzero positive element. Then there exists a nonzero projection $p \in B$ such that p is Murray-von Neumann equivalent in $C^*(\mathbb{Z}, X, h)$ to a projection in $\overline{c C^*(\mathbb{Z}, X, h) c}$.

Proof. Let $E \colon C^*(\mathbb{Z}, X, h) \to C(X)$ be the standard conditional expectation (Definition 9.18). It follows from Proposition 9.16(4) and Exercise 9.17(3) that $E(c)$ is a nonzero positive element of $C(X)$. Set $\delta = \frac{1}{7} \| E(c) \|$. Let $f \in C(X)$ be the pointwise minimum $f(x) = \min \big(6 \delta, \, E(c)(x) \big)$. Then

$$\| f - E(c) \| = \delta. \tag{17.20}$$

Set

$$U_0 = \{ x \in X \colon E(c)(x) > 6 \delta \},$$

which is a nonempty open set such that $f(x) = 6 \delta$ for all $x \in U_0$.

Choose a finite sum $a = \sum_{n = -N}^{N} a_n u^n \in C^*(\mathbb{Z}, X, h)$ with $a_n \in C(X)$ for $n = -N, \, -N + 1, \, \ldots, \, N$ such that $\| a - c \| < \delta$. Let ρ be the metric on X. Choose $\varepsilon > 0$ so small that whenever $x_1, x_2 \in X$ satisfy $\rho(x_1, x_2) < \varepsilon$, then

$$|a_n(x_1) - a_n(x_2)| < \frac{\delta}{2N + 1}$$

for $n = -N, \, -N + 1, \, \ldots, \, N$. Using freeness of the action of $\mathbb{Z}$ induced by h, choose open subsets $U, V \subset X$ such that $U \neq \varnothing$, such that $U \subset \overline{U} \subset V \subset \overline{V} \subset U_0$, such that the sets $h^{-N}(V), \, h^{-N+1}(V), \, \ldots, \, h^{N}(V)$ are disjoint, and such that $\overline{V}$ has diameter less than ε. Then there are $b_{-N}, \, b_{-N+1}, \, \ldots, \, b_N \in C(X)$ such that b_n is constant on $\overline{U}$ and

$$\|b_n - a_n\| < \frac{\delta}{2N+1}$$

for $n = -N, \, -N+1, \, \ldots, \, N$. Set $b = \sum_{n=-N}^{N} b_n u^n$. Then $\|b - a\| < \delta$. So

$$\|b - c\| < 2\delta. \tag{17.21}$$

Choose continuous functions $g_0, g_1 \colon X \to [0,1]$ such that

$$\operatorname{supp}(g_0) \subset U, \qquad g_0 g_1 = g_1, \qquad \text{and} \qquad g_1 \neq 0.$$

Use the hypotheses on B to choose a nonzero projection $p \in \overline{g_1 B g_1}$. We clearly have $g_0 p = p g_0 = p$. It follows that

$$fp = f g_0 p = 6\delta g_0 p = 6\delta p, \tag{17.22}$$

and similarly

$$pf = 6\delta p. \tag{17.23}$$

The same reasoning shows that $p b_n = b_n p$ for $n = -N, \, -N+1, \, \ldots, \, N$. Also, for $n \in \{-N, \, -N+1, \, \ldots, \, N\} \setminus \{0\}$, the disjointness condition implies that $g_0 u^n g_0 = 0$, whence $p u^n p = p g_0 u^n g_0 p = 0$. It follows that

$$pbp = \sum_{n=-N}^{N} p b_n u^n p = \sum_{n=-N}^{N} b_n p u^n p = b_0 p = pE(b)p.$$

Using (17.22) and (17.23) at the first step, this last equation at the second step, and (17.20), (17.21), and Exercise 9.17(4) at the third step, we get

$$\begin{aligned} \|pcp - 6\delta p\| &= \|pcp - pfp\| \\ &\leq \|pcp - pbp\| + \|pE(b)p - pE(c)p\| + \|pE(c)p - pfp\| \\ &\leq 2\|c - b\| + \|E(c) - f\| < 5\delta. \end{aligned}$$

It follows that pcp is invertible in $pC^*(\mathbb{Z}, X, h)p$. Let $d = (pcp)^{-1/2}$, calculated in $pC^*(\mathbb{Z}, X, h)p$. Set $v = dpc^{1/2}$. Then

$$vv^* = dpcpd = (pcp)^{-1/2}(pcp)(pcp)^{-1/2} = p$$

and

$$v^* v = c^{1/2} p d^2 p c^{1/2} \in \overline{cC^*(\mathbb{Z}, X, h)c}.$$

This completes the proof. ☐

For the full statement of Theorem 17.1. (involving the tensor product with a UHF algebra D), one needs a generalization of Lemma 17.32, which we omit.

We can now describe the proof of Theorem 17.1. For simplicity, we omit D, thus really dealing only with Theorem 16.1. The main part is the substitute for Lemma 16.23, but we also refer to the proof of the Cantor set case of Theorem 16.1, given at the end of Section 16.

Combining the hypothesis and Corollary 17.27, for any $y \in X$, $\rho\big(K_0(D \otimes C^*(\mathbb{Z}, X, h)_{\{y\}})\big)$ is dense in $\operatorname{Aff}\big(\mathrm{T}(D \otimes C^*(\mathbb{Z}, X, h)_{\{y\}})\big)$. Since $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is simple and infinite dimensional (Proposition 17.21) and a direct limit of a direct systems of recursive subhomogeneous algebras with no dimension growth, classification results imply it has tracial rank zero (Definition 11.35). (Actually, in [157],

$y \in X$ is chosen with a little care, to allow a direct proof that $C^*(\mathbb{Z}, X, h)_{\{y\}}$ has tracial rank zero.)

This algebra usually isn't AF, so Lemma 16.16 must be replaced as follows, allowing a simple subalgebra with tracial rank zero in place of an AF algebra. The algebra B in the statement takes the place of the AF algebra pBp in Lemma 16.16.

Lemma 17.33 (Lemma 4.4 of [157])**.** Let A be a simple unital C*-algebra. Suppose that for every finite subset $F \subset A$, every $\varepsilon > 0$, and every nonzero positive element $c \in A$, there exists a nonzero projection $p \in A$ and a simple unital subalgebra $B \subset pAp$ with tracial rank zero such that:

(1) $\| [a, p] \| < \varepsilon$ for all $a \in F$.
(2) $\mathrm{dist}(pap, B) < \varepsilon$ for all $a \in F$.
(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{cAc}$.

Then A has tracial rank zero (Definition 11.35).

The word "nonzero" is missing in Lemma 4.4 of [157]. This leads to the same issue as discussed after Definition 11.35, although this condition is not needed if A is already known to be finite.

We must therefore verify the hypotheses of Lemma 17.33. This is Lemma 4.2 of [157]. Following the proof of the Cantor set case of Theorem 16.1, and using the analog of Exercise 16.17, we take the finite set in Lemma 17.33 to be $F_0 \cup \{u\}$ for a finite subset $F_0 \subset C(X)$. The role of the compact open set U used in the proof of the proof of the Cantor set case of Theorem 16.1 will be played by a nonzero projection $r \in C^*(\mathbb{Z}, X, h)_{\{y\}}$ gotten from Lemma 17.32, but getting $1 - p \precsim r$ will require a different argument.

In the replacement for the proof of Lemma 16.23, we will not use $C^*(\mathbb{Z}, X, h)_Y$ (only $C^*(\mathbb{Z}, X, h)_{\{y\}}$), so we choose Y to be a small open set containing y. (This set is called U in the proof of Lemma 4.2 of [157].) To specify how small, we choose N_0 and N as in the proof of Lemma 16.23, and require that conditions (1), (2), and (3) in that proof hold, plus a substitute (see below) for condition (4). (This substitute will depend only on ε and the projection r.)

We don't have anything like χ_Y, so we proceed as follows. Choose continuous functions $g_0, g_1, g_2, f_0 \colon X \to [0, 1]$ such that

$$g_0(y) = 1, \qquad g_1 g_0 = g_0, \qquad g_2 g_1 = g_1, \qquad f_0 g_2 = g_2, \qquad \text{and} \qquad \mathrm{supp}(f_0) \subset Y.$$

Since $C^*(\mathbb{Z}, X, h)_{\{y\}}$ has real rank zero (a consequence of tracial rank zero, by Theorem 11.38), one can find a projection $q_0 \in C^*(\mathbb{Z}, X, h)_{\{y\}}$ such that

$$g_1 q_0 = q_0 g_1 = g_1 \qquad \text{and} \qquad f_0 q_0 = q_0 f_0 = q_0.$$

(See Lemma 4.1 of [157]; the main part of the proof actually comes from Theorem 1 of [34].) We still get orthogonality for the same list of projections as in the proof of Lemma 16.23. The projection q_0 must commute with any function $f \in C(X)$ which is constant on Y, which is good enough for the parts of the proof of Lemma 16.23 involving commutators with and approximation of functions in F_0. The part about commutators with and approximation of u needs little change. Moreover, the relation $q_0 g_1 = g_1$ says, heuristically, that q_0 dominates the characteristic function of a neighborhood of Y, and this relation is in fact good enough to prove that the required cutdowns by the new version of the projection e in the proof of Lemma 16.23 are in fact in $C^*(\mathbb{Z}, X, h)_{\{y\}}$. Also, $pC^*(\mathbb{Z}, X, h)_{\{y\}}p$ has tracial rank zero by Lemma 11.40.

It remains only to show how to arrange to get $1 - p \precsim r$. Define

$$\beta = \inf\left(\left\{\tau(r) \colon \tau \in \mathrm{T}(C^*(\mathbb{Z}, X, h)_{\{y\}})\right\}\right).$$

Then $\beta > 0$ because $\mathrm{T}(C^*(\mathbb{Z}, X, h)_{\{y\}})$ is weak* compact (Remark 16.12), $\tau \mapsto \tau(r)$ is weak* continuous, and $\tau(r)$ can never by zero (Lemma 11.32). Choose $R \in \mathbb{Z}_{>0}$ such that $N/R < \beta$. In place of condition (4) in the proof of Lemma 16.23, we require that the sets

$$Y,\ h(Y),\ h^2(Y),\ \ldots,\ h^R(Y)$$

be disjoint. It follows that for every h-invariant Borel probability measure μ on X, we have $\mu(Y) < 1/R$, so $\int_X f_0\, d\mu < 1/R$. Using Lemma 17.29, we deduce that $\tau(f_0) < 1/R$ for all $\tau \in \mathrm{T}(C^*(\mathbb{Z}, X, h)_{\{y\}})$, so $\tau(q_0) < 1/R$ for all $\tau \in \mathrm{T}(C^*(\mathbb{Z}, X, h)_{\{y\}})$. The projection $1 - p$ is the sum of N projections which are Murray-von Neumann equivalent to q_0 in $C^*(\mathbb{Z}, X, h)$. Every tracial state on $C^*(\mathbb{Z}, X, h)$ therefore takes the same value on all of them. By Lemma 17.22, every tracial state on $C^*(\mathbb{Z}, X, h)_{\{y\}}$ takes the same value on all of them. It follows that for all $\tau \in \mathrm{T}(C^*(\mathbb{Z}, X, h)_{\{y\}})$ we have $\tau(1 - p) < N/R \leq \beta$. By Theorem 11.38, tracial rank zero implies that the order on projections is determined by traces as in Definition 11.34. So $1 - p \precsim r$ in $C^*(\mathbb{Z}, X, h)_{\{y\}}$, and thus also in $C^*(\mathbb{Z}, X, h)$.

Part 5. An Introduction to Large Subalgebras and Applications to Crossed Products

18. The Cuntz Semigroup

In this part, we give an introduction to large subalgebras of C*-algebras and some applications. Much of the text of this part is taken directly from [215], which is a survey of applications of large subalgebras based on lectures given at the University of Wyoming in the summer of 2015. That survey assumes much more background than these notes (it starts with the material here), there are some differences in the organization, and it contains some open problems and other discussion omitted here because they are too far off the topic of these notes.

Large subalgebras are an abstraction of the Putnam subalgebras $C^*(\mathbb{Z}, X, h)_{\{y\}}$ (see Definition 16.18) used in the proof of Theorem 16.1 (and in other places). This abstraction was first introduced in [213]. We give some very brief motivation here, but postpone a more systematic discussion to the beginning of Section 19. The applications discussed in these notes mostly involve $C^*(\mathbb{Z}, X, h)_{\{y\}}$ and $C^*(\mathbb{Z}, X, h)_Y$ for other subsets $Y \subset X$ such that $h^n(Y) \cap Y = \varnothing$ for every $n \in \mathbb{Z} \setminus \{0\}$. However, the real motivation for the abstraction (given very short shrift in these notes) is the construction of analogous subalgebras in C*-algebras such as $C^*(\mathbb{Z}^d, X)$ for a free minimal action of $\mathbb{Z}^d$ on X. Such subalgebras are used in a crucial way in [202] when X is the Cantor set, although with an axiomatization useful only for actions on the Cantor set. In general, there seems to be no useful concrete formula for such subalgebras. Instead, one specifies a list of properties and proves the existence of a subalgebra which has these properties and is otherwise accessible (perhaps being a direct limit of the recursive subhomogeneous C*-algebras of Definition 17.10). The two lists of properties which have been most useful so far make up the definitions of a large subalgebra (Definition 19.1) and of a centrally large subalgebra (Definition 19.2).

The definitions and proofs of the theorems make essential use of Cuntz comparison, and to a lesser extent of the Cuntz semigroup. We therefore begin with a summary of what we need to know about Cuntz comparison and the Cuntz semigroup. The reader is encouraged to just read the introductory discussion and basic definitions, and then skip to Section 19, referring back to this section later as needed. (In particular, there is nothing about dynamics in this section.) We refer to [4] for an extensive introduction (which does not include all the results that we need). The material we need is either summarized or proved in the first two sections of [213].

The Cuntz semigroup can be thought of as being a version of the K_0-group based on positive elements instead of projections. For that reason, we will occasionally make comparisons with K-theory. The reader not familiar with K-theory can ignore these remarks.

For a C*-algebra A, we let $M_\infty(A)$ denote the algebraic direct limit of the system $(M_n(A))_{n=1}^{\infty}$ using the usual embeddings $M_n(A) \to M_{n+1}(A)$, given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

If $a \in M_m(A)$ and $b \in M_n(A)$, we write $a \oplus b$ for the diagonal direct sum

$$a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

By abuse of notation, we will also write $a \oplus b$ when $a, b \in M_\infty(A)$ and we do not care about the precise choice of m and n with $a \in M_m(A)$ and $b \in M_n(A)$.

Parts (1) and (2) of the following definition are originally from [47]. Since we will frequently need to relate Cuntz subequivalence in a C*-algebra B to Cuntz subequivalence in a C*-algebra A containing B, we include (contrary to the usual convention) the algebra A in the notation. The notation $a \sim_A b$ in Definition 18.1(2) conflicts with the notation $p \sim q$ in Notation 11.5, except that in the context of Cuntz subequivalence we include the algebra A as a subscript.

Definition 18.1. Let A be a C*-algebra.

(1) For $a, b \in (K \otimes A)_+$, we say that a is *Cuntz subequivalent to b over A*, written $a \precsim_A b$, if there is a sequence $(v_n)_{n=1}^{\infty}$ in $K \otimes A$ such that $\lim_{n\to\infty} v_n b v_n^* = a$. This relation is transitive: $a \precsim_A b$ and $b \precsim_A c$ imply $a \precsim_A c$.

(2) We say that a and b are *Cuntz equivalent over A*, written $a \sim_A b$, if $a \precsim_A b$ and $b \precsim_A a$. This relation is an equivalence relation, and we write $\langle a \rangle_A$ for the equivalence class of a.

(3) The *Cuntz semigroup* of A is

$$\mathrm{Cu}(A) = (K \otimes A)_+ / \sim_A,$$

together with the commutative semigroup operation, gotten from an isomorphism $M_2(K) \to K$,

$$\langle a \rangle_A + \langle b \rangle_A = \langle a \oplus b \rangle_A$$

(the class does not depend on the choice of the isomorphism) and the partial order

$$\langle a \rangle_A \leq \langle b \rangle_A \Longleftrightarrow a \precsim_A b.$$

It is taken to be an object of the category **Cu** given in Definition 4.1 of [4]. We write 0 for $\langle 0 \rangle_A$.

(4) We also define the subsemigroup

$$W(A) = M_\infty(A)_+ / \sim_A,$$

with the same operations and order. (It will follow from Remark 18.2 that the obvious map $W(A) \to \mathrm{Cu}(A)$ is injective.)

(5) Let A and B be C*-algebras, and let $\varphi\colon A \to B$ be a homomorphism. We use the same letter for the induced maps $M_n(A) \to M_n(B)$ for $n \in \mathbb{Z}_{>0}$, $M_\infty(A) \to M_\infty(B)$, and $K \otimes A \to K \otimes B$. We define $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ and $W(\varphi)\colon W(A) \to W(B)$ by $\langle a \rangle_A \mapsto \langle \varphi(a) \rangle_B$ for $a \in (K \otimes A)_+$ or $M_\infty(A)_+$ as appropriate.

It is easy to check that the maps $\mathrm{Cu}(\varphi)$ and $W(\varphi)$ are well defined homomorphisms of ordered semigroups which send 0 to 0. Also, it follows from Lemma 18.4(14) below that if $\eta_1, \eta_2, \mu_1, \mu_2 \in \mathrm{Cu}(A)$ satisfy $\eta_1 \leq \mu_1$ and $\eta_2 \leq \mu_2$, then $\eta_1 + \eta_2 \leq \mu_1 + \mu_2$.

The semigroup $\mathrm{Cu}(A)$ generally has better properties than $W(A)$. For example, certain supremums exist (Theorem 4.19 of [4]), and, when understood as an object of the category **Cu**, it behaves properly with respect to direct limits (Theorem 4.35 of [4]). In this exposition, we mainly use $W(A)$ because, when A is unital, the dimension function d_τ associated to a normalized quasitrace τ (Definition 18.7 below) is finite on $W(A)$ but usually not on $\mathrm{Cu}(A)$. In particular, the radius of comparison (Definition 21.2 below) is easier to deal with in terms of $W(A)$.

We will not need the definition of the category **Cu**.

Remark 18.2. We make the usual identifications

$$A \subset M_n(A) \subset M_\infty(A) \subset K \otimes A. \tag{18.1}$$

It is easy to check, by cutting down to corners, that if $a, b \in (K \otimes A)_+$ satisfy $a \precsim_A b$, then the sequence $(v_n)_{n=1}^\infty$ such that $\lim_{n\to\infty} v_n b v_n^* = a$ (as in Definition 18.1(1)) can be taken to be in the smallest of the algebras in (18.1) which contains both a and b. See Remark 1.2 of [213] for details.

The Cuntz semigroup of a separable C*-algebra can be very roughly thought of as K-theory using open projections in matrices over A'', that is, open supports of positive elements in matrices over A, instead of projections in matrices over A. As justification for this heuristic, we note that if X is a compact Hausdorff space and $f, g \in C(X)_+$, then $f \precsim_{C(X)} g$ if and only if

$$\big\{x \in X \colon f(x) > 0\big\} \subset \big\{x \in X \colon g(x) > 0\big\}.$$

A version of this can be made rigorous, at least in the separable case. See [183].

There is a description of $\mathrm{Cu}(A)$ using Hilbert modules over A in place of finitely generated projective modules as for K-theory. See [43].

Unlike K-theory, the Cuntz semigroup is not discrete. If $p, q \in A$ are projections such that $\|p - q\| < 1$, then p and q are Murray-von Neumann equivalent (Lemma 11.7). However, for $a, b \in A_+$, the relation $\|a - b\| < \varepsilon$ says nothing about the classes of a and b in $\mathrm{Cu}(A)$ or $W(A)$, however small $\varepsilon > 0$ is. We can see this in $\mathrm{Cu}(C(X))$. Even if $\big\{x \in X \colon g(x) > 0\big\}$ is a very small subset of X, for every $\varepsilon > 0$ the function $f = g + \frac{\varepsilon}{2}$ has $\langle f \rangle_{C(X)} = \langle 1 \rangle_{C(X)}$. What is true when $\|f - g\| < \varepsilon$ is that

$$\big\{x \in X \colon f(x) > \varepsilon\big\} \subset \big\{x \in X \colon g(x) > 0\big\},$$

so that the function $\max(f-\varepsilon,\, 0)$ satisfies $\max(f-\varepsilon,\, 0) \precsim_{C(X)} g$. This motivates the systematic use of the elements $(a-\varepsilon)_+$, defined as follows.

Definition 18.3. Let A be a C*-algebra, let $a \in A_+$, and let $\varepsilon \geq 0$. Let $f\colon [0,\infty) \to [0,\infty)$ be the function

$$f(\lambda) = (\lambda-\varepsilon)_+ = \begin{cases} 0 & 0 \leq \lambda \leq \varepsilon \\ \lambda - \varepsilon & \varepsilon < \lambda. \end{cases}$$

Then define $(a-\varepsilon)_+ = f(a)$ (using continuous functional calculus).

One must still be much more careful than with K-theory. First, $a \leq b$ does not imply $(a-\varepsilon)_+ \leq (b-\varepsilon)_+$ (although one does get $(a-\varepsilon)_+ \precsim_A (b-\varepsilon)_+$; see Lemma 18.4(17) below). Second, $a \precsim_A b$ does not imply any relation between $(a-\varepsilon)_+$ and $(b-\varepsilon)_+$. For example, if $A = C([0,1])$ and $a \in C([0,1])$ is $a(t) = t$ for $t \in [0,1]$, then for any $\varepsilon \in (0,1)$ the element $b = \varepsilon a$ satisfies $a \precsim_A b$. But $(a-\varepsilon)_+ \not\precsim_A (b-\varepsilon)_+$, since $(a-\varepsilon)_+$ has open support $(\varepsilon, 1]$ while $(b-\varepsilon)_+ = 0$. The best one can do is in Lemma 18.4(11) below.

We now list a collection of basic results about Cuntz comparison and the Cuntz semigroup. There are very few such results about projections and the K_0-group, the main ones being that if $\|p-q\| < 1$, then p and q are Murray-von Neumann equivalent; that $p \leq q$ if and only if $pq = p$; the relations between homotopy, unitary equivalence, and Murray-von Neumann equivalence; and the fact that addition of equivalence classes respects orthogonal sums. There are many more for Cuntz comparison. We will not use all the facts listed below in these notes (although they are all used in [213]); we include them all so as to give a fuller picture of Cuntz comparison.

Parts (1) through (14) of Lemma 18.4 are mostly taken from [138], with some from [46], [75], [195], and [245], and are summarized in Lemma 1.4 of [213]; we refer to [213] for more on the attributions (although not all the attributions there are to the original sources). Part (15) is Lemma 1.5 of [213]; part (16) is Corollary 1.6 of [213]; part (17) is Lemma 1.7 of [213]; and part (18) is Lemma 1.9 of [213].

As we have done earlier, we denote by A^+ the unitization of a C*-algebra A. (We add a new unit even if A is already unital.)

Lemma 18.4. Let A be a C*-algebra.

(1) Let $a, b \in A_+$. Suppose $a \in \overline{bAb}$. Then $a \precsim_A b$.
(2) Let $a \in A_+$ and let $f\colon [0, \|a\|] \to [0,\infty)$ be a continuous function such that $f(0) = 0$. Then $f(a) \precsim_A a$.
(3) Let $a \in A_+$ and let $f\colon [0, \|a\|] \to [0,\infty)$ be a continuous function such that $f(0) = 0$ and $f(\lambda) > 0$ for $\lambda > 0$. Then $f(a) \sim_A a$.
(4) Let $c \in A$. Then $c^*c \sim_A cc^*$.
(5) Let $a \in A_+$, and let $u \in A^+$ be unitary. Then $uau^* \sim_A a$.
(6) Let $c \in A$ and let $\alpha > 0$. Then $(c^*c - \alpha)_+ \sim_A (cc^* - \alpha)_+$.
(7) Let $v \in A$. Then there is an isomorphism $\varphi\colon \overline{v^*vAv^*v} \to \overline{vv^*Avv^*}$ such that, for every positive element $z \in \overline{v^*vAv^*v}$, we have $z \sim_A \varphi(z)$.
(8) Let $a \in A_+$ and let $\varepsilon_1, \varepsilon_2 > 0$. Then
$$\big((a-\varepsilon_1)_+ - \varepsilon_2\big)_+ = \big(a - (\varepsilon_1+\varepsilon_2)\big)_+.$$
(9) Let $a, b \in A_+$ satisfy $a \precsim_A b$ and let $\delta > 0$. Then there is $v \in A$ such that $v^*v = (a-\delta)_+$ and $vv^* \in \overline{bAb}$.

(10) Let $a, b \in A_+$. Then $\|a - b\| < \varepsilon$ implies $(a - \varepsilon)_+ \precsim_A b$.
(11) Let $a, b \in A_+$. Then the following are equivalent:
  (a) $a \precsim_A b$.
  (b) $(a - \varepsilon)_+ \precsim_A b$ for all $\varepsilon > 0$.
  (c) For every $\varepsilon > 0$ there is $\delta > 0$ such that $(a - \varepsilon)_+ \precsim_A (b - \delta)_+$.
  (d) For every $\varepsilon > 0$ there are $\delta > 0$ and $v \in A$ such that
$$(a - \varepsilon)_+ = v[(b - \delta)_+]v^*.$$
(12) Let $a, b \in A_+$. Then $a + b \precsim_A a \oplus b$.
(13) Let $a, b \in A_+$ be orthogonal (that is, $ab = 0$). Then $a + b \sim_A a \oplus b$.
(14) Let $a_1, a_2, b_1, b_2 \in A_+$, and suppose that $a_1 \precsim_A a_2$ and $b_1 \precsim_A b_2$. Then $a_1 \oplus b_1 \precsim_A a_2 \oplus b_2$.
(15) Let $a, b \in A$ be positive, and let $\alpha, \beta \geq 0$. Then
$$\big((a + b - (\alpha + \beta)\big)_+ \precsim_A (a - \alpha)_+ + (b - \beta)_+ \precsim_A (a - \alpha)_+ \oplus (b - \beta)_+.$$
(16) Let $\varepsilon > 0$ and $\lambda \geq 0$. Let $a, b \in A$ satisfy $\|a - b\| < \varepsilon$. Then $(a - \lambda - \varepsilon)_+ \precsim_A (b - \lambda)_+$.
(17) Let $a, b \in A$ satisfy $0 \leq a \leq b$. Let $\varepsilon > 0$. Then $(a - \varepsilon)_+ \precsim_A (b - \varepsilon)_+$.
(18) Let $a \in (K \otimes A)_+$. Then for every $\varepsilon > 0$ there are $n \in \mathbb{Z}_{>0}$ and $b \in (M_n \otimes A)_+$ such that $(a - \varepsilon)_+ \sim_A b$.

The following result is sufficiently closely tied to the ideas behind large subalgebras that we include the proof.

Lemma 18.5 (Lemma 1.8 of [213])**.** Let A be a C*-algebra, let $a \in A_+$, let $g \in A_+$ satisfy $0 \leq g \leq 1$, and let $\varepsilon \geq 0$. Then
$$(a - \varepsilon)_+ \precsim_A \big[(1 - g)a(1 - g) - \varepsilon\big]_+ \oplus g.$$

Proof. Set $h = 2g - g^2$, so that $(1 - g)^2 = 1 - h$. We claim that $h \sim_A g$. Since $0 \leq g \leq 1$, this follows from Lemma 18.4(3), using the continuous function $\lambda \mapsto 2\lambda - \lambda^2$ on $[0, 1]$.

Set $b = \big[(1 - g)a(1 - g) - \varepsilon\big]_+$. Using Lemma 18.4(15) at the second step, Lemma 18.4(6) and Lemma 18.4(4) at the third step, and Lemma 18.4(14) at the last step, we get
$$\begin{aligned}
(a - \varepsilon)_+ &= \big[a^{1/2}(1 - h)a^{1/2} + a^{1/2}ha^{1/2} - \varepsilon\big]_+ \\
&\precsim_A \big[a^{1/2}(1 - h)a^{1/2} - \varepsilon\big]_+ \oplus a^{1/2}ha^{1/2} \\
&\sim_A \big[(1 - g)a(1 - g) - \varepsilon\big]_+ \oplus h^{1/2}ah^{1/2} \\
&= b \oplus h^{1/2}ah^{1/2} \leq b \oplus \|a\|h \precsim_A b \oplus g.
\end{aligned}$$
This completes the proof. □

The definition of the radius of comparison (Definition 21.2 below) is stated in terms of quasitraces. We don't discuss quasitraces here. Instead, we refer to the fact (Theorem 5.11 of [102]) that all normalized 2-quasitraces on exact C*-algebras are tracial states. For the purpose of all the applications discussed in these notes, the reader can therefore substitute tracial states for quasitraces, and the tracial state space $\mathrm{T}(A)$ for the space $\mathrm{QT}(A)$ defined below.

It is still open whether every quasitrace on any C*-algebra is a trace.

Notation 18.6. For a unital C*-algebra A, we denote by $\mathrm{QT}(A)$ the set of normalized 2-quasitraces on A (Definition II.1.1 of [25]; Definition 2.31 of [4]).

Definition 18.7. Let A be a stably finite unital C*-algebra, and let $\tau \in \mathrm{QT}(A)$. Define $d_\tau \colon M_\infty(A)_+ \to [0, \infty)$ by $d_\tau(a) = \lim_{n \to \infty} \tau(a^{1/n})$ for $a \in M_\infty(A)_+$. Further (the use of the same notation should cause no confusion) define $d_\tau \colon (K \otimes A)_+ \to [0, \infty]$ by the same formula, but now for $a \in (K \otimes A)_+$. We also use the same notation for the corresponding functions on $\mathrm{Cu}(A)$ and $W(A)$, as in Proposition 18.8 below.

Proposition 18.8. Let A be a stably finite unital C*-algebra, and let $\tau \in \mathrm{QT}(A)$. Then d_τ as in Definition 18.7 is well defined on $\mathrm{Cu}(A)$ and $W(A)$. That is, if $a, b \in (K \otimes A)_+$ satisfy $a \sim_A b$, then $d_\tau(a) = d_\tau(b)$.

Proof. This is part of Proposition 4.2 of [75]. $\square$

Also see the beginning of Section 2.6 of [4], especially the proof of Theorem 2.32 there. It follows that d_τ defines a state on $W(A)$. Thus (see Theorem II.2.2 of [25], which gives the corresponding bijection between 2-quasitraces and dimension functions which are not necessarily normalized but are finite everywhere), the map $\tau \mapsto d_\tau$ is a bijection from $\mathrm{QT}(A)$ to the set of lower semicontinuous dimension functions on A.

We now present some results related to Cuntz comparison specifically for simple C*-algebras.

Lemma 18.9 (Proposition 4.10 of [138])**.** Let A be a C*-algebra which is not of type I and let $n \in \mathbb{Z}_{>0}$. Then there exists an injective homomorphism from the cone CM_n over M_n to A.

The proof uses heavy machinery, namely Glimm's result that there is a subalgebra $B \subset A$ and an ideal $I \subset B$ such that the 2^∞ UHF algebra embeds in B/I. Some of what we use this result for can be proved by more elementary methods, but for Lemma 18.13 we don't know such a proof.

Lemma 18.10 (Lemma 2.1 of [213])**.** Let A be a simple C*-algebra which is not of type I. Let $a \in A_+ \setminus \{0\}$, and let $l \in \mathbb{Z}_{>0}$. Then there exist $b_1, b_2, \ldots, b_l \in A_+ \setminus \{0\}$ such that $b_1 \sim_A b_2 \sim_A \cdots \sim_A b_l$, such that $b_j b_k = 0$ for $j \neq k$, and such that $b_1 + b_2 + \cdots + b_l \in \overline{aAa}$.

Proof. Replacing A by $\overline{aAa}$, we can ignore the requirement $b_1 + b_2 + \cdots + b_l \in \overline{aAa}$ of the conclusion. Now fix $n \in \mathbb{Z}_{>0}$. For $j, k = 1, 2, \ldots, n$, we let $e_{j,k} \in M_n$ be the standard matrix unit. In

$$CM_n = \big\{ f \in C([0,1], \, M_n) \colon f(0) = 0 \big\},$$

take $b_j(\lambda) = \lambda e_{j,j}$ for $\lambda \in [0,1]$ and $j = 1, 2, \ldots, n$. Use Lemma 18.9 to embed CM_n in A. $\square$

This lemma has the following corollary.

Corollary 18.11 (Corollary 2.2 of [213])**.** Let A be a simple unital infinite dimensional C*-algebra. Then for every $\varepsilon > 0$ there is $a \in A_+ \setminus \{0\}$ such that for all $\tau \in \mathrm{QT}(A)$ we have $d_\tau(a) < \varepsilon$.

Lemma 18.12 (Lemma 2.4 of [213])**.** Let A be a simple C*-algebra, and let $B \subset A$ be a nonzero hereditary subalgebra. Let $n \in \mathbb{Z}_{>0}$, and let $a_1, a_2, \ldots, a_n \in A_+ \setminus \{0\}$. Then there exists $b \in B_+ \setminus \{0\}$ such that $b \precsim_A a_j$ for $j = 1, 2, \ldots, n$.

Sketch of proof. The proof is by induction. The case $n = 0$ is trivial. The induction step requires that for $a, b_0 \in A_+ \setminus \{0\}$ one find $b \in A_+ \setminus \{0\}$ such that $b \in \overline{b_0 A b_0}$ (so that $b \precsim_A b_0$ by Lemma 18.4(1)) and $b \precsim_A a$. Use simplicity to find $x \in A$ such that the element $y = b_0 x a$ is nonzero, and take $b = y y^* \in \overline{b_0 A b_0}$. Using Lemma 18.4(5) and Lemma 18.4(1), we get $b \sim_A y^* y \precsim_A a$. $\square$

The following lemma says, roughly, that a nonzero element of $W(A)$ can be approximated arbitrarily well by elements of $W(A)$ which are strictly smaller.

Lemma 18.13 (Lemma 2.3 of [213])**.** Let A be a simple infinite dimensional C*-algebra which is not of type I. Let $b \in A_+ \setminus \{0\}$, let $\varepsilon > 0$, and let $n \in \mathbb{Z}_{>0}$. Then there are $c \in A_+$ and $y \in A_+ \setminus \{0\}$ such that, in $W(A)$, we have

$$n\langle (b-\varepsilon)_+ \rangle_A \leq (n+1)\langle c \rangle_A \qquad \text{and} \qquad \langle c \rangle_A + \langle y \rangle_A \leq \langle b \rangle_A.$$

Sketch of proof. We divide the proof into two cases. First assume that $\mathrm{sp}(b) \cap (0, \varepsilon) \neq \varnothing$. Then there is a continuous function $f \colon [0, \infty) \to [0, \infty)$ which is zero on $\{0\} \cup [\varepsilon, \infty)$ and such that $f(b) \neq 0$. We take $c = (b - \varepsilon)_+$ and $y = f(b)$.

Now suppose that $\mathrm{sp}(b) \cap (0, \varepsilon) = \varnothing$. In this case, we might as well assume that b is a projection, and that $\langle (b-\varepsilon)_+ \rangle_A$, which is always dominated by $\langle b \rangle_A$, is equal to $\langle b \rangle_A$. Cutting down by b, we can assume that $b = 1$ (in particular, A is unital), and it is enough to find $c \in A_+$ and $y \in A_+ \setminus \{0\}$ such that $n\langle 1 \rangle_A \leq (n+1)\langle c \rangle_A$ and $\langle c \rangle_A + \langle y \rangle_A \leq \langle 1 \rangle_A$.

Take the unitized cone over M_{n+1} to be $C = (CM_{n+1})^+ = [C_0((0,1]) \otimes M_{n+1}]^+$, and use the usual notation for matrix units. By Lemma 18.9, we can assume that $C \subset A$. Let $t \in C_0((0,1])$ be the function $t(\lambda) = \lambda$ for $\lambda \in (0,1]$. Choose continuous functions $g_1, g_2, g_3 \in C([0,1])$ such that

$$0 \leq g_3 \leq g_2 \leq g_1 \leq 1, \quad g_1(0) = 0, \quad g_3(1) = 1, \quad g_1 g_2 = g_2, \quad \text{and} \quad g_2 g_3 = g_3.$$

Define

$$x = g_2 \otimes e_{1,1}, \quad c = 1 - x, \qquad \text{and} \qquad y = g_3 \otimes e_{1,1}.$$

Then $xy = y$ so $cy = 0$. It follows from Lemma 18.4(13) that $\langle c \rangle_A + \langle y \rangle_A \leq \langle 1 \rangle_A$.

It remains to prove that $n\langle 1 \rangle_A \leq (n+1)\langle c \rangle_A$, and it is enough to prove that in $W(C)$ we have $n\langle 1 \rangle_C \leq (n+1)\langle 1 - g_2 \otimes e_{1,1} \rangle_C$, that is, in $M_{n+1}(C)$,

$$\mathrm{diag}(1, 1, \ldots, 1, 0) \precsim_C \mathrm{diag}\big(1 - g_2 \otimes e_{1,1},\, 1 - g_2 \otimes e_{1,1},\, \ldots,\, 1 - g_2 \otimes e_{1,1}\big). \tag{18.2}$$

To see why this should be true, view $M_{n+1}(C)$ as a set of functions from $[0,1]$ to $M_{(n+1)^2}$ with restrictions on the value at zero. Since $g_1 g_2 = g_2$, the function $1 - g_2 \otimes e_{1,1}$ is constant equal to 1 on a neighborhood U of 0, and at $\lambda \in U$ the right hand side of (18.2) therefore dominates the left hand side. Elsewhere, both sides of (18.2) are diagonal, with the right hand side being a constant projection of rank $n(n+1)$ and the left hand side dominating

$$\mathrm{diag}\big(1 - e_{1,1},\, 1 - e_{1,1},\, \ldots,\, 1 - e_{1,1}\big),$$

which is a (different) constant projection of rank $n(n+1)$. It is not hard to construct an explicit formula for a unitary $v \in M_{n+1}(C)$ such that

$$\mathrm{diag}(1, 1, \ldots, 1, 0) \leq v \cdot \mathrm{diag}\big(1 - g_2 \otimes e_{1,1},\, 1 - g_2 \otimes e_{1,1},\, \ldots,\, 1 - g_2 \otimes e_{1,1}\big) \cdot v^*.$$

See [213] for the details (arranged a little differently). □

19. Large Subalgebras

Large and centrally large subalgebras are a technical tool which has played a key role in work on the structure of the C*-algebras of minimal dynamical systems and some related algebras. In this section, we outline some old and new applications as motivation. We then give the definitions and several useful reformulations of them. We next state some general theorems (for some of which we give partial proofs in Section 20 and Section 21). Finally, we give further information on some recent applications.

Large subalgebras are a generalization and abstraction of a construction introduced by Putnam in [229] (see Definition 16.18), where it was used to prove that if h is a minimal homeomorphism of the Cantor set X, then $K_0(C^*(\mathbb{Z}, X, h))$ is order isomorphic to the K_0-group of a simple AF algebra (Theorem 4.1 and Corollary 5.6 of [229]). Putnam's construction and some generalizations (almost all of which are centrally large subalgebras in our sense) also played key roles in proofs of other many other results. We list some of the them, starting with older ones (which in many cases have been superseded, and which were proved before there was a formal definition of a large subalgebra). We then give some recent results for which no proofs not using large subalgebras are known.

Here is a selection of the older results.

- Let $h\colon X \to X$ be a minimal homeomorphism of the Cantor set. Then $C^*(\mathbb{Z}, X, h)$ is an AT algebra. (Local approximation by circle algebras was proved in Section 2 of [230]. Direct limit decomposition follows from semiprojectivity of circle algebras.)
- Let $h\colon X \to X$ be a minimal homeomorphism of a finite dimensional compact metric space. Then $C^*(\mathbb{Z}, X, h)$ satisfies the following K-theoretic version of Definition 11.34 (Blackadar's Second Fundamental Comparability Question): if $\eta \in K_0(A)$ satisfies $\tau_*(\eta) > 0$ for all tracial states τ on A, then there is a projection $p \in M_\infty(A)$ such that $\eta = [p]$. (See [160] and Theorem 4.5(1) of [204]).
- Let X be a finite dimensional infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism such that the map
$$\rho\colon K_0\big(C^*(\mathbb{Z}, X, h)\big) \to \operatorname{Aff}\big(\mathrm{T}\big(C^*(\mathbb{Z}, X, h)\big)\big)$$
of Definition 16.14 has dense range. Then $C^*(\mathbb{Z}, X, h)$ has tracial rank zero (Definition 11.35). This was proved in [157]; much of the method is described in Section 16 and Section 17.
- Let X be the Cantor set and let $h\colon X \times S^1 \to X \times S^1$ be a minimal homeomorphism. For any $x \in X$, the set $Y = \{x\} \times S^1$ intersects each orbit at most once. The algebra $C^*(\mathbb{Z}, X \times S^1, h)_Y$ (see Definition 16.18 for the notation) is introduced before Proposition 3.3 of [154], where it is called A_x. It is a centrally large subalgebra which plays a key role in the proofs of some of the results there. For example, the proofs that the crossed products considered there have stable rank one (as in Definition 11.1; see Theorem 3.12 of [154]) and order on projections determined by traces (as in Definition 11.34; see Theorem 3.13 of [154]) rely directly on the use of this subalgebra.

- A similar construction, with $X \times S^1 \times S^1$ in place of $X \times S^1$ and with $Y = \{x\} \times S^1 \times S^1$, appears in Section 1 of [268]. It plays a role in that paper similar to the role of the algebra $C^*(\mathbb{Z},\, X \times S^1,\, h)_Y$ in the previous item.
- Let $h \colon X \to X$ be a minimal homeomorphism of an infinite compact metric space. The large subalgebras $C^*(\mathbb{Z}, X, h)_Y$ of $C^*(\mathbb{Z}, X, h)$ (as in Definition 16.18), with several choices of Y (several one point sets as well as $\{x_1, x_2\}$ with x_1 and x_2 on different orbits), have been used by Toms and Winter [286] to prove that $C^*(\mathbb{Z}, X, h)$ has finite decomposition rank.

Here are some newer results. For most of them, we give more information later in this section.

- The extended irrational rotation algebras are AF (Elliott and Niu [73]; Theorem 19.18).
- Let X be an infinite compact metric space, and let $h \colon X \to X$ be a minimal homeomorphism with mean dimension zero. Then $C^*(\mathbb{Z}, X, h)$ is Z-stable (Elliott and Niu [74]; Theorem 19.19).
- Let X be a compact metric space such that there is a continuous surjection from X to the Cantor set. Then $\mathrm{rc}\big(C^*(\mathbb{Z}, X, h)\big) \leq \frac{1}{2}\mathrm{mdim}(h)$ ([110]; Theorem 19.15).
- Let X be a compact metric space such that there is a continuous surjection from X to the Cantor set. Then $C^*(\mathbb{Z}, X, h)$ has stable rank one (Theorem 7.1 of [8]; Theorem 19.17). (There are examples in which this holds but $C^*(\mathbb{Z}, X, h)$ does not have strict comparison of positive elements and is not Z-stable.)
- Let X be a compact metric space and let $h \colon X \to X$ be a minimal homeomorphism. Then $\mathrm{rc}\big(C^*(\mathbb{Z}, X, h)\big) \leq 1 + 2 \cdot \mathrm{mdim}(h)$ (Theorem 19.16; this strengthens Corollary 4.8 in the current version of of [214]).
- We give an example involving a crossed product $C^*(\mathbb{Z},\, C(X, D),\, \alpha)$ in which D is simple and α "lies over" a minimal homeomorphism of X. Let F_∞ be the free group on generators indexed by $\mathbb{Z}$, and for $n \in \mathbb{Z}$ let $u_n \in C^*_{\mathrm{r}}(F_\infty)$ be the unitary which is the image of the corresponding generator of F_∞. Let $h \colon X \to X$ be the restriction of a Denjoy homeomorphism (a nonminimal homeomorphism of the circle whose rotation number is irrational) to its unique minimal set. (See [234].) Thus X is homeomorphic to the Cantor set, and there are $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and a surjective map $\zeta \colon X \to S^1$ such that $\zeta(h(x)) = e^{2\pi i\theta}\zeta(x)$ for all $x \in X$. For $x \in X$ let $\alpha_x \in \mathrm{Aut}(C^*_{\mathrm{r}}(F_\infty))$ be determined by $\alpha_x(u_n) = \zeta(x)u_n$ for $n \in \mathbb{Z}$. Define a kind of noncommutative Furstenberg transformation $\alpha \in \mathrm{Aut}\big(C(X,\, C^*_{\mathrm{r}}(F_\infty))\big)$ by $\alpha(a)(x) = \alpha_x(a(x))$ for $a \in C(X,\, C^*_{\mathrm{r}}(F_\infty))$ and $x \in X$. Then $C^*\big(\mathbb{Z},\, C(X,\, C^*_{\mathrm{r}}(F_\infty)),\, \alpha\big)$ has stable rank one.

Large subalgebras were also used to give the first proof that if X is a finite dimensional compact metric space with a free minimal action of $\mathbb{Z}^d$, then $C^*(\mathbb{Z}^d, X)$ has strict comparison of positive elements.

Almost all the examples above involve actions of $\mathbb{Z}$ (although not necessarily on an algebra of the form $C(X)$). In the known applications of this type, there are explicit formulas for the large subalgebras involved. See Definition 16.18 and Definition 22.3. The real importance of the abstraction of the idea is in applications to actions of groups such as $\mathbb{Z}^d$, in which there are no known formulas for useful large subalgebras. Instead, subalgebras with useful properties must be shown to exist by more abstract methods. These applications are barely touched on in these notes.

There is a competing approach, the method of Rokhlin dimension of group actions [114], which can be used for some of the same problems large subalgebras are good for. When it applies, it often gives stronger results. For example, Szabó has used this method successfully for free minimal actions of $\mathbb{Z}^d$ on finite dimensional compact metric spaces [273]. For many problems involving crossed products for which large subalgebras are a plausible approach, Rokhlin dimension methods should also be considered. Rokhlin dimension has also been successfully applied to problems involving actions on simple C*-algebras, a context in which no useful large subalgebras are known. (But see [185] and [182], where what might be called large systems of subalgebras are used effectively.) On the other hand, finite Rokhlin dimension requires freeness of the action (in a suitable heuristic sense when the algebra is simple), while some form of essential freeness seems likely to be good enough for large subalgebra methods. (This is suggested by the examples in [182].) Finite Rokhlin dimension also requires some form of topological finite dimensionality.

It seems plausible that there might be a generalization of finite Rokhlin dimension which captures actions on infinite dimensional spaces which have mean dimension zero. Such a generalization might be similar to the progression from the study of simple AH algebras with no dimension growth to those with slow dimension growth. It looks much less likely that Rokhlin dimension methods can be usefully applied to minimal homeomorphisms which do not have mean dimension zero. Large subalgebras have been used to estimate the radius of comparison of $C^*(\mathbb{Z}, X, h)$ when h does not have mean dimension zero (and the radius of comparison is nonzero); see Theorem 19.15 and Theorem 19.16, both discussed in Section 23. These results do not seem to be accessible via Rokhlin dimension methods. Rokhlin dimension methods can also potentially be used to prove regularity properties of crossed products $C^*\big(\mathbb{Z}, C(X, D), \alpha\big)$ when D is simple, the automorphism $\alpha \in \mathrm{Aut}(C(X, D))$ "lies over" a minimal homeomorphism of X with large mean dimension, and the regularity properties of the crossed product come from D rather than from the action of $\mathbb{Z}$ on X. See [38].

Unfortunately, we are not able to discuss Rokhlin dimension here.

In these notes, we mostly limit ourselves to applications to crossed products by minimal homeomorphisms.

By convention, if we say that B is a unital subalgebra of a C*-algebra A, we mean that B contains the identity of A.

Definition 19.1 (Definition 4.1 of [213])**.** Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *large* in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
(4) $g \precsim_B y$ and $g \precsim_A x$.
(5) $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

We emphasize that the Cuntz subequivalence involving y in (4) is relative to B, not A.

Condition (5) is needed to avoid triviality when A is purely infinite and simple. With $B = \mathbb{C} \cdot 1$, we could then satisfy all the other conditions by taking $g = 1$. In the stably finite case, we can dispense with (5) (see Proposition 20.3 below), but we still need $g \precsim_A x$ in (4). Otherwise, even if we require that B be simple and that the restriction maps $\mathrm{T}(A) \to \mathrm{T}(B)$ and $\mathrm{QT}(A) \to \mathrm{QT}(B)$ on traces and quasitraces be bijective, we can take A to be any UHF algebra and take $B = \mathbb{C} \cdot 1$. The choice $g = 1$ would always work.

It is crucial to the usefulness of large subalgebras that g in Definition 19.1 need not be a projection. Also, one can do a lot without any kind of approximate commutation condition. Such a condition does seem to be needed for some results. Here is the relevant definition, although we will not make full use of it in these notes.

Definition 19.2 (Definition 3.2 of [8])**.** Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *centrally large* in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
(4) $g \precsim_B y$ and $g \precsim_A x$.
(5) $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.
(6) For $j = 1, 2, \ldots, m$ we have $\|g a_j - a_j g\| < \varepsilon$.

The difference between Definition 19.2 and Definition 19.1 is the approximate commutation condition in Definition 19.2(6).

The following strengthening of Definition 19.2 will be more important in these notes.

Definition 19.3 (Definition 5.1 of [213])**.** Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *stably large* in A if $M_n(B)$ is large in $M_n(A)$ for all $n \in \mathbb{Z}_{>0}$.

Proposition 19.4 (Proposition 5.6 of [213])**.** Let A_1 and A_2 be infinite dimensional simple unital C*-algebras, and let $B_1 \subset A_1$ and $B_2 \subset A_2$ be large subalgebras. Assume that $A_1 \otimes_{\min} A_2$ is finite. Then $B_1 \otimes_{\min} B_2$ is a large subalgebra of $A_1 \otimes_{\min} A_2$.

In particular, if A is stably finite and $B \subset A$ is large, then B is stably large. We will give a direct proof (Proposition 20.11 below). We don't know whether stable finiteness of A is needed (Question 24.2 below).

The main example used in these notes is the Y-orbit breaking subalgebra (generalized Putnam subalgebra)

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\, C_0(X \setminus Y)u\big) \subset C^*(\mathbb{Z}, X, h).$$

of Definition 16.18, for a compact metric space X, a minimal homeomorphism $h \colon X \to X$, and a "sufficiently small" nonempty closed subset $Y \subset X$.

Theorem 19.5. Let X be an infinite compact Hausdorff space and let $h \colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a compact subset such that

$h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then $C^*(\mathbb{Z}, X, h)_Y$ (as above) is a centrally large subalgebra of $C^*(\mathbb{Z}, X, h)$ in the sense of Definition 19.2.

We give a proof in Section 22, along with proofs or sketches of proofs of the lemmas which go into the proof.

The key fact about $C^*(\mathbb{Z}, X, h)_Y$ which makes this theorem useful is that it is a direct limit of recursive subhomogeneous C*-algebras (as in Definition 1.1 of [203]) whose base spaces are closed subsets of X. This follows from Theorem 17.14 (or Theorem 17.19) and Remark 17.20. The structure of $C^*(\mathbb{Z}, X, h)_Y$ is therefore much more accessible than the structure of crossed products.

We now state the main known results about large subalgebras and some recent applications.

Proposition 19.6 (Proposition 5.2 and Proposition 5.5 of [213])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then B is simple and infinite dimensional.

The special case $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is stated without proof as Proposition 17.21. In the next section, we prove the simplicity statement (see Proposition 20.7 below) and the stably finite case of the infinite dimensionality statement (see Proposition 20.10 below).

Theorem 19.7 (Theorem 6.2 and Proposition 6.9 of [213])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then the restriction maps $\mathrm{T}(A) \to \mathrm{T}(B)$ and $\mathrm{QT}(A) \to \mathrm{QT}(B)$, on traces and quasitraces (see Definition 11.23 and Notation 18.6), are bijective.

The special case involving $\mathrm{T}\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big)$ is in Lemma 17.22, but the proof given for Lemma 17.22 is quite different. The proofs for $\mathrm{T}(A)$ and for $\mathrm{QT}(A)$ are very different. We prove that $\mathrm{T}(A) \to \mathrm{T}(B)$ is bijective below (Theorem 20.12).

Let A be a C*-algebra. Recall the Cuntz semigroup $\mathrm{Cu}(A)$ from Definition 18.1(3). Let $\mathrm{Cu}_+(A)$ denote the set of elements $\eta \in \mathrm{Cu}(A)$ which are not the classes of projections. (Such elements are sometimes called *purely positive*.)

Theorem 19.8 (Theorem 6.8 of [213])**.** Let A be a stably finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $\iota \colon B \to A$ be the inclusion map. Then $\mathrm{Cu}(\iota)$ defines an order and semigroup isomorphism from $\mathrm{Cu}_+(B) \cup \{0\}$ to $\mathrm{Cu}_+(A) \cup \{0\}$.

It is not true that $\mathrm{Cu}(\iota)$ defines an isomorphism from $\mathrm{Cu}(B)$ to $\mathrm{Cu}(A)$. Example 7.13 of [213] shows that $\mathrm{Cu}(\iota) \colon \mathrm{Cu}(B) \to \mathrm{Cu}(A)$ need not be injective. We suppose this map can also fail to be surjective, but we don't know an example.

Theorem 19.9 (Theorem 6.14 of [213])**.** Let A be an infinite dimensional stably finite simple separable unital C*-algebra. Let $B \subset A$ be a large subalgebra. Let $\mathrm{rc}(-)$ be the radius of comparison (Definition 21.2 below). Then $\mathrm{rc}(A) = \mathrm{rc}(B)$.

We will prove this result in Section 21 when A is exact. See Theorem 21.3 below.

Proposition 19.10 (Proposition 6.15, Corollary 6.16, and Proposition 6.17 of [213])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then:

(1) A is finite if and only if B is finite.

(2) If B is stably large in A, then A is stably finite if and only if B is stably finite.
(3) A is purely infinite if and only if B is purely infinite.

Proposition 19.11 (Theorem 6.18 of [213])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Suppose that B has property (SP) (Definition 11.4). Then A has property (SP).

Theorem 19.12 (Theorem 6.3 and Theorem 6.4 of [8])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a centrally large subalgebra. Then:

(1) If B has stable rank one (Definition 11.1), then so does A.
(2) If B has real rank zero (Definition 11.3) and stable rank one, then so does A.

In the next theorem, Z is the Jiang-Su algebra (briefly described in Example 3.33). The condition that a given C*-algebra A tensorially absorb the Jiang-Su algebra, that is, $Z \otimes A \cong A$ (A is said to be "Z-stable" or "Z-absorbing"), is one of the regularity conditions in the Toms-Winter conjecture. For simple separable nuclear C*-algebras it is hoped, and known in many cases, that Z-stability implies classifiability in the sense of the Elliott program.

Theorem 19.13 (Theorem 2.3 of [6])**.** Let A be an infinite dimensional simple nuclear unital C*-algebra, and let $B \subset A$ be a centrally large subalgebra. If B tensorially absorbs the Jiang-Su algebra Z, then so does A.

If A isn't nuclear, the best we can say so far is that A is tracially Z-absorbing in the sense of Definition 2.1 of [111].

The following two key technical results are behind many of the theorems stated above. In particular, they are the basis for proving Theorem 19.8, which is used to prove many of the other results.

Lemma 19.14 (Lemmas 6.3 and 6.5 of [213])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a stably large subalgebra.

(1) Let $a, b, x \in (K \otimes A)_+$ satisfy $x \neq 0$ and $a \oplus x \precsim_A b$. Then for every $\varepsilon > 0$ there are $n \in \mathbb{Z}_{>0}$, $c \in (M_n \otimes B)_+$, and $\delta > 0$ such that $(a - \varepsilon)_+ \precsim_A c \precsim_A (b - \delta)_+$.
(2) Let $a, b \in (K \otimes B)_+$ and $c, x \in (K \otimes A)_+$ satisfy $x \neq 0$, $a \precsim_A c$, and $c \oplus x \precsim_A b$. Then $a \precsim_B b$.

We state some of the applications. In the following theorem, $\mathrm{rc}(A)$ is the radius of comparison of A (see Definition 21.2 below), and $\mathrm{mdim}(h)$ is the mean dimension of h (see Definition 23.3 below).

Theorem 19.15 ([110])**.** Let X be a compact metric space. Assume that there is a continuous surjective map from X to the Cantor set. Let $h \colon X \to X$ be a minimal homeomorphism. Then $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) \leq \frac{1}{2}\mathrm{mdim}(h)$.

It is conjectured that $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) = \frac{1}{2}\mathrm{mdim}(h)$ for all minimal homeomorphisms. In [110], we also prove that $\mathrm{rc}(C^*(\mathbb{Z}, X, h)) \geq \frac{1}{2}\mathrm{mdim}(h)$ for a reasonably large class of homomorphisms constructed using the methods of Giol and Kerr [93], including the ones in that paper. For all minimal homeomorphisms of this type, there is a continuous surjective map from the space to the Cantor set.

The proof of Theorem 19.15 uses Theorem 19.5, Theorem 19.9, the fact that we can arrange that $C^*(\mathbb{Z}, X, h)_Y$ be the direct limit of an AH system with diagonal maps, and methods of [176] (see especially Theorem 6.2 there) to estimate radius of comparison of simple direct limits of AH systems with diagonal maps. We would like to use Theorem 6.2 of [176] directly. Unfortunately, the definition of mean dimension of an AH direct system in [176] requires that the base spaces be connected. See Definition 3.6 of [176], which refers to the setup described after Lemma 3.4 of [176].

Theorem 19.16. Let X be a compact metric space. Let $h\colon X \to X$ be a minimal homeomorphism. Then $\operatorname{rc}(C^*(\mathbb{Z}, X, h)) \leq 1 + 2 \cdot \operatorname{mdim}(h)$.

Corollary 4.8 of [214] states that $\operatorname{rc}(C^*(\mathbb{Z}, X, h)) \leq 1+36 \cdot \operatorname{mdim}(h)$. A key ingredient is Theorem 5.1 of [163], an embedding result for minimal homeomorphisms in shifts on cubes, the dimension of the cube depending on the mean dimension of the homeomorphism. The improvement, to appear in a revised version of [214], is based on the use of a stronger embedding result for minimal dynamical systems, Theorem 1.4 of [101]. We really want $\operatorname{rc}(C^*(\mathbb{Z}, X, h)) \leq \frac{1}{2}\operatorname{mdim}(h)$, as in Theorem 19.15.

Theorem 19.17 (Theorem 7.1 of [8])**.** Let X be a compact metric space. Assume that there is a continuous surjective map from X to the Cantor set. Let $h\colon X \to X$ be a minimal homeomorphism. Then $C^*(\mathbb{Z}, X, h)$ has stable rank one (Definition 11.1).

There is no finite dimensionality assumption on X. We don't even assume that h has mean dimension zero. In particular, this theorem holds for the examples of Giol and Kerr [93], for which the crossed products are known not to be Z-stable and not to have strict comparison of positive elements. (For such systems, it is shown in [110] that $\operatorname{rc}(C^*(\mathbb{Z}, X, h)) = \frac{1}{2}\operatorname{mdim}(h)$, and in [93] that $\operatorname{mdim}(h) \neq 0$. See the discussion in Section 7 of [8] for details.)

The proof uses Theorem 19.5, Theorem 19.12(1), the fact that we can arrange that $C^*(\mathbb{Z}, X, h)_Y$ be the direct limit of an AH system with diagonal maps, and Theorem 4.1 of [69], according to which simple direct limits of AH systems with diagonal maps always have stable rank one, without any dimension growth hypotheses.

Theorem 19.18 (Elliott and Niu [73])**.** The "extended" irrational rotation algebras, obtained by "cutting" each of the standard unitary generators at one or more points in its spectrum, are AF algebras.

We omit the precise descriptions of these algebras.

If one cuts just one of the generators, the resulting algebra is a crossed product by a minimal homeomorphism of the Cantor set, with the other unitary playing the role of the image of a generator of the group $\mathbb{Z}$. If both are cut, the algebra is no longer an obvious crossed product.

Theorem 19.19 (Elliott and Niu [74])**.** Let X be an infinite compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism with mean dimension zero. Then $C^*(\mathbb{Z}, X, h)$ is Z-stable.

20. Basic Properties of Large Subalgebras

In this section, we give some equivalent versions of the definition of a large subalgebra. Then we state some of the basic properties of large subalgebras. Recall that, by convention, if we say that B is a unital subalgebra of a C*-algebra A, we mean that B contains the identity of A. The change from the definition in the following lemma is that we only require the usual conclusions of Definition 19.1 to hold for $a_1, a_2, \ldots, a_m$ in a subset of A whose linear span is dense.

Lemma 20.1. Let A be an infinite dimensional simple unital C*-algebra, let $B \subset A$ be a unital subalgebra, and let $S \subset A$ be a subset whose linear span is dense in A. Suppose that for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in S$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
(4) $g \precsim_B y$ and $g \precsim_A x$.
(5) $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$.

Then B is a large subalgebra of A in the sense of Definition 19.1.

As before, the Cuntz subequivalence involving y in (4) is relative to B, not A.

Exercise 20.2. Prove Lemma 20.1.

Unlike other approximation properties (such as tracial rank), it seems not to be possible to take S in Lemma 20.1 to be a generating subset, or even a selfadjoint generating subset. (We can do this for the definition of a centrally large subalgebra, Definition 19.2. See Proposition 3.10 of [8].)

By Proposition 4.4 of [213], in Definition 19.1 we can omit mention of $c_1, c_2, \ldots, c_m$, and replace (2) and (3) by the requirement that $\operatorname{dist}\big((1 - g)a_j,\, B\big) < \varepsilon$ for $j = 1, 2, \ldots, m$. So far, however, most verifications of Definition 19.1 proceed by constructing elements $c_1, c_2, \ldots, c_m$ as in Definition 19.1.

When A is finite, we do not need condition (5) of Definition 19.1.

Proposition 20.3 (Proposition 4.5 of [213])**.** Let A be a finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a unital subalgebra. Suppose that for $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+ \setminus \{0\}$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
(4) $g \precsim_B y$ and $g \precsim_A x$.

Then B is large in A.

The proof of Proposition 20.3 needs Lemma 20.5 below, which is a version for Cuntz comparison of Lemma 1.15 of [208].

We describe the idea of the proof of Proposition 20.3. (Most of the details are given below.) Given $x \in A_+$ with $\|x\| = 1$, we want $x_0 \in A_+ \setminus \{0\}$ such that $g \precsim_A x_0$ and otherwise as above implies $\|(1 - g)x(1 - g)\| > 1 - \varepsilon$. (We then use x_0 in place of x in the definition of a large subalgebra.) Choose a sufficiently small number $\varepsilon_0 > 0$. (It will be much smaller than ε.) Choose $f \colon [0, 1] \to [0, 1]$ such that $f = 0$ on $[0,\, 1-\varepsilon_0]$ and $f(1) = 1$. Construct $a, b_1, b_2, c_1, c_2, d_1, d_2 \in \overline{f(x)Af(x)}$ such that for $j = 1, 2$ we have

$$0 \leq d_j \leq c_j \leq b_j \leq a \leq 1, \quad ab_j = b_j, \quad b_j c_j = c_j, \quad c_j d_j = d_j, \qquad \text{and} \qquad d_j \neq 0,$$

and $b_1 b_2 = 0$. Take $x_0 = d_1$. If ε_0 is small enough, $g \precsim_A d_1$, and $\|(1-g)x(1-g)\| \leq 1 - \varepsilon$, this gives

$$\|(1-g)(b_1+b_2)(1-g)\| < 1 - \frac{\varepsilon}{3}.$$

One then gets $c_1 + c_2 \precsim_A d_1$. (This is the calculation (20.1) in the proof below.) Now $r = (1-c_1-c_2)+d_1$ satisfies $r \precsim_A 1$, so there is $v \in A$ such that $\|vrv^* - 1\| < \frac{1}{2}$. Then $vr^{1/2}$ is right invertible, but $vr^{1/2}d_2 = 0$, so $vr^{1/2}$ is not left invertible. This contradicts finiteness of A.

We now give a more detailed argument.

Lemma 20.4 (Lemma 2.5 of [213])**.** Let A be a C*-algebra, let $x \in A_+$ satisfy $\|x\| = 1$, and let $\varepsilon > 0$. Then there are positive elements $a, b \in \overline{xAx}$ with $\|a\| = \|b\| = 1$, such that $ab = b$, and such that whenever $c \in \overline{bAb}$ satisfies $\|c\| \leq 1$, then $\|xc - c\| < \varepsilon$.

Sketch of proof. Choose continuous functions $f_0, f_1 \colon [0,1] \to [0,1]$ such that $f_1(1) = 1$, f_1 is supported near 1, $|f_0(\lambda) - \lambda| < \varepsilon$ for all $\lambda \in [0,1]$, and $f_0 = 1$ near 1 (so that $f_0 f_1 = f_1$). Take $a = f_0(x)$ and $b = f_1(x)$. Then $\|x - a\| < \varepsilon$ and $ab = b$. $\square$

Lemma 20.5 (Lemma 2.6 of [213])**.** Let A be a finite simple infinite dimensional unital C*-algebra. Let $x \in A_+$ satisfy $\|x\| = 1$. Then for every $\varepsilon > 0$ there is $x_0 \in \left(\overline{xAx}\right)_+ \setminus \{0\}$ such that whenever $g \in A_+$ satisfies $0 \leq g \leq 1$ and $g \precsim_A x_0$, then $\|(1-g)x(1-g)\| > 1 - \varepsilon$.

Proof. Choose positive elements $a, b \in \overline{x^{1/2}Ax^{1/2}}$ as in Lemma 20.4, with $x^{1/2}$ in place of x and $\frac{\varepsilon}{3}$ in place of ε. Then $a, b \in \overline{xAx}$ since $\overline{x^{1/2}Ax^{1/2}} = \overline{xAx}$. Since $b \neq 0$, Lemma 18.10 provides nonzero positive orthogonal elements $z_1, z_2 \in \overline{bAb}$ (with $z_1 \sim_A z_2$). We may require $\|z_1\| = \|z_2\| = 1$.

Choose continuous functions $f_0, f_1, f_2 \colon [0, \infty) \to [0,1]$ such that

$$f_0(0) = 0, \quad f_0 f_1 = f_1, \quad f_1 f_2 = f_2, \qquad \text{and} \qquad f_2(1) = 1.$$

For $j = 1, 2$ define

$$b_j = f_0(z_j), \quad c_j = f_1(z_j), \qquad \text{and} \qquad d_j = f_2(z_j).$$

Then

$$0 \leq d_j \leq c_j \leq b_j \leq 1, \quad ab_j = b_j, \quad b_j c_j = c_j, \quad c_j d_j = d_j, \qquad \text{and} \qquad d_j \neq 0.$$

Also $b_1 b_2 = 0$. Define $x_0 = d_1$. Then $x_0 \in \left(\overline{xAx}\right)_+$.

Let $g \in A_+$ satisfy $0 \leq g \leq 1$ and $g \precsim_A x_0$. We want to show that

$$\|(1-g)x(1-g)\| > 1 - \varepsilon,$$

so suppose that $\|(1-g)x(1-g)\| \leq 1 - \varepsilon$. The choice of a and b, and the relations $(b_1 + b_2)^{1/2} \in \overline{bAb}$ and $\left\|(b_1+b_2)^{1/2}\right\| = 1$, imply that

$$\left\|x^{1/2}(b_1+b_2)^{1/2} - (b_1+b_2)^{1/2}\right\| < \frac{\varepsilon}{3}.$$

Using this relation and its adjoint at the second step, we get

$$\begin{aligned}\big\|(1-g)(b_1+b_2)(1-g)\big\| &= \big\|(b_1+b_2)^{1/2}(1-g)^2(b_1+b_2)^{1/2}\big\| \\ &< \big\|(b_1+b_2)^{1/2}x^{1/2}(1-g)^2x^{1/2}(b_1+b_2)^{1/2}\big\| + \frac{2\varepsilon}{3} \\ &\leq \big\|x^{1/2}(1-g)^2x^{1/2}\big\| + \frac{2\varepsilon}{3} \\ &= \|(1-g)x(1-g)\| + \frac{2\varepsilon}{3} \leq 1 - \frac{\varepsilon}{3}.\end{aligned}$$

Using the equation $(b_1+b_2)(c_1+c_2) = c_1+c_2$ and taking C to be the commutative C*-algebra generated by b_1+b_2 and c_1+c_2, one easily sees that for every $\beta \in [0,1)$ we have $c_1+c_2 \precsim_C [(b_1+b_2)-\beta]_+$. Take $\beta = 1-\frac{\varepsilon}{3}$, use this fact and Lemma 18.5 at the first step, use the estimate above at the second step, and use $g \precsim_A x_0 = d_1$ at the third step, to get

$$c_1+c_2 \precsim_A \big[(1-g)(b_1+b_2)(1-g)-\beta\big]_+ \oplus g = 0 \oplus g \precsim_A d_1. \tag{20.1}$$

Set $r = (1-c_1-c_2)+d_1$. Use Lemma 18.4(12) at the first step, (20.1) at the second step, and Lemma 18.4(13) and $d_1(1-c_1-c_2) = 0$ at the third step, to get

$$1 \precsim_A (1-c_1-c_2) \oplus (c_1+c_2) \precsim_A (1-c_1-c_2) \oplus d_1 \sim_A (1-c_1-c_2)+d_1 = r.$$

Thus there is $v \in A$ such that $\|vrv^*-1\| < \frac{1}{2}$. It follows that $vr^{1/2}$ has a right inverse. But $vr^{1/2}d_2 = 0$, so $vr^{1/2}$ is not invertible. We have contradicted finiteness of A, and thus proved the lemma. □

Proof of Proposition 20.3. Let $a_1, a_2, \ldots, a_m \in A$, let $\varepsilon > 0$, let $x \in A_+ \setminus \{0\}$, and let $y \in B_+ \setminus \{0\}$. Without loss of generality $\|x\| = 1$.

Apply Lemma 20.5, obtaining $x_0 \in \big(\overline{xAx}\big)_+ \setminus \{0\}$ such that whenever $g \in A_+$ satisfies $0 \leq g \leq 1$ and $g \precsim_A x_0$, then $\|(1-g)x(1-g)\| > 1-\varepsilon$. Apply the hypothesis with x_0 in place of x and everything else as given, getting $c_1, c_2, \ldots, c_m \in A$ and $g \in B$. We need only prove that $\|(1-g)x(1-g)\| > 1-\varepsilon$. But this is immediate from the choice of x_0. □

The following strengthening of the definition is often convenient. First, we can always require $\|c_j\| \leq \|a_j\|$. Second, if we cut down on both sides instead of on one side, and the elements a_j are positive, then we may take the elements c_j to be positive.

Lemma 20.6 (Lemma 4.8 of [213])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $m, n \in \mathbb{Z}_{\geq 0}$, let $a_1, a_2, \ldots, a_m \in A$, let $b_1, b_2, \ldots, b_n \in A_+$, let $\varepsilon > 0$, let $x \in A_+$ satisfy $\|x\| = 1$, and let $y \in B_+ \setminus \{0\}$. Then there are $c_1, c_2, \ldots, c_m \in A$, $d_1, d_2, \ldots, d_n \in A_+$, and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$, and for $j = 1, 2, \ldots, n$ we have $\|d_j - b_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $\|c_j\| \leq \|a_j\|$, and for $j = 1, 2, \ldots, n$ we have $\|d_j\| \leq \|b_j\|$.
(4) For $j = 1, 2, \ldots, m$ we have $(1-g)c_j \in B$, and for $j = 1, 2, \ldots, n$ we have $(1-g)d_j(1-g) \in B$.
(5) $g \precsim_B y$ and $g \precsim_A x$.

(6) $\|(1-g)x(1-g)\| > 1-\varepsilon$.

Sketch of proof. To get $\|c_j\| \leq \|a_j\|$ for $j = 1, 2, \ldots, m$, one takes $\varepsilon > 0$ to be a bit smaller in the definition, and scales down c_j for any j for which $\|c_j\|$ is too big. Given that one can do this, following the definition, approximate

$$a_1,\, a_2,\, \ldots,\, a_m,\, b_1^{1/2},\, b_2^{1/2},\, \ldots,\, b_n^{1/2}$$

sufficiently well by

$$c_1, c_2, \ldots, c_m, r_1, r_2, \ldots, r_n,$$

and take $d_j = r_j r_j^*$ for $j = 1, 2, \ldots, n$. □

In Definition 4.9 of [213] we defined a "large subalgebra of crossed product type", a strengthening of the definition of a large subalgebra, and in Proposition 4.11 of [213] we gave a convenient way to verify that a subalgebra is a large subalgebra of crossed product type. The large subalgebras we have constructed in crossed products are of crossed product type. Theorem 4.6 of [8] shows that a large subalgebra of crossed product type is in fact centrally large. We will show directly (proof of Theorem 19.5, in Section 22 below) that if X is an infinite compact Hausdorff space, $h\colon X \to X$ is a minimal homeomorphism, and $Y \subset X$ is a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$, then the orbit breaking subalgebra $C^*(\mathbb{Z}, X, h)_Y$ of Definition 16.18 is centrally large in $C^*(\mathbb{Z}, X, h)$. This procedure is easier than using large subalgebras of crossed product type. The abstract version is more useful for subalgebras of crossed products by more complicated groups, but we don't consider these in these notes.

We now give proofs of two of the basic properties of large subalgebras above: if B is large in A, then B is simple (part of Proposition 19.6) and has the "same" tracial states as A (part of Theorem 19.7).

We start with the simplicity statement in Proposition 19.6.

Proposition 20.7 (Proposition 5.2 of [213])**.** Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then B is simple.

We need some preliminary work.

Lemma 20.8 (Lemma 1.12 of [213])**.** Let A be a C*-algebra, let $n \in \mathbb{Z}_{>0}$, and let $a_1, a_2, \ldots, a_n \in A$. Set $a = \sum_{k=1}^n a_k$. Then $a^*a \leq 2^{n-1} \sum_{k=1}^n a_k^* a_k$.

Proof. We prove this by induction on n. For $n = 1$, the statement is immediate. Suppose it is known for n; we prove it for $n+1$. Set $x = \sum_{k=1}^n a_k$. Then, expanding and cancelling at the third step, using the induction hypothesis at the fourth step, and using $n \geq 1$ at the fifth step, we get

$$\begin{aligned} a^*a &= (x + a_{n+1})^*(x + a_{n+1}) \leq (x + a_{n+1})^*(x + a_{n+1}) + (x - a_{n+1})^*(x - a_{n+1}) \\ &= 2x^*x + 2a_{n+1}^* a_{n+1} \leq 2^n \sum_{k=1}^n a_k^* a_k + 2a_{n+1}^* a_{n+1} \leq 2^n \sum_{k=1}^{n+1} a_k^* a_k. \end{aligned}$$

This completes the induction step and the proof. □

Lemma 20.9 (Lemma 1.13 of [213])**.** Let A be a C*-algebra and let $a \in A_+$. Let $b \in \overline{AaA}$ be positive. Then for every $\varepsilon > 0$ there exist $n \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_n \in A$ such that $\left\| b - \sum_{k=1}^n x_k^* a x_k \right\| < \varepsilon$.

This result is used without proof in the proof of Proposition 2.7(v) of [138]. We prove it when A is unital and $b = 1$, which is the case needed here. In this case, we can get $\sum_{k=1}^{n} x_k^* a x_k = 1$. In particular, we get Corollary 1.14 of [213] this way. (This result can also be obtained from Proposition 1.10 of [46], as pointed out after the proof of that proposition.)

Proof of Lemma 20.9 when $b = 1$. Choose

$$n \in \mathbb{Z}_{>0} \qquad \text{and} \qquad y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_n \in A$$

such that the element $c = \sum_{k=1}^{n} y_k a z_k$ satisfies $\|c - 1\| < 1$. Set

$$r = \sum_{k=1}^{n} z_k^* a y_k^* y_k a z_k, \quad M = \max\big(\|y_1\|, \|y_2\|, \ldots, \|y_n\|\big), \quad \text{and} \quad s = M^2 \sum_{k=1}^{n} z_k^* a^2 z_k.$$

Lemma 20.8 implies that $c^* c \in \overline{rAr}$. The relation $\|c - 1\| < 1$ implies that c is invertible, so r is invertible. Since $r \leq s$, it follows that s is invertible. Set $x_k = M a^{1/2} z_k s^{-1/2}$ for $k = 1, 2, \ldots, n$. Then $\sum_{k=1}^{n} x_k^* a x_k = s^{-1/2} s s^{-1/2} = 1$. $\square$

Sketch of proof of Proposition 20.7. Let $b \in B_+ \setminus \{0\}$. We show that there are $n \in \mathbb{Z}_{>0}$ and $r_1, r_2, \ldots, r_n \in B$ such that $\sum_{k=1}^{n} r_k b r_k^*$ is invertible.

Since A is simple, Lemma 20.9 provides $m \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_m \in A$ such that $\sum_{k=1}^{m} x_k b x_k^* = 1$. Set

$$M = \max\big(1, \|x_1\|, \|x_2\|, \ldots, \|x_m\|, \|b\|\big) \qquad \text{and} \qquad \delta = \min\left(1, \frac{1}{3mM(2M+1)}\right).$$

By definition, there are $y_1, y_2, \ldots, y_m \in A$ and $g \in B_+$ such that $0 \leq g \leq 1$, such that $\|y_j - x_j\| < \delta$ and $(1-g)y_j \in B$ for $j = 1, 2, \ldots, m$, and such that $g \precsim_B b$. Set $z = \sum_{k=1}^{m} y_j b y_j^*$. The number δ has been chosen to ensure that $\|z - 1\| < \frac{1}{3}$; the estimate is carried out in [213]. It follows that $\big\|(1-g)z(1-g) - (1-g)^2\big\| < \frac{1}{3}$.

Set $h = 2g - g^2$. Lemma 18.4(3), applied to the function $\lambda \mapsto 2\lambda - \lambda^2$, implies that $h \sim_B g$. Therefore $h \precsim_B b$. So there is $v \in B$ such that $\|vbv^* - h\| < \frac{1}{3}$. Now take $n = m + 1$, take $r_j = (1-g)y_j$ for $j = 1, 2, \ldots, m$, and take $r_{m+1} = v$. Then $r_1, r_2, \ldots, r_n \in B$. One can now check, using $(1-g)^2 + h = 1$, that $\|1 - \sum_{k=1}^{n} r_k b r_k^*\| < \frac{2}{3}$. Therefore $\sum_{k=1}^{n} r_k b r_k^*$ is invertible, as desired. $\square$

The following is a special case of the infinite dimensionality statement in Proposition 19.6 (Proposition 5.5 of [213]), which is easier to prove.

Proposition 20.10 (Stably finite case of Proposition 5.5 of [213])**.** Let A be a stably finite infinite dimensional simple unital C*-algebra and let $B \subset A$ be a large subalgebra. Then B is infinite dimensional.

Proof. Suppose B is finite dimensional. Proposition 20.7 tells us that B is simple, so there is $n \in \mathbb{Z}_{>0}$ such that $B \cong M_n$. It follows from the discussion after Theorem 3.3 of [29] that there is a quasitrace τ on A. Apply Corollary 18.11 to get $x \in A_+ \setminus \{0\}$ such that $d_\tau(x) < (n+1)^{-1}$. We may assume that $\|x\| = 1$. Clearly $B \neq A$, so there is $a \in A$ such that $\operatorname{dist}(a, B) > 1$. Apply Definition 19.1, getting $g \in B$ and $c \in A$ such that

$$0 \leq g \leq 1, \qquad \|a - c\| < \frac{1}{2}, \qquad (1-g)c \in B, \qquad \text{and} \qquad g \precsim_A x.$$

Then $c \notin B$, so $g \neq 0$. Also, $d_\tau(g) \leq d_\tau(x) < (n+1)^{-1}$. Now $\sigma = \tau|_B$ is a quasitrace on B, so must be the normalized trace on B, and $0 < d_\sigma(g) = d_\tau(g) < (n+1)^{-1}$.

There are no elements $g \in (M_n)_+$ with $0 < d_\sigma(g) < (n+1)^{-1}$, so we have a contradiction. $\square$

Proposition 20.11 (Corollary 5.8 of [213])**.** Let A be a stably finite infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $n \in \mathbb{Z}_{>0}$. Then $M_n(B)$ is large in $M_n(A)$.

In [213], this result is obtained as a corollary of a more general result (Proposition 19.4 here). A direct proof is easier, and we give it here.

Proof of Proposition 20.11. Let $m \in \mathbb{Z}_{>0}$, let $a_1, a_2, \ldots, a_m \in M_n(A)$, let $\varepsilon > 0$, let $x \in M_n(A)_+ \setminus \{0\}$, and let $y \in M_n(B)_+ \setminus \{0\}$. There are $b_{k,l} \in A$ for $k, l = 1, 2, \ldots, n$ such that

$$x^{1/2} = \sum_{k,l=1}^{n} e_{k,l} \otimes b_{k,l} \in M_n \otimes A.$$

Choose $k, l \in \{1, 2, \ldots, n\}$ such that $b_{k,l} \neq 0$. Set $x_0 = b_{k,l}^* b_{k,l} \in A_+ \setminus \{0\}$. Using selfadjointness of $x^{1/2}$, we find that

$$e_{1,1} \otimes x_0 = (e_{l,1} \otimes 1)^* x^{1/2} (e_{k,k} \otimes 1) x^{1/2} (e_{l,1} \otimes 1) \leq (e_{l,1} \otimes 1)^* x (e_{l,1} \otimes 1) \precsim_A x.$$

Similarly, there is $y_0 \in B_+ \setminus \{0\}$ such that $e_{1,1} \otimes y_0 \precsim_B y$.

Use Lemma 18.10 and simplicity (Proposition 20.7) and infinite dimensionality (Proposition 20.10) of B to find systems of nonzero mutually orthogonal and mutually Cuntz equivalent positive elements

$$x_1, x_2, \ldots, x_n \in \overline{x_0 A x_0} \qquad \text{and} \qquad y_1, y_2, \ldots, y_n \in \overline{y_0 B y_0}.$$

For $j = 1, 2, \ldots, m$, choose elements $a_{j,k,l} \in A$ for $k, l = 1, 2, \ldots, n$ such that

$$a_j = \sum_{k,l=1}^{n} e_{k,l} \otimes a_{j,k,l} \in M_n \otimes A.$$

Apply Proposition 20.3 with mn^2 in place of m, with the elements $a_{j,k,l}$ in place of $a_1, a_2, \ldots, a_m$, with ε/n^2 in place of ε, with x_1 in place of x, and with y_1 in place of y, getting $g_0 \in A_+$ and $c_{j,k,l} \in A$ for $j = 1, 2, \ldots, m$ and $k, l = 1, 2, \ldots, n$. Define $c_j = \sum_{k,l=1}^{n} e_{k,l} \otimes c_{j,k,l}$ for $j = 1, 2, \ldots, m$ and define $g = 1 \otimes g_0$. It is clear that $0 \leq g \leq 1$, that $\|c_j - a_j\| < \varepsilon$ and $(1-g)c_j \in M_n(B)$ for $j = 1, 2, \ldots, m$. We have $g \precsim_A 1 \otimes x_1$ and $g \precsim_B 1 \otimes y_1$, so Lemma 18.4(1) and Lemma 18.4(13) imply that $g \precsim_A x_0$ and $g \precsim_B y_0$. Therefore $g \precsim_A x$ and $g \precsim_B y$. $\square$

We prove the statement about traces in Theorem 19.7, assuming that the algebras are stably finite (the interesting case).

Theorem 20.12 (Stably finite case of Theorem 6.2 of [213])**.** Let A be an infinite dimensional stably finite simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then the restriction map $\mathrm{T}(A) \to \mathrm{T}(B)$ is bijective.

Again, we need a lemma.

Lemma 20.13. Let A be an infinite dimensional simple unital C*-algebra, and let $B \subset A$ be a large subalgebra. Let $\tau \in \mathrm{T}(B)$. Then there exists a unique state ω on A such that $\omega|_B = \tau$.

Proof. Existence of ω follows from the Hahn-Banach Theorem.

For uniqueness, let ω_1 and ω_2 be states on A such that $\omega_1|_B = \omega_2|_B = \tau$, let $a \in A_+$, and let $\varepsilon > 0$. We prove that $|\omega_1(a) - \omega_2(a)| < \varepsilon$. Without loss of generality $\|a\| \leq 1$.

It follows from Proposition 20.7 and Proposition 20.10 that B is simple and infinite dimensional. So Corollary 18.11 provides $y \in B_+ \setminus \{0\}$ such that $d_\tau(y) < \frac{\varepsilon^2}{64}$ (for the particular choice of τ we are using). Use Lemma 20.6 to find $c \in A_+$ and $g \in B_+$ such that

$$\|c\| \leq 1, \quad \|g\| \leq 1, \quad \|c - a\| < \frac{\varepsilon}{4}, \quad (1-g)c(1-g) \in B, \qquad \text{and} \qquad g \precsim_B y.$$

For $j = 1, 2$, the Cauchy-Schwarz inequality gives

$$|\omega_j(rs)| \leq \omega_j(rr^*)^{1/2}\omega_j(s^*s)^{1/2} \tag{20.2}$$

for all $r, s \in A$. Also, by Lemma 18.4(3) we have $g^2 \sim_B g \precsim_B y$. Since $\|g^2\| \leq 1$ and $\omega_j|_B = \tau$ is a tracial state, it follows that $\omega_j(g^2) \leq d_\tau(y) < \frac{\varepsilon^2}{64}$. Using $\|c\| \leq 1$ and the Cauchy-Schwarz inequality, we then get

$$|\omega_j(gc)| \leq \omega_j(g^2)^{1/2}\omega_j(c^2)^{1/2} < \frac{\varepsilon}{8}$$

and

$$|\omega_j((1-g)cg)| \leq \omega_j\big((1-g)c^2(1-g)\big)^{1/2}\omega_j(g^2)^{1/2} < \frac{\varepsilon}{8}.$$

So

$$\begin{aligned}\big|\omega_j(c) - \tau((1-g)c(1-g))\big| &= \big|\omega_j(c) - \omega_j((1-g)c(1-g))\big| \\ &\leq |\omega_j(gc)| + |\omega_j((1-g)cg)| < \frac{\varepsilon}{4}.\end{aligned}$$

Also $|\omega_j(c) - \omega_j(a)| < \frac{\varepsilon}{4}$. So

$$\big|\omega_j(a) - \tau((1-g)c(1-g))\big| < \frac{\varepsilon}{2}.$$

Thus $|\omega_1(a) - \omega_2(a)| < \varepsilon$. ∎

The uniqueness statement in Lemma 20.13 is used to prove that the restriction map $\mathrm{T}(A) \to \mathrm{T}(B)$ is injective.

One might hope that Lemma 20.13 would enable the following idea for the proof that $\mathrm{T}(A) \to \mathrm{T}(B)$ is surjective.

We first observe that a state ω is tracial whenever $\omega(uau^*) = \omega(u)$ for all $a \in A$ and all unitaries $u \in A$. Indeed, putting au for a gives $\omega(ua) = \omega(au)$ for all $a \in A$ and all unitaries $u \in A$. Since A is the linear span of its unitaries, it follows that $\omega(ba) = \omega(ab)$ for all $a, b \in A$.

Now let A and B be as in Theorem 20.12, let $\tau \in \mathrm{T}(B)$, and $u \in A$. Let ω be the unique state on A which extends τ (Lemma 20.13). We would like to argue that the state $\rho(a) = \omega(uau^*)$ for $a \in A$ is equal to ω because it also extends τ. The first thing which goes wrong is that if $b \in B$ and $u \in A$ is unitary, then ubu^* need not even be in B. So the is no immediate reason to think that ρ extends τ.

If the unitary u is actually in B, then ρ does indeed extend ω. Thus, the uniqueness statement in Lemma 20.13 implies that $\omega(uau^*) = \omega(a)$ for all $a \in A$ and all unitaries $u \in B$. We can still replace a by au as above, and deduce that $\omega(ba) = \omega(ab)$ for all $a \in A$ and $b \in B$. In particular, $\omega(vb) = \omega(bv)$ for all $b \in B$

and unitaries $v \in A$. But to get $\omega(vbv^*) = \omega(b)$ from this requires putting bv^* in place of b, and bv^* isn't in B.

Proof of Theorem 20.12. Let $\tau \in \mathrm{T}(B)$. We show that there is a unique $\omega \in \mathrm{T}(A)$ such that $\omega|_B = \tau$. Lemma 20.13 shows that there is a unique state ω on A such that $\omega|_B = \tau$, and it suffices to show that ω is a trace. Thus let $a_1, a_2 \in A$ satisfy $\|a_1\| \leq 1$ and $\|a_2\| \leq 1$, and let $\varepsilon > 0$. We show that $|\omega(a_1 a_2) - \omega(a_2 a_1)| < \varepsilon$.

It follows from Proposition 20.7 and Proposition 20.10 (without stable finiteness, we must appeal to Proposition 5.5 of [213]) that B is simple and infinite dimensional. So Corollary 18.11 provides $y \in B_+ \setminus \{0\}$ such that $d_\tau(y) < \frac{\varepsilon^2}{64}$. Use Lemma 20.6 to find $c_1, c_2 \in A$ and $g \in B_+$ such that

$$\|c_j\| \leq 1, \quad \|c_j - a_j\| < \frac{\varepsilon}{8}, \qquad \text{and} \qquad (1-g)c_j \in B$$

for $j = 1, 2$, and such that $\|g\| \leq 1$ and $g \precsim_B y$. By Lemma 18.4(3), we have $g^2 \sim g \precsim_B y$. Since $\|g^2\| \leq 1$ and $\omega|_B = \tau$ is a tracial state, it follows that $\omega(g^2) \leq d_\tau(y) < \frac{\varepsilon^2}{64}$.

We claim that

$$\big|\omega((1-g)c_1(1-g)c_2) - \omega(c_1 c_2)\big| < \frac{\varepsilon}{4}.$$

Using the Cauchy-Schwarz inequality ((20.2) in the previous proof), we get

$$|\omega(g c_1 c_2)| \leq \omega(g^2)^{1/2} \omega(c_2^* c_1^* c_1 c_2)^{1/2} \leq \omega(g^2)^{1/2} < \frac{\varepsilon}{8}.$$

Similarly, and also at the second step using $\|c_2\| \leq 1$, $(1-g)c_1 g \in B$, and the fact that $\omega|_B$ is a tracial state,

$$\begin{aligned}\big|\omega((1-g)c_1 g c_2)\big| &\leq \omega\big((1-g)c_1 g^2 c_1^*(1-g)\big)^{1/2} \omega(c_2^* c_2)^{1/2} \\ &\leq \omega\big(g c_1^*(1-g)^2 c_1 g\big)^{1/2} \leq \omega(g^2)^{1/2} < \frac{\varepsilon}{8}.\end{aligned}$$

The claim now follows from the estimate

$$\big|\omega((1-g)c_1(1-g)c_2) - \omega(c_1 c_2)\big| \leq \big|\omega((1-g)c_1 g c_2)\big| + |\omega(g c_1 c_2)|.$$

Similarly

$$\big|\omega((1-g)c_2(1-g)c_1) - \omega(c_2 c_1)\big| < \frac{\varepsilon}{4}.$$

Since $(1-g)c_1$, $(1-g)c_2 \in B$ and $\omega|_B$ is a tracial state, we get

$$\omega((1-g)c_1(1-g)c_2) = \omega((1-g)c_2(1-g)c_1).$$

Therefore $|\omega(c_1 c_2) - \omega(c_2 c_1)| < \frac{\varepsilon}{2}$.

One checks that $\|c_1 c_2 - a_1 a_2\| < \frac{\varepsilon}{4}$ and $\|c_2 c_1 - a_2 a_1\| < \frac{\varepsilon}{4}$. It now follows that $|\omega(a_1 a_2) - \omega(a_2 a_1)| < \varepsilon$. ∎

21. Large Subalgebras and the Radius of Comparison

Let A be a simple unital C*-algebra. Recall (Definition 11.34) that the order on projections over A is determined by traces if, as happens for type II_1 factors, whenever $p, q \in M_\infty(A)$ are projections such that for all $\tau \in \mathrm{T}(A)$ we have $\tau(p) < \tau(q)$, then p is Murray-von Neumann equivalent to a subprojection of q. Without knowing whether every quasitrace is a trace (see the discussion before Notation 18.6), it is more appropriate to use a condition involving quasitraces. For exact C*-algebras, every quasitrace is a trace (Theorem 5.11 of [102]), so it makes no difference.

Simple C*-algebras need not have very many projections, so a more definitive version of this condition is to ask for the condition in the following definition.

Definition 21.1. Let A be a simple unital C*-algebra. Then A has *strict comparison of positive elements* if whenever $a, b \in M_\infty(A)$ satisfy $d_\tau(a) < d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$, then $a \precsim_A b$.

By Proposition 6.12 of [213], one can use $K \otimes A$ in place of $M_\infty(A)$, but this is not as easy to see as with projections.

Simple AH algebras with slow dimension growth have strict comparison, but other simple AH algebras need not. (For example, see [284].) Strict comparison seems to be necessary for any reasonable hope of classification in the sense of the Elliott program. According to the Toms-Winter Conjecture, when A is simple, separable, nuclear, unital, and stably finite, strict comparison should imply Z-stability, and this is known to hold in a number of cases.

The radius of comparison $\mathrm{rc}(A)$ of A (for a C*-algebra which is unital and stably finite but not necessarily simple) measures the failure of strict comparison. (See [28] for what to do in more general C*-algebras.) For additional context, we point out the following special case of Theorem 5.1 of [285] (which will be needed in Section 23, where it is restated as Theorem 23.28): if X is a compact metric space and $n \in \mathbb{Z}_{>0}$, then

$$\mathrm{rc}(M_n \otimes C(X)) \leq \frac{\dim(X) - 1}{2n}.$$

Under some conditions on X (being a finite complex is enough), this inequality is at least approximately an equality. See [72].

The following definition of the radius of comparison is adapted from Definition 6.1 of [283].

Definition 21.2. Let A be a stably finite unital C*-algebra.

(1) Let $r \in [0, \infty)$. We say that A has *r-comparison* if whenever $a, b \in M_\infty(A)_+$ satisfy $d_\tau(a) + r < d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$, then $a \precsim_A b$.

(2) The *radius of comparison* of A, denoted $\mathrm{rc}(A)$, is

$$\mathrm{rc}(A) = \inf\big(\big\{r \in [0, \infty) \colon A \text{ has } r\text{-comparison}\big\}\big).$$

(We take $\mathrm{rc}(A) = \infty$ if there is no r such that A has r-comparison.)

Definition 6.1 of [283] actually uses lower semicontinuous dimension functions on A instead of d_τ for $\tau \in \mathrm{QT}(A)$, but these are the same functions by Theorem II.2.2 of [25]. It is also stated in terms of the order on the Cuntz semigroup $W(A)$ rather than in terms of Cuntz subequivalence; this is clearly equivalent.

We also note (Proposition 6.3 of [283]) that if every element of $\mathrm{QT}(A)$ is faithful, then the infimum in Definition 21.2(2) is attained, that is, A has $\mathrm{rc}(A)$-comparison. In particular, this is true when A is simple. (See Lemma 1.23 of [213].)

We warn that r-comparison and $\mathrm{rc}(A)$ are sometimes defined using tracial states rather than quasitraces.

It is equivalent to use $K \otimes A$ in place of $M_\infty(A)$. See Proposition 6.12 of [213].

We prove here the following special case of Theorem 19.9.

Theorem 21.3. Let A be an infinite dimensional stably finite simple separable unital exact C*-algebra. Let $B \subset A$ be a large subalgebra. Then $\mathrm{rc}(A) = \mathrm{rc}(B)$.

The extra assumption is that A is exact, so that every quasitrace is a trace by Theorem 5.11 of [102].

We will give a proof directly from the definition of a large subalgebra. We describe the heuristic argument, using the following simplifications:

(1) The algebra A, and therefore also B, has a unique tracial state τ.
(2) We consider elements of A_+ and B_+ instead of elements of $M_\infty(A)_+$ and $M_\infty(B)_+$.
(3) For $a \in A_+$, when applying the definition of a large subalgebra (Definition 19.1), instead of getting $(1-g)c(1-g) \in B$ for some $c \in A_+$ which is close to a, we can actually get $(1-g)a(1-g) \in B$. Similarly, for $a \in A$ we can get $(1-g)a \in B$.
(4) For $a, b \in A_+$ with $a \precsim_A b$, we can find $v \in A$ such that $v^*bv = a$ (not just such that $\|v^*bv - a\|$ is small).
(5) None of the elements we encounter are Cuntz equivalent to projections, that is, we never encounter anything for which 0 is an isolated point of, or not in, the spectrum.

The most drastic simplification is (3). In the actual proof, to compensate for the fact that we only get approximation, we will need to make systematic use of elements $(a-\varepsilon)_+$ for carefully chosen, and varying, values of $\varepsilon > 0$. Avoiding this complication gives a much better view of the idea behind the argument, and the usefulness of large subalgebras in general.

We first consider the inequality $\mathrm{rc}(A) \leq \mathrm{rc}(B)$. So let $a, b \in A_+$ satisfy $d_\tau(a) + \mathrm{rc}(B) < d_\tau(b)$. The essential idea is to replace b by something slightly smaller which is in B_+, say y, and replace a by something slightly larger which is in B_+, say x, in such a way that we still have $d_\tau(x) + \mathrm{rc}(B) < d_\tau(y)$. Then use the definition of $\mathrm{rc}(B)$. With g sufficiently small in the sense of Cuntz comparison, we will take $y = (1-g)b(1-g)$ and (following Lemma 18.5) $x = (1-g)a(1-g) \oplus g$.

Choose $\delta > 0$ such that

$$d_\tau(a) + \mathrm{rc}(B) + \delta \leq d_\tau(b). \tag{21.1}$$

Applying (3) of our simplification, we can find $g \in B$ with $0 \leq g \leq 1$, such that

$$(1-g)a(1-g) \in B \qquad \text{and} \qquad (1-g)b(1-g) \in B,$$

and so small in $W(A)$ that

$$d_\tau(g) < \frac{\delta}{3}. \tag{21.2}$$

Using Lemma 18.4(4) at the first step, we get

$$(1-g)b(1-g) \sim_A b^{1/2}(1-g)^2 b^{1/2} \leq b,$$

so

$$(1-g)b(1-g) \precsim_A b. \tag{21.3}$$

Similarly, $(1-g)a(1-g) \precsim_A a$, and this relation implies

$$d_\tau\big((1-g)a(1-g)\big) \leq d_\tau(a). \tag{21.4}$$

Also, $b \precsim_A (1-g)b(1-g) \oplus g$ by Lemma 18.5, so

$$d_\tau\big((1-g)b(1-g)\big) + d_\tau(g) \geq d_\tau(b). \tag{21.5}$$

Using (21.4) at the first step, using (21.1) at the second step, using (21.5) at the third step, and using (21.2) at the fourth step, we get

$$\begin{aligned} d_\tau\big((1-g)a(1-g)\oplus g\big) + \mathrm{rc}(B) + \frac{\delta}{3} &\leq d_\tau(a) + d_\tau(g) + \mathrm{rc}(B) + \frac{\delta}{3} \\ &\leq d_\tau(b) + d_\tau(g) - \frac{2\delta}{3} \\ &\leq d_\tau\big((1-g)b(1-g)\big) + 2d_\tau(g) - \frac{2\delta}{3} \\ &\leq d_\tau\big((1-g)b(1-g)\big). \end{aligned}$$

So, by the definition of $\mathrm{rc}(B)$,

$$(1-g)a(1-g)\oplus g \precsim_B (1-g)b(1-g).$$

Therefore, using Lemma 18.5 at the first step and (21.3) at the third step, we get

$$a \precsim_A (1-g)a(1-g)\oplus g \precsim_B (1-g)b(1-g) \precsim_A b,$$

that is, $a \precsim_A b$, as desired.

Now we consider the inequality $\mathrm{rc}(A) \geq \mathrm{rc}(B)$. Let $a, b \in B_+$ satisfy $d_\tau(a) + \mathrm{rc}(A) < d_\tau(b)$. Choose $\delta > 0$ such that $d_\tau(a) + \mathrm{rc}(A) + \delta \leq d_\tau(b)$. By lower semicontinuity of d_τ, we always have

$$d_\tau(b) = \sup_{\varepsilon>0} d_\tau\big((b-\varepsilon)_+\big).$$

So there is $\varepsilon > 0$ such that

$$d_\tau\big((b-\varepsilon)_+\big) > d_\tau(a) + \mathrm{rc}(A). \tag{21.6}$$

Define a continuous function $f\colon [0,\infty) \to [0,\infty)$ by $f(\lambda) = \max(0, \varepsilon^{-1}\lambda(\varepsilon-\lambda))$ for $\lambda \in [0,\infty)$. Then $f(b)$ and $(b-\varepsilon)_+$ are orthogonal positive elements such that $f(b) \neq 0$ (by (5)) and $f(b) + (b-\varepsilon)_+ \leq b$. We have $a \precsim_A (b-\varepsilon)_+$ by (21.6) and the definition of $\mathrm{rc}(A)$. Applying (4) of our simplification, we can find $v \in A$ such that $v^*(b-\varepsilon)_+ v = a$. Applying (3) of our simplification, we can find $g \in B$ with $0 \leq g \leq 1$ such that $(1-g)v^* \in B$ and $g \precsim_B f(b)$. Since

$$v(1-g) \in B \qquad \text{and} \qquad [v(1-g)]^*(b-\varepsilon)_+[v(1-g)] = (1-g)a(1-g),$$

we get $(1-g)a(1-g) \precsim_B (b-\varepsilon)_+$. Therefore, using Lemma 18.5 at the first step,

$$a \precsim_B (1-g)a(1-g)\oplus g \precsim_B (b-\varepsilon)_+ \oplus g \precsim_B (b-\varepsilon)_+ \oplus f(b) \precsim_B b,$$

as desired.

The original proof of Theorem 21.3 followed the heuristic arguments above, and this is the proof we give below. The proof in [213] uses the same basic ideas, but gives much more. The heuristic arguments above are the basis for the technical results in Lemma 19.14. In [213], these are used to prove Theorem 19.8, which states that, after deleting the classes of the nonzero projections from the Cuntz semigroups $\mathrm{Cu}(B)$ and $\mathrm{Cu}(A)$, the inclusion of B in A is an order isomorphism on what remains. (The inclusion need not be an isomorphism if the classes of the nonzero projections are included. See Example 7.13 of [213].) In Section 3 of [213], it is shown that, in our situation, the part of the Cuntz semigroup without the classes of the nonzero projections is enough to determine the quasitraces, so that the restriction map $\mathrm{QT}(A) \to \mathrm{QT}(B)$ is bijective. It follows that the radius of comparison in this part of the Cuntz semigroup is the same for both A and B, and it turns out that the radius of comparison in this part of the Cuntz semigroup is the same as in the entire Cuntz semigroup.

We will use the characterizations of $\mathrm{rc}(A)$ in the following theorem, which is a special case of results in [28]. The difference between (1) and (2) is that (2) has $n+1$ in one of the places where (1) has n. This result substitutes for the observation that if $a, b \in A_{+}$ satisfy $\tau(a) < \tau(b)$ for all $\tau \in \mathrm{QT}(A)$, then, by compactness of $\mathrm{QT}(A)$ and continuity, we have $\inf_{\tau \in \mathrm{QT}(A)} [\tau(b) - \tau(a)] > 0$. The difficulty is that we need an analog using d_{τ} instead of τ, and $\tau \mapsto d_{\tau}(a)$ is in general only lower semicontinuous, so that $\tau \mapsto d_{\tau}(b) - d_{\tau}(a)$ may be neither upper nor lower semicontinuous.

Unfortunately, the results in [28] are stated in terms of $\mathrm{Cu}(A)$ rather than $W(A)$.

Theorem 21.4. Let A be a stably finite simple unital C*-algebra. Then:

(1) The radius of comparison $\mathrm{rc}(A)$ is the least number $s \in [0, \infty]$ such that whenever $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > s$, and $a, b \in M_{\infty}(A)_{+}$ satisfy
$$n \langle a \rangle_A + m \langle 1 \rangle_A \leq n \langle b \rangle_A$$
in $W(A)$, then $a \precsim_A b$.

(2) The radius of comparison $\mathrm{rc}(A)$ is the least number $t \in [0, \infty]$ such that whenever $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > t$, and $a, b \in M_{\infty}(A)_{+}$ satisfy
$$(n+1) \langle a \rangle_A + m \langle 1 \rangle_A \leq n \langle b \rangle_A$$
in $W(A)$, then $a \precsim_A b$.

Proof. It is easy to check that there is in fact a least $s \in [0, \infty]$ satisfying the condition in (1), and similarly that there is a least $t \in [0, \infty]$ as in (2).

We will first prove this for $K \otimes A$ and $\mathrm{Cu}(A)$ in place of $M_{\infty}(A)$ and $W(A)$. So let s_0 and t_0 be the numbers defined as in (1) and (2), except with $K \otimes A$ and $\mathrm{Cu}(A)$ in place of $M_{\infty}(A)$ and $W(A)$. Again, it is clear that there are least such numbers s_0 and t_0. Clearly $s_0 \geq t_0$. Since A is simple and stably finite and $\langle 1 \rangle_A$ is a full element of $\mathrm{Cu}(A)$, Proposition 3.2.3 of [28], the preceding discussion in [28], and Definition 3.2.2 of [28] give $t_0 = \mathrm{rc}(A)$. So we need to show that $s_0 \leq t_0$.

We thus assume $m, n \in \mathbb{Z}_{>0}$ and $m/n > t_0$, and that $a, b \in (K \otimes A)_{+}$ satisfy $n \langle a \rangle_A + m \langle 1 \rangle_A \leq n \langle b \rangle_A$ in $\mathrm{Cu}(A)$. We must prove that $a \precsim_A b$. For any functional ω on $\mathrm{Cu}(A)$ (as at the beginning of Section 2.4 of [28]), we have $n\omega(\langle a \rangle_A) + m\omega(\langle 1 \rangle_A) \leq n\omega(\langle b \rangle_A)$, so $\omega(\langle a \rangle_A) + (m/n)\omega(\langle 1 \rangle_A) \leq \omega(\langle b \rangle_A)$. Since $m/n > t_0$, Proposition 3.2.1 of [28] implies that $a \precsim_A b$.

It remains to prove that $s_0 = s$ and $t_0 = t$. We prove that $s_0 = s$; the proof that $t_0 = t$ is the same. Let $m, n \in \mathbb{Z}_{>0}$. We have to prove the following. Suppose that m and n have the property that whenever $a, b \in M_{\infty}(A)_{+}$ satisfy
$$n \langle a \rangle_A + m \langle 1 \rangle_A \leq n \langle b \rangle_A$$
in $W(A)$, then $a \precsim_A b$. Then whenever $a, b \in (K \otimes A)_{+}$ satisfy
$$n \langle a \rangle_{K \otimes A} + m \langle 1 \rangle_{K \otimes A} \leq n \langle b \rangle_{K \otimes A}$$
in $\mathrm{Cu}(A)$, we have $a \precsim_{K \otimes A} b$. We also need to prove the reverse implication.

The reverse implication is easy, so we prove the forwards implication. Let $a, b \in (K \otimes A)_{+}$ satisfy
$$n \langle a \rangle_A + m \langle 1 \rangle_A \leq n \langle b \rangle_A$$

in $\mathrm{Cu}(A)$. Let $\varepsilon > 0$; by Lemma 18.4(11), it suffices to prove that $(a-\varepsilon)_+ \precsim_{K\otimes A} b$. We may clearly assume that $\varepsilon < 1$. Using an isomorphism $K \otimes K \to K$, let $x \in (K \otimes A)_+$ be the direct sum of n copies of a, let $y \in (K \otimes A)_+$ be the direct sum of n copies of b, and let $q \in (K \otimes A)_+$ be the direct sum of m copies of the identity of A. The relation $n\langle a\rangle_{K\otimes A} + m\langle 1\rangle_{K\otimes A} \leq n\langle b\rangle_{K\otimes A}$ means that $x \oplus q \precsim_{K\otimes A} y$. By Lemma 18.4(11c), there exists $\delta > 0$ such that

$$\big((x \oplus q) - \varepsilon\big)_+ \precsim_{K\otimes A} (y - \delta)_+.$$

Since $\varepsilon < 1$ and q is a projection, this relation is equivalent to

$$(x - \varepsilon)_+ \oplus q \precsim_{K\otimes A} (y - \delta)_+.$$

Since $(x - \varepsilon)_+$ is the direct sum of n copies of $(a - \varepsilon)_+$ and $(y - \delta)_+$ is the direct sum of n copies of $(b - \delta)_+$, we therefore have

$$n\langle (a - \varepsilon)_+\rangle_{K\otimes A} + m\langle 1\rangle_{K\otimes A} \leq n\langle (b - \delta)_+\rangle_{K\otimes A}.$$

It follows from Lemma 1.9 of [213] that $\langle (a-\varepsilon)_+\rangle_{K\otimes A}$ and $\langle (b-\delta)_+\rangle_{K\otimes A}$ are actually classes of elements $c, d \in M_\infty(A)_+$, and it is easy to check that inequalities among classes in $W(A)$ which hold in $\mathrm{Cu}(A)$ must also hold in $W(A)$. The assumption therefore implies that $c \precsim_A d$. Thus

$$(a - \varepsilon)_+ \sim_{K\otimes A} c \precsim_A d \sim_{K\otimes A} (b - \delta)_+ \leq b,$$

whence $(a - \varepsilon)_+ \precsim_{K\otimes A} b$, as desired. ☐

Lemma 21.5. Let $M \in (0, \infty)$, let $f\colon [0, \infty) \to \mathbb{C}$ be a continuous function such that $f(0) = 0$, and let $\varepsilon > 0$. Then there is $\delta > 0$ such that whenever A is a C*-algebra and $a, b \in A_{\mathrm{sa}}$ satisfy $\|a\| \leq M$ and $\|a - b\| < \delta$, then $\|f(a) - f(b)\| < \varepsilon$.

This is a standard polynomial approximation argument. We have not found it written in the literature. There are similar arguments in [213] and many other places. It is also stated (in a slightly different form) as Lemma 2.5.11(2) of [152]; the proof there is left to the reader (although a related proof is given). We therefore give it for completeness.

Proof of Lemma 21.5. Choose $n \in \mathbb{Z}_{>0}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ such that the polynomial function $g(\lambda) = \sum_{k=1}^{n} \alpha_k \lambda^k$ satisfies $|g(\lambda) - f(\lambda)| < \frac{\varepsilon}{3}$ for $\lambda \in [-M-1, M+1]$. Define

$$\delta = \min\left(1, \frac{\varepsilon}{1 + 3\sum_{k=1}^{n} |\alpha_k| k (M+1)^{k-1}}\right).$$

Now let A be a C*-algebra and let $a, b \in A_{\mathrm{sa}}$ satisfy $\|a\| \leq M$ and $\|a - b\| < \delta$. Then $\|b\| \leq M + 1$. So for $m \in \mathbb{Z}_{>0}$ we have

$$\|a^m - b^m\| \leq \sum_{k=1}^{m} \|a^{k-1}\| \cdot \|a - b\| \cdot \|b^{m-k}\| < m(M+1)^{m-1}\delta.$$

Therefore

$$\|g(a) - g(b)\| \leq \sum_{k=1}^{n} |\alpha_k| k (M+1)^{k-1}\delta < \frac{\varepsilon}{3}.$$

So

$$\|f(a) - f(b)\| \leq \|f(a) - g(a)\| + \|g(a) - g(b)\| + \|g(b) - f(b)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This completes the proof. ☐

Proposition 21.6. Let A be an infinite dimensional stably finite simple separable unital exact C*-algebra. Let $B \subset A$ be a large subalgebra. Then $\mathrm{rc}(A) \leq \mathrm{rc}(B)$.

Proof. We use the criterion of Theorem 21.4(1). Thus, let $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > \mathrm{rc}(B)$, and let $a, b \in M_{\infty}(A)_{+}$ satisfy $n\langle a\rangle_A + m\langle 1\rangle_A \leq n\langle b\rangle_A$ in $W(A)$. We want to prove that $a \precsim_A b$. Without loss of generality $\|a\|, \|b\| \leq 1$. It suffices to prove that $(a-\varepsilon)_{+} \precsim_A b$ for every $\varepsilon > 0$.

So let $\varepsilon > 0$. We may assume $\varepsilon < 1$. Let $x \in M_{\infty}(A)_{+}$ be the direct sum of n copies of a, let $y \in M_{\infty}(A)_{+}$ be the direct sum of n copies of b, and let $q \in M_{\infty}(A)_{+}$ be the direct sum of m copies of the identity of A. The relation $n\langle a\rangle_A + m\langle 1\rangle_A \leq n\langle b\rangle_A$ means that $x \oplus q \precsim_A y$. By Lemma 18.4(11b), there exists $\delta > 0$ such that

$$\left((x \oplus q) - \tfrac{1}{3}\varepsilon\right)_{+} \precsim_A (y-\delta)_{+}.$$

Since $\varepsilon < 3$, this is equivalent to

$$\left(x - \tfrac{1}{3}\varepsilon\right)_{+} \oplus q \precsim_A (y-\delta)_{+}. \tag{21.7}$$

Choose $l \in \mathbb{Z}_{>0}$ so large that $a, b \in M_l \otimes A$. Since $m/n > \mathrm{rc}(B)$, there is $k \in \mathbb{Z}_{>0}$ such that

$$\mathrm{rc}(B) < \frac{m}{n} - \frac{2}{k}.$$

Set

$$\varepsilon_0 = \min\left(\tfrac{1}{3}\varepsilon, \tfrac{1}{2}\delta\right).$$

Using Lemma 21.5, choose $\varepsilon_1 > 0$ with $\varepsilon_1 \leq \varepsilon_0$ and so small that whenever D is a C*-algebra and $z \in D_{+}$ satisfies $\|z\| \leq 1$, then $\|z_0 - z\| < \varepsilon_1$ implies

$$\|(z_0-\varepsilon_0)_{+} - (z-\varepsilon_0)_{+}\| < \varepsilon_0, \quad \left\|\left(z_0 - \tfrac{1}{3}\varepsilon\right)_{+} - \left(z - \tfrac{1}{3}\varepsilon\right)_{+}\right\| < \varepsilon_0,$$

and

$$\left\|\left(z_0 - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right)_{+} - \left(z - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right)_{+}\right\| < \varepsilon_0.$$

Since A is infinite dimensional and simple, Lemma 18.10 provides $z \in A_{+} \setminus \{0\}$ such that $(k+1)\langle z\rangle_A \leq \langle 1\rangle_A$. Using Proposition 20.11 and Lemma 20.6, choose $g \in M_l(B)_{+}$ and $a_0, b_0 \in M_l(A)_{+}$ satisfying

$$0 \leq g, a_0, b_0 \leq 1, \quad \|a_0 - a\| < \varepsilon_1, \quad \|b_0 - b\| < \varepsilon_1, \quad g \precsim_A z,$$

and such that

$$(1-g)a_0(1-g),\ (1-g)b_0(1-g) \in M_l \otimes B.$$

From $g \precsim_A z$ and $(k+1)\langle z\rangle_A \leq \langle 1\rangle_A$ we get

$$\sup_{\tau \in \mathrm{T}(A)} d_{\tau}(g) < \frac{1}{k}. \tag{21.8}$$

Set

$$a_1 = \left[(1-g)a_0(1-g) - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_{+} \quad \text{and} \quad b_1 = \left[(1-g)b_0(1-g) - \varepsilon_0\right]_{+},$$

which are in $M_l \otimes B$. We claim that a_0, a_1, b_0, and b_1 have the following properties:

(1) $(a-\varepsilon)_{+} \precsim_A \left[a_0 - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_{+}$.
(2) $\left[a_0 - \left(\varepsilon_0 + \tfrac{1}{3}\varepsilon\right)\right]_{+} \precsim_B a_1 \oplus g$.
(3) $a_1 \precsim_A \left(a - \tfrac{1}{3}\varepsilon\right)_{+}$.
(4) $(b-\delta)_{+} \precsim_A (b_0 - \varepsilon_0)_{+}$.
(5) $(b_0 - \varepsilon_0)_{+} \precsim_B b_1 \oplus g$.
(6) $b_1 \precsim_A b$.

We give full details of the proofs for (1), (2), and (3) (involving a_0 and a_1). The proofs for (4), (5), and (6) (involving b_0 and b_1) are a bit more sketchy.

We prove (1). Since $\|a_0 - a\| < \varepsilon_1$, the choice of ε_1 implies

$$\left\| \left[a_0 - \left(\tfrac{1}{3}\varepsilon + \varepsilon_0\right)\right]_+ - \left[a - \left(\tfrac{1}{3}\varepsilon + \varepsilon_0\right)\right]_+ \right\| < \varepsilon_0 \leq \tfrac{1}{3}\varepsilon.$$

At the last step in the following computation use this inequality and Lemma 18.4(10), at the first step use $\varepsilon_0 \leq \frac{1}{3}\varepsilon$, and at the second step use Lemma 18.4(8), to get

$$\begin{aligned} (a-\varepsilon)_+ &\leq \left[a - \left(\tfrac{2}{3}\varepsilon + \varepsilon_0\right)\right]_+ \\ &= \left[\left(a - \left(\tfrac{1}{3}\varepsilon + \varepsilon_0\right)\right)_+ - \tfrac{1}{3}\varepsilon\right]_+ \precsim_A \left[a_0 - \left(\tfrac{1}{3}\varepsilon + \varepsilon_0\right)\right]_+. \end{aligned}$$

For (4) (the corresponding argument for b_0), we use $\varepsilon_0 \leq \frac{1}{2}\delta$ at the first step; since

$$\|(b - \varepsilon_0)_+ - (b_0 - \varepsilon_0)_+\| < \varepsilon_0,$$

we get

$$(b-\delta)_+ \leq (b - 2\varepsilon_0)_+ = \left[(b-\varepsilon_0)_+ - \varepsilon_0\right]_+ \precsim_A (b_0 - \varepsilon_0)_+.$$

For (2), we use Lemma 18.5 with a_0 in place of a and with $\frac{1}{3}\varepsilon + \varepsilon_0$ in place of ε. For (5), we use Lemma 18.5 with b_0 in place of a and with ε_0 in place of ε.

For (3), begin by recalling that $\|a_0 - a\| < \varepsilon_1$, whence

$$\|(1-g)a_0(1-g) - (1-g)a(1-g)\| < \varepsilon_1.$$

Therefore

$$\left\| \left[(1-g)a_0(1-g) - \tfrac{1}{3}\varepsilon\right]_+ - \left[(1-g)a(1-g) - \tfrac{1}{3}\varepsilon\right]_+ \right\| < \varepsilon_0.$$

Using Lemma 18.4(8) at the first step, this fact and Lemma 18.4(10) at the second step, Lemma 18.4(6) at the third step, and Lemma 18.4(17) and $a^{1/2}(1-g)^2 a^{1/2} \leq a$ at the last step, we get

$$\begin{aligned} a_1 &= \left[\left[(1-g)a_0(1-g) - \tfrac{1}{3}\varepsilon\right]_+ - \varepsilon_0\right]_+ \\ &\precsim_A \left[(1-g)a(1-g) - \tfrac{1}{3}\varepsilon\right]_+ \sim_A \left[a^{1/2}(1-g)^2 a^{1/2} - \tfrac{1}{3}\varepsilon\right]_+ \precsim_A \left(a - \tfrac{1}{3}\varepsilon\right)_+, \end{aligned}$$

as desired.

For (6) (the corresponding part involving b_1), just use

$$\|(1-g)b_0(1-g) - (1-g)b(1-g)\| < \varepsilon_1 \leq \varepsilon_0$$

to get, using Lemma 18.4(4) at the second step,

$$b_1 \precsim_A (1-g)b(1-g) \sim_A b^{1/2}(1-g)^2 b^{1/2} \leq b.$$

The claims (1)–(6) are now proved.

Now let $\tau \in \mathrm{T}(A)$. Recall that x and y are the direct sums of n copies of a and b. Therefore $\left(x - \frac{1}{3}\varepsilon\right)_+$ is the direct sum of n copies of $\left(a - \frac{1}{3}\varepsilon\right)_+$ and $(y-\delta)_+$ is the direct sum of n copies of $(b-\delta)_+$. So the relation (21.7) implies

$$n \cdot d_\tau\left(\left(a - \tfrac{1}{3}\varepsilon\right)_+\right) + m \leq n \cdot d_\tau\left((b-\delta)_+\right). \tag{21.9}$$

Using (4) and (5) at the first step and (21.8) at the third step, we get the estimate

$$d_\tau\left((b-\delta)_+\right) \leq d_\tau(b_1) + d_\tau(g) < d_\tau(b_1) + k^{-1}. \tag{21.10}$$

The relation (3) implies

$$d_\tau(a_1) \leq d_\tau\left(\left(a - \tfrac{1}{3}\varepsilon\right)_+\right). \tag{21.11}$$

Using (21.11) and (21.8) at the second step, (21.9) at the third step, and (21.10) at the fourth step, we get

$$\begin{aligned} n \cdot d_\tau(a_1 \oplus g) + m &= n \cdot d_\tau(a_1) + m + n \cdot d_\tau(g) \\ &\leq n \cdot d_\tau\big(\big(a - \tfrac{1}{3}\varepsilon\big)_+\big) + m + nk^{-1} \\ &\leq n \cdot d_\tau\big((b - \delta)_+\big) + nk^{-1} \\ &\leq n \cdot d_\tau(b_1) + 2nk^{-1}. \end{aligned}$$

It follows that

$$d_\tau(a_1 \oplus g) + \frac{m}{n} - \frac{2}{k} \leq d_\tau(b_1).$$

This holds for all $\tau \in \mathrm{T}(A)$, and therefore, by Theorem 20.12, for all $\tau \in \mathrm{T}(B)$.

Subalgebras of exact C*-algebras are exact (by Proposition 7.1(1) of [137]), so Theorem 5.11 of [102] implies that $\mathrm{QT}(B) = \mathrm{T}(B)$. Since

$$\frac{m}{n} - \frac{2}{k} > \mathrm{rc}(B),$$

and since $a_1, b_1, g \in M_l \otimes B$, it follows that $a_1 \oplus g \precsim_B b_1$. Using this relation at the third step, (1) at the first step, (2) at the second step, and (6) at the last step, we then get

$$(a - \varepsilon)_+ \precsim_A \big[a_0 - \big(\varepsilon_0 + \tfrac{1}{3}\varepsilon\big)\big]_+ \precsim_A a_1 \oplus g \precsim_B b_1 \precsim_A b.$$

This completes the proof that $\mathrm{rc}(A) \leq \mathrm{rc}(B)$. □

Proposition 21.7. Let A be an infinite dimensional stably finite simple separable unital exact C*-algebra. Let $B \subset A$ be a large subalgebra. Then $\mathrm{rc}(A) \geq \mathrm{rc}(B)$.

Proof. We use Theorem 21.4(2). Thus, let $m, n \in \mathbb{Z}_{>0}$ satisfy $m/n > \mathrm{rc}(A)$. Let $l \in \mathbb{Z}_{>0}$, and let $a, b \in (M_l \otimes B)_+$ satisfy

$$(n+1)\langle a \rangle_B + m\langle 1 \rangle_B \leq n\langle b \rangle_B$$

in $W(B)$. We must prove that $a \precsim_B b$. Without loss of generality $\|a\| \leq 1$. Moreover, by Lemma 18.4(11), it is enough to show that for every $\varepsilon > 0$ we have $(a - \varepsilon)_+ \precsim_B b$. So let $\varepsilon > 0$. Without loss of generality $\varepsilon < 1$.

Choose $k \in \mathbb{Z}_{>0}$ such that

$$\frac{km}{kn + 1} > \mathrm{rc}(A).$$

Then in $W(B)$ we have

$$(kn + 1)\langle a \rangle_B + km\langle 1 \rangle_B \leq k(n+1)\langle a \rangle_B + km\langle 1 \rangle_B \leq kn\langle b \rangle_B.$$

Let $x \in M_\infty(B)_+$ be the direct sum of $kn + 1$ copies of a, let $z \in M_\infty(B)_+$ be the direct sum of kn copies of b, and let $q \in M_\infty(B)_+$ be the direct sum of km copies of 1. Then, by definition, $x \oplus q \precsim_B z$. Therefore Lemma 18.4(11) provides $\delta > 0$ such that $\big(x \oplus q - \frac{1}{4}\varepsilon\big)_+ \precsim_B (z - \delta)_+$. Since $\varepsilon < 4$, we have

$$\big(x \oplus q - \tfrac{1}{4}\varepsilon\big)_+ = \big(x - \tfrac{1}{4}\varepsilon\big)_+ \oplus \big(q - \tfrac{1}{4}\varepsilon\big)_+ \sim_B \big(x - \tfrac{1}{4}\varepsilon\big)_+ \oplus q,$$

so

$$(kn + 1)\big\langle \big(a - \tfrac{1}{4}\varepsilon\big)_+ \big\rangle_B + km\langle 1 \rangle_B \leq kn\langle (b - \delta)_+ \rangle_B.$$

Lemma 18.13 provides $c \in (M_l \otimes B)_+$ and $y \in (M_l \otimes B)_+ \setminus \{0\}$ such that

$$kn\langle (b - \delta)_+ \rangle_B \leq (kn + 1)\langle c \rangle_B \qquad \text{and} \qquad \langle c \rangle_B + \langle y \rangle_B \leq \langle b \rangle_B \tag{21.12}$$

in $W(B)$. Then

$$(kn+1)\big\langle \big(a - \tfrac{1}{4}\varepsilon\big)_{+} \big\rangle_B + km\langle 1\rangle_B \leq (kn+1)\langle c\rangle_B.$$

Applying the map $W(A) \to W(B)$, we get

$$(kn+1)\big\langle \big(a - \tfrac{1}{4}\varepsilon\big)_{+} \big\rangle_A + km\langle 1\rangle_A \leq (kn+1)\langle c\rangle_A.$$

For $\tau \in \mathrm{T}(A)$, we apply d_τ and divide by $kn+1$ to get

$$d_\tau\big(\big\langle \big(a - \tfrac{1}{4}\varepsilon\big)_{+} \big\rangle\big) + \frac{km}{kn+1} \leq d_\tau(c).$$

Since $\mathrm{QT}(A) = \mathrm{T}(A)$ (by Theorem 5.11 of [102]) and

$$\frac{km}{kn+1} > \mathrm{rc}(A),$$

it follows that $\big(a - \frac{1}{4}\varepsilon\big)_{+} \precsim_A c$. In particular, there is $v \in M_l \otimes A$ such that

$$\big\| vcv^* - \big(a - \tfrac{1}{4}\varepsilon\big)_{+} \big\| < \tfrac{1}{4}\varepsilon.$$

Since B is large in A, we can apply Proposition 20.11 and Lemma 20.6 to find $v_0 \in M_l \otimes A$ and $g \in M_l \otimes B$ with $0 \leq g \leq 1$ and such that

$$g \precsim_B y, \quad \|v_0\| \leq \|v\|, \quad \|v_0 - v\| < \frac{\varepsilon}{4\|v\|\|c\| + 1}, \qquad \text{and} \qquad (1-g)v_0 \in M_l \otimes B.$$

It follows that $\|v_0^* c v_0 - v^* c v\| < \frac{\varepsilon}{2}$, so

$$\big\| (1-g)v_0 c[(1-g)v_0]^* - (1-g)\big(a - \tfrac{1}{4}\varepsilon\big)_{+}(1-g) \big\| < \tfrac{3}{4}\varepsilon.$$

Therefore, using Lemma 18.4(10) at the first step,

$$\big[(1-g)\big(a - \tfrac{1}{4}\varepsilon\big)_{+}(1-g) - \tfrac{3}{4}\varepsilon\big]_{+} \precsim_B (1-g)v_0 c[(1-g)v_0]^* \precsim_B c. \tag{21.13}$$

Using Lemma 18.5 at the first step, with $\big(a - \frac{1}{4}\varepsilon\big)_{+}$ in place of a and $\frac{3}{4}\varepsilon$ in place of ε, as well as Lemma 18.4(8), using (21.13) at the second step, using the choice of g at the third step, and using the second part of (21.12) at the fourth step, we get

$$(a-\varepsilon)_{+} \precsim_B \big[(1-g)\big(a - \tfrac{1}{4}\varepsilon\big)_{+}(1-g) - \tfrac{3}{4}\varepsilon\big]_{+} \oplus g \precsim_B c \oplus g \precsim_B c \oplus y \precsim_B b.$$

This is the relation we need, and the proof is complete. ☐

Proof of Theorem 21.3. Combine Proposition 21.6 and Proposition 21.7. ☐

22. Large Subalgebras in Crossed Products by $\mathbb{Z}$

In this section, we prove that if X is an infinite compact metric space, $h\colon X \to X$ is a minimal homeomorphism, and $Y \subset X$ is closed and intersects each orbit of h at most once, then the Y-orbit breaking subalgebra $C^*(\mathbb{Z}, X, h)_Y$ of Definition 16.18 is a centrally large subalgebra of $C^*(\mathbb{Z}, X, h)$. For easy reference, we summarize the relevant crossed product notation. This summary combines parts of Definition 1.5, Notation 8.7, and Definition 9.18.

Notation 22.1. Let X be a compact metric space, and let $h\colon X \to X$ be a homeomorphism. We take the corresponding automorphism $\alpha \in \mathrm{Aut}(C(X))$ to be given by $\alpha(f)(x) = f(h^{-1}(x))$ for $f \in C(X)$ and $x \in X$. The crossed product is $C^*(\mathbb{Z}, X, h)$. (Since $\mathbb{Z}$ is amenable, the full and reduced crossed products are the same, by Theorem 9.7.) We let $u \in C^*(\mathbb{Z}, X, h)$ be the standard unitary corresponding to the generator $1 \in \mathbb{Z}$. Thus, $ufu^* = f \circ h^{-1}$ for all $f \in C(X)$, and for $n \in \mathbb{Z}$ the standard unitary u_n (following Notation 8.7) is $u_n = u^n$. The dense subalgebra $C(X)[\mathbb{Z}]$ is the set of all finite sums

$$a = \sum_{k=-n}^{n} f_k u^k \tag{22.1}$$

with $n \in \mathbb{Z}_{\geq 0}$ and $f_{-n}, f_{-n+1}, \ldots, f_n \in C(X)$. We identify $C(X)$ with a subalgebra of $C^*(\mathbb{Z}, X, h)$ in the standard way (as discussed after Remark 8.12): it is all a as in (22.1) such that $f_k = 0$ for $k \neq 0$. The standard conditional expectation $E\colon C^*(\mathbb{Z}, X, h) \to C(X)$ is given on $C(X)[\mathbb{Z}]$ by $E(a) = f_0$ when a is as in (22.1).

In order to state more general results, we generalize the construction of Definition 16.18. Notation 22.2 and Definition 22.3 below differ from Notation 22.1 and Definition 16.18 in that they consider $C_0(X, D)$ for a C*-algebra D instead of just $C_0(X)$.

Notation 22.2. For a locally compact Hausdorff space X, a C*-algebra D, and an open subset $U \subset X$, we use the abbreviation

$$C_0(U, D) = \big\{ f \in C_0(X, D) \colon f(x) = 0 \text{ for all } x \in X \setminus U \big\} \subset C_0(X, D).$$

This subalgebra is of course canonically isomorphic to the usual algebra $C_0(U, D)$ when U is considered as a locally compact Hausdorff space in its own right. If $D = \mathbb{C}$ we omit it from the notation.

In particular, if $Y \subset X$ is closed, then

$$C_0(X \setminus Y,\, D) = \big\{ f \in C_0(X, D) \colon f(x) = 0 \text{ for all } x \in Y \big\}. \tag{22.2}$$

Definition 22.3. Let X be a locally compact Hausdorff space, let D be a unital C*-algebra, and let $h\colon X \to X$ be a homeomorphism. Let $\alpha \in \mathrm{Aut}(C(X, D))$ be an automorphism which "lies over h", in the sense that there exists a function $x \mapsto \alpha_x$ from X to $\mathrm{Aut}(D)$ such that $\alpha(a)(x) = \alpha_x\big(a(h^{-1}(x))\big)$ for all $x \in X$ and $a \in C_0(X, D)$. Let $Y \subset X$ be a nonempty closed subset, and, following (22.2), define

$$C^*\big(\mathbb{Z},\, C_0(X, D),\, \alpha\big)_Y = C^*\big(C_0(X, D),\, C_0(X \setminus Y,\, D)u\big) \subset C^*\big(\mathbb{Z},\, C_0(X, D),\, \alpha\big).$$

We call it the *Y-orbit breaking subalgebra* of $C^*\big(\mathbb{Z},\, C_0(X, D),\, \alpha\big)$.

We describe the proof of Theorem 19.5, namely that if $h\colon X \to X$ is a minimal homeomorphism and $Y \subset X$ is a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$, then $C^*(\mathbb{Z}, X, h)_Y$ is a centrally large subalgebra of $C^*(\mathbb{Z}, X, h)$ in the sense of Definition 19.2. Our presentation differs from that of [213] and [8] in that we prove the result directly rather than via large subalgebras of crossed product type.

Under some technical conditions on α and D, similar methods can be used to prove the analogous result for $C^*\big(\mathbb{Z},\, C(X, D),\, \alpha\big)_Y$. The following theorem is a consequence of results in [7].

Theorem 22.4 ([7])**.** Let X be an infinite compact metric space, let $h\colon X \to X$ be a minimal homeomorphism, let D be a simple unital C*-algebra which has a tracial state, and let $\alpha \in \mathrm{Aut}(C(X, D))$ lie over h. Assume that D has strict comparison of positive elements, or that the automorphisms α_x in Definition 22.3 are all approximately inner. Let $Y \subset X$ be a compact subset such that $h^n(Y) \cap Y =$

$\varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then $C^*\big(\mathbb{Z}, C(X, D), \alpha\big)_Y$ is a centrally large subalgebra of $C^*\big(\mathbb{Z}, C(X, D), \alpha\big)$ in the sense of Definition 19.2.

The ideas of the proof of Theorem 19.5 are all used in the proof of the general theorem behind Theorem 22.4, but additional work is needed to deal with the presence of D.

We now describe the proof of Theorem 19.5, omitting a few details. We will let X be an infinite compact Hausdorff space with a minimal homeomorphism $h\colon X \to X$. We follow Notation 22.1. We will fix a nonempty closed subset $Y \subset X$. For $n \in \mathbb{Z}$, set

$$Z_n = \begin{cases} \bigcup_{j=0}^{n-1} h^j(Y) & n > 0 \\ \varnothing & n = 0 \\ \bigcup_{j=1}^{-n} h^{-j}(Y) & n < 0. \end{cases}$$

Recall from Proposition 17.6 that

$$C^*(\mathbb{Z}, X, h)_Y = \big\{ a \in C^*(\mathbb{Z}, X, h)\colon E(au^{-n}) \in C_0(X \setminus Z_n) \text{ for all } n \in \mathbb{Z} \big\}$$

and

$$\overline{C^*(\mathbb{Z}, X, h)_Y \cap C(X)[\mathbb{Z}]} = C^*(\mathbb{Z}, X, h)_Y.$$

Lemma 22.5 (Corollary 7.6 of [213])**.** Let X be a compact Hausdorff space and let $h\colon X \to X$ be a homeomorphism. Let $Y \subset X$ be a nonempty closed subset. Let $u \in C^*(\mathbb{Z}, X, h)$ be the standard unitary, as in Notation 22.1, and let $v \in C^*(\mathbb{Z}, X, h^{-1})$ be the analogous standard unitary in $C^*(\mathbb{Z}, X, h^{-1})$. Then there exists a unique homomorphism $\varphi\colon C^*(\mathbb{Z}, X, h^{-1}) \to C^*(\mathbb{Z}, X, h)$ such that $\varphi(f) = f$ for $f \in C(X)$ and $\varphi(v) = u^*$, the map φ is an isomorphism, and

$$\varphi\big(C^*(\mathbb{Z}, X, h^{-1})_{h^{-1}(Y)}\big) = C^*(\mathbb{Z}, X, h)_Y.$$

See [213] for the straightforward proof, based on Proposition 17.6.

Lemma 22.6 (Lemma 7.4 of [213])**.** Let X be an infinite compact Hausdorff space and let $h\colon X \to X$ be a minimal homeomorphism. Let $K \subset X$ be a compact set such that $h^n(K) \cap K = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Let $U \subset X$ be a nonempty open subset. Then there exist $l \in \mathbb{Z}_{\geq 0}$, compact sets $K_1, K_2, \ldots, K_l \subset X$, and $n_1, n_2, \ldots, n_l \in \mathbb{Z}_{>0}$, such that $K \subset \bigcup_{j=1}^{l} K_j$ and such that $h^{n_1}(K_1), h^{n_2}(K_2), \ldots, h^{n_l}(K_l)$ are disjoint subsets of U.

Sketch of proof. Choose a nonempty open subset $V \subset X$ such that $\overline{V}$ is compact and contained in U. Use minimality of h to cover K with the images of V under finitely many negative powers of h, say $h^{-n_1}(V), h^{-n_2}(V), \ldots, h^{-n_l}(V)$. Set $K_j = h^{-n_j}\big(\overline{V}\big) \cap K$ for $j = 1, 2, \ldots, l$. □

The next lemma is straightforward if one only asks that $f \precsim_{C^*(\mathbb{Z},X,h)} g$ (Cuntz subequivalence in the crossed product), and then doing it for one value of n is equivalent to doing it for any other. Getting $f \precsim_{C^*(\mathbb{Z},X,h)_Y} g$ for both positive n and negative n is a key step in showing $C^*(\mathbb{Z}, X, h)_Y$ a large subalgebra of $C^*(\mathbb{Z}, X, h)$. This result is related to the statement about equivalence of projections in Lemma 16.22.

Lemma 22.7 (Lemma 7.7 of [213])**.** Let X be an infinite compact Hausdorff space and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a compact subset such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Let $U \subset X$ be a nonempty open subset and let $n \in \mathbb{Z}$. Then there exist $f, g \in C(X)_+$ such that

$$f|_{h^n(Y)} = 1, \quad 0 \leq f \leq 1, \quad \operatorname{supp}(g) \subset U, \qquad \text{and} \qquad f \precsim_{C^*(\mathbb{Z},X,h)_Y} g.$$

Proof. We first prove this when $n = 0$.

Apply Lemma 22.6 with Y in place of K, obtaining $l \in \mathbb{Z}_{\geq 0}$, compact sets $Y_1, Y_2, \ldots, Y_l \subset X$, and $n_1, n_2, \ldots, n_l \in \mathbb{Z}_{>0}$. Set $N = \max(n_1, n_2, \ldots, n_l)$. Choose disjoint open sets $V_1, V_2, \ldots, V_l \subset U$ such that $h^{n_j}(Y_j) \subset V_j$ for $j = 1, 2, \ldots, l$. Then $Y_j \subset h^{-n_j}(V_j)$, so the sets $h^{-n_1}(V_1)$, $h^{-n_2}(V_2)$, $\ldots$, $h^{-n_l}(V_l)$ cover Y. For $j = 1, 2, \ldots, l$, define

$$W_j = h^{-n_j}(V_j) \cap \left(X \setminus \bigcup_{n=1}^{N} h^{-n}(Y) \right).$$

Then $W_1, W_2, \ldots, W_l$ form an open cover of Y. Therefore there are $f_1, f_2, \ldots, f_l \in C(X)_+$ such that for $j = 1, 2, \ldots, l$ we have $\operatorname{supp}(f_j) \subset W_j$ and $0 \leq f_j \leq 1$, and such that the function $f = \sum_{j=1}^{l} f_j$ satisfies $f(x) = 1$ for all $x \in Y$ and $0 \leq f \leq 1$. Further define $g = \sum_{j=1}^{l} f_j \circ h^{-n_j}$. Then $\operatorname{supp}(g) \subset U$.

Let $u \in C^*(\mathbb{Z}, X, h)$ be as in Notation 22.1. For $j = 1, 2, \ldots, l$, set $a_j = f_j^{1/2} u^{-n_j}$. Since f_j vanishes on $\bigcup_{n=1}^{n_j} h^{-n}(Y)$, Proposition 17.6 implies that $a_j \in C^*(\mathbb{Z}, X, h)_Y$. Therefore, in $C^*(\mathbb{Z}, X, h)_Y$ we have

$$f_j \circ h^{-n_j} = a_j^* a_j \sim_{C^*(\mathbb{Z},X,h)_Y} a_j a_j^* = f_j.$$

Consequently, using Lemma 18.4(12) at the second step and Lemma 18.4(13) and disjointness of the supports of the functions $f_j \circ h^{-n_j}$ at the last step, we have

$$f = \sum_{j=1}^{l} f_j \precsim_{C^*(\mathbb{Z},X,h)_Y} \bigoplus_{j=1}^{l} f_j \sim_{C^*(\mathbb{Z},X,h)_Y} \bigoplus_{j=1}^{l} f_j \circ h^{-n_j} \sim_{C^*(\mathbb{Z},X,h)_Y} g.$$

This completes the proof for $n = 0$.

Now suppose that $n > 0$. Choose functions f and g for the case $n = 0$, and call them f_0 and g. Since $f_0(x) = 1$ for all $x \in Y$, and since $Y \cap \bigcup_{l=1}^{n} h^{-l}(Y) = \varnothing$, there is $f_1 \in C(X)$ with $0 \leq f_1 \leq f_0$, $f_1(x) = 1$ for all $x \in Y$, and $f_1(x) = 0$ for $x \in \bigcup_{l=1}^{n} h^{-l}(Y)$. Set $v = f_1^{1/2} u^{-n}$ and $f = f_1 \circ h^{-n}$. Then $f(x) = 1$ for all $x \in h^n(Y)$ and $0 \leq f \leq 1$. Proposition 17.6 implies that $v \in C^*(\mathbb{Z}, X, h)_Y$. We have

$$v^* v = u^n f_1 u^{-n} = f_1 \circ h^{-n} = f \qquad \text{and} \qquad v v^* = f_1.$$

Using Lemma 18.4(4), we thus get

$$f \sim_{C^*(\mathbb{Z},X,h)_Y} f_1 \leq f_0 \precsim_{C^*(\mathbb{Z},X,h)_Y} g.$$

This completes the proof for the case $n > 0$.

Finally, we consider the case $n < 0$. In this case, we have $-n-1 \geq 0$. Apply the cases already done with h^{-1} in place of h. We get $f, g \in C^*(\mathbb{Z}, X, h^{-1})_{h^{-1}(Y)}$ such that $f(x) = 1$ for all $x \in (h^{-1})^{-n-1}(h^{-1}(Y)) = h^n(Y)$, such that $0 \leq f \leq 1$, such that $\operatorname{supp}(g) \subset U$, and such that $f \precsim_{C^*(\mathbb{Z},X,h^{-1})_{h^{-1}(Y)}} g$. Let $\varphi \colon C^*(\mathbb{Z}, X, h^{-1}) \to C^*(\mathbb{Z}, X, h)$ be the isomorphism of Lemma 22.5. Then

$$\varphi(f) = f, \quad \varphi(g) = g, \qquad \text{and} \qquad \varphi\big(C^*(\mathbb{Z}, X, h^{-1})_{h^{-1}(Y)}\big) = C^*(\mathbb{Z}, X, h)_Y.$$

Therefore $f \precsim_{C^*(\mathbb{Z},X,h)_Y} g$. $\square$

The following result is a special case of Lemma 7.9 of [213]. The basic idea has been used frequently; related arguments can be found, for example, in the proofs of Theorem 3.2 of [67], Lemma 2 and Theorem 1 in [5], Lemma 10 of [144], and Lemma 3.2 of [195]. (The papers listed are not claimed to be representative or to be the original sources; they are ones I happen to know of.)

Lemma 22.8. Let X be an infinite compact space, and let $h\colon X \to X$ be a minimal homeomorphism. Let $B \subset C^*(\mathbb{Z}, X, h)$ be a unital subalgebra such that $C(X) \subset B$ and $B \cap C(X)[\mathbb{Z}]$ is dense in B. Let $a \in B_+ \setminus \{0\}$. Then there exists $b \in C(X)_+ \setminus \{0\}$ such that $b \precsim_B a$.

Sketch of proof. Without loss of generality $\|a\| \leq 1$. The conditional expectation $E_\alpha\colon C^*_{\mathrm{r}}(G, X) \to C(X)$ is faithful. Therefore $E_\alpha(a) \in C(X)$ is a nonzero positive element. Set $\varepsilon = \frac{1}{6}\|E_\alpha(a)\|$. Choose $c \in B \cap C(X)[\mathbb{Z}]$ such that $\|c - a^{1/2}\| < \varepsilon$ and $\|c\| \leq 1$. One can check that $\|E_\alpha(c^*c)\| > 4\varepsilon$. There are $n \in \mathbb{Z}_{\geq 0}$ and $g_{-n}, g_{-n+1}, \ldots, g_n \in C(X)$ such that $c^*c = \sum_{k=-n}^{n} g_k u^k$. We have $g_0 = E_\alpha(c^*c) \in C(X)_+$ and $\|g_0\| > 4\varepsilon$. Therefore there is $x \in X$ such that $g_0(x) > 4\varepsilon$. Choose $f \in C(X)$ such that $0 \leq f \leq 1$, $f(x) = 1$, and the sets $h^k(\mathrm{supp}(f))$ are disjoint for $k = -n, -n+1, \ldots, n$. One can then check that $fc^*cf = fg_0f$, so that $\|fc^*cf\| > 4\varepsilon$. Therefore $(fc^*cf - 2\varepsilon)_+$ is a nonzero element of $C(X)$. Using Lemma 18.4(6) at the first step, Lemma 18.4(17) and $cf^2c^* \leq cc^*$ at the second step, and Lemma 18.4(10) and $\|cc^* - a\| < 2\varepsilon$ at the last step, we then have

$$(fc^*cf - 2\varepsilon)_+ \sim_B (cf^2c^* - 2\varepsilon)_+ \precsim_B (cc^* - 2\varepsilon)_+ \precsim_B a.$$

This completes the proof. $\square$

Corollary 22.9. Let X be an infinite compact Hausdorff space, and let $h\colon X \to X$ be a minimal homeomorphism. Let $B \subset C^*(\mathbb{Z}, X, h)$ be a unital subalgebra such that $C(X) \subset B$ and $B \cap C(X)[\mathbb{Z}]$ is dense in B. Let $a \in A_+ \setminus \{0\}$ and let $b \in B_+ \setminus \{0\}$. Then there exists $f \in C(X)_+ \setminus \{0\}$ such that $f \precsim_{C^*(\mathbb{Z},X,h)} a$ and $f \precsim_B b$.

Proof. Applying Lemma 22.8 to both a (with $C^*(\mathbb{Z}, X, h)$ in place of B) and b (with B as given), we see that it is enough to prove the corollary for $a, b \in C(X)_+ \setminus \{0\}$. Also, without loss of generality $\|a\| \leq 1$.

Choose $x_0 \in X$ such that $b(x_0) \neq 0$. Since the orbit of x_0 is dense, there is $n \in \mathbb{Z}$ such that $a(h^n(x_0)) \neq 0$. Define $f \in C(X)$ by $f(x) = b(x)a(h^n(x))$ for $x \in X$. Then $f \neq 0$ since $f(x_0) \neq 0$. We have $f \precsim_B b$ because $\|a\| \leq 1$ implies $f \leq b$. Also, $f = (b^{1/2}u^{-n})a(b^{1/2}u^{-n})^*$ so $f \precsim_{C^*(\mathbb{Z},X,h)} a$. $\square$

The next result is a standard type of approximation lemma.

Lemma 22.10. Let A be a C*-algebra, and let $S \subset A$ be a subset which generates A as a C*-algebra and such that $a \in S$ implies $a^* \in S$. Then for every finite subset $F \subset A$ and every $\varepsilon > 0$ there are a finite subset $T \subset S$ and $\delta > 0$ such that whenever $c \in A$ satisfies $\|c\| \leq 1$ and $\|cb - bc\| < \delta$ for all $b \in T$, then $\|ca - ac\| < \varepsilon$ for all $a \in F$.

Proof. Let $B \subset A$ be the set of all $a \in A$ such that for every $\varepsilon > 0$ there are $T(a, \varepsilon) \subset S$ and $\delta(a, \varepsilon) > 0$ as in the statement of the lemma, that is, $T(a, \varepsilon)$ is finite and whenever $c \in A$ satisfies $\|c\| \leq 1$ and $\|cb - bc\| < \delta(a, \varepsilon)$ for all $b \in T(a, \varepsilon)$, then $\|ca - ac\| < \varepsilon$.

We have $S \subset B$, as is seen by taking $T(a, \varepsilon) = \{a\}$ and $\delta(a, \varepsilon) = \varepsilon$. If $a \in B$, then also $a^* \in B$, as is seen by taking

$$T(a^*, \varepsilon) = \{b^* : b \in T(a, \varepsilon)\} \qquad \text{and} \qquad \delta(a^*, \varepsilon) = \delta(a, \varepsilon).$$

We show that B is closed under addition. So let $a_1, a_2 \in B$ and let $\varepsilon > 0$. Define

$$T = T\big(a_1, \tfrac{\varepsilon}{2}\big) \cup T\big(a_2, \tfrac{\varepsilon}{2}\big) \qquad \text{and} \qquad \delta = \min\big(\delta\big(a_1, \tfrac{\varepsilon}{2}\big),\, \delta\big(a_2, \tfrac{\varepsilon}{2}\big)\big).$$

Suppose $c \in A$ satisfies $\|c\| \leq 1$ and $\|cb - bc\| < \delta$ for all $b \in T$. Then

$$\|ca_1 - a_1c\| < \tfrac{\varepsilon}{2} \qquad \text{and} \qquad \|ca_2 - a_2c\| < \tfrac{\varepsilon}{2},$$

so

$$\|c(a_1 + a_2) - (a_1 + a_2)c\| \leq \|ca_1 - a_1c\| + \|ca_2 - a_2c\| < \varepsilon.$$

This shows that $a_1 + a_2 \in B$. To show that if $a_1, a_2 \in B$ then $a_1a_2 \in B$, we use a similar argument, taking

$$\varepsilon_0 = \frac{\varepsilon}{1 + \|a_1\| + \|a_2\|}$$

and using the choices

$$T = T(a_1, \varepsilon_0) \cup T(a_2, \varepsilon_0) \qquad \text{and} \qquad \delta = \min\big(\delta(a_1, \varepsilon_0),\, \delta(a_2, \varepsilon_0)\big),$$

and the estimate

$$\|ca_1a_2 - a_1a_2c\| \leq \|ca_1 - a_1c\|\|a_2\| + \|a_1\|\|ca_2 - a_2c\|.$$

Finally, we claim that B is closed. So let $a \in \overline{B}$ and let $\varepsilon > 0$. Choose $a_0 \in B$ such that $\|a - a_0\| < \frac{\varepsilon}{3}$. Define

$$T = T\big(a_0, \tfrac{\varepsilon}{3}\big) \qquad \text{and} \qquad \delta = \delta\big(a_0, \tfrac{\varepsilon}{3}\big).$$

Suppose $c \in A$ satisfies $\|c\| \leq 1$ and $\|cb - bc\| < \delta$ for all $b \in T$. Then $\|ca_0 - a_0c\| < \frac{\varepsilon}{3}$, so

$$\|ca - ac\| \leq 2\|c\|\|a - a_0\| + \|ca_0 - a_0c\| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

The claim is proved.

Since S generates A as a C*-algebra, we have $B = A$. Now let $F \subset A$ be finite and let $\varepsilon > 0$. The conclusion of the lemma follows by taking

$$T = \bigcup_{a \in F} T(a, \varepsilon) \qquad \text{and} \qquad \delta = \min_{a \in F} \delta(a, \varepsilon).$$

This completes the proof. $\square$

Proof of Theorem 19.5. Set $A = C^*(\mathbb{Z}, X, h)$ and $B = C^*(\mathbb{Z}, X, h)_Y$. Since h is minimal, it is well known that A is simple and finite. Also clearly A is infinite dimensional.

We claim that the following holds. Let $m \in \mathbb{Z}_{>0}$, let $a_1, a_2, \ldots, a_m \in A$, let $\varepsilon > 0$, and let $f \in C(X)_+ \setminus \{0\}$. Then there are $c_1, c_2, \ldots, c_m \in A$ and $g \in C(X)$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j \in B$.
(4) $g \precsim_B f$.
(5) $\|gu - ug\| < \varepsilon$.

Suppose the claim has been proved; we show that the theorem follows. Let $m \in \mathbb{Z}_{>0}$, let $a_1, a_2, \ldots, a_m \in A$, let $\varepsilon > 0$, let $r \in A_+ \setminus \{0\}$, and let $s \in B_+ \setminus \{0\}$. (The elements r and s play the roles of x and y in Definition 19.2. Here, we use x and y for elements of X.) Apply Lemma 20.5 with r in place of x, getting $r_0 \in A_+ \setminus \{0\}$. In Lemma 22.10, take $S = C(X) \cup \{u, u^*\}$, take $F = \{a_1, a_2, \ldots, a_m\}$, and let $\varepsilon > 0$ be as given. Let the finite set $T \subset S$ and $\delta > 0$ be as in the conclusion. We may assume that $u, u^* \in T$. Apply Corollary 22.9 with r_0 in place of a and s in place of b, getting $f \in C(X)_+ \setminus \{0\}$ such that $f \precsim_A r_0$ and $f \precsim_B s$. Apply the claim with $a_1, a_2, \ldots, a_m$ as given, and with $\min(\varepsilon, \delta)$ in place of ε.

We can now verify the conditions of Definition 19.2. Conditions (1), (2), and (3) there are conditions (1), (2), and (3) here. Condition (4) there follows from condition (4) here and the relations $f \precsim_A r_0 \precsim_A r$ and $f \precsim_B s$. Condition (5) there follows from $g \precsim_A r_0$ and the choice of r_0. It remains only to verify condition (6) there, namely $\|g a_j - a_j g\| < \varepsilon$ for $j = 1, 2, \ldots, m$. It suffices to check that $\|gb - bg\| < \delta$ for all $b \in T$. We have $\|gu - ug\| < \delta$ by construction. Also, $gu^* - u^* g = -u^*(gu - ug)u^*$, so $\|gu^* - u^* g\| < \delta$. Finally, if $b \in T$ is any element other than u or u^*, then $b \in C(X)$, so $gb = bg$. This completes the proof that the claim implies the conclusion of the theorem.

We now prove the claim. Choose $c_1, c_2, \ldots, c_m \in C(X)[\mathbb{Z}]$ such that $\|c_j - a_j\| < \varepsilon$ for $j = 1, 2, \ldots, m$. (This estimate is condition (2).) Choose $N \in \mathbb{Z}_{>0}$ such that there are $c_{j,l} \in C(X)$ for $j = 1, 2, \ldots, m$ and $l = -N, \, -N+1, \, \ldots, \, N-1, \, N$ with

$$c_j = \sum_{l=-N}^{N} c_{j,l} u^l.$$

Choose $N_0 \in \mathbb{Z}_{>0}$ such that $\frac{1}{N_0} < \varepsilon$. Define

$$I = \big\{ -N - N_0, \, -N - N_0 + 1, \, \ldots, \, N + N_0 - 1, \, N + N_0 \big\}.$$

Set $U = \{x \in X \colon f(x) \neq 0\}$, and choose nonempty disjoint open sets $U_l \subset U$ for $l \in I$. For each such l, use Lemma 22.7 to choose $f_l, r_l \in C(X)_+$ such that $r_l(x) = 1$ for all $x \in h^l(Y)$, such that $0 \leq r_l \leq 1$, such that $\operatorname{supp}(f_l) \subset U_l$, and such that $r_l \precsim_B f_l$.

Choose an open set W containing Y such that the sets $h^l(W)$ are disjoint for $l \in I$, and choose $r \in C(X)$ such that

$$0 \leq r \leq 1, \quad r|_Y = 1, \qquad \text{and} \qquad \operatorname{supp}(r) \subset W.$$

Set

$$g_0 = r \cdot \prod_{l \in I} r_l \circ h^l.$$

Set $g_l = g_0 \circ h^{-l}$ for $l \in I$. Then $0 \leq g_l \leq r_l \leq 1$. Define λ_l for $l \in I$ by

$$\lambda_{-N-N_0} = 0, \quad \lambda_{-N-N_0+1} = \frac{1}{N_0}, \quad \lambda_{-N-N_0+2} = \frac{2}{N_0}, \quad \ldots, \quad \lambda_{-N-1} = 1 - \frac{1}{N_0},$$

$$\lambda_{-N} = \lambda_{-N+1} = \cdots = \lambda_{N-1} = \lambda_N = 1,$$

$$\lambda_{N+1} = 1 - \frac{1}{N_0}, \quad \lambda_{N+2} = 1 - \frac{2}{N_0}, \quad \ldots, \quad \lambda_{N+N_0-1} = \frac{1}{N_0}, \quad \lambda_{N+N_0} = 0.$$

Set $g = \sum_{l \in I} \lambda_l g_l$. The supports of the functions g_l are disjoint, so $0 \leq g \leq 1$. This is condition (1). Using Lemma 18.4(13) at the first and fourth steps and

Lemma 18.4(14) at the third step, we get

$$g \precsim_B \bigoplus_{l\in I} g_l \leq \bigoplus_{l\in I} r_l \precsim_B \bigoplus_{l\in I} f_l \sim_{C(X)} \sum_{l\in I} f_l \precsim_{C(X)} f.$$

This is condition (4).

We check condition (5). We have

$$\|gu - ug\| = \|g - ugu^*\| = \|g - g\circ h^{-1}\| = \left\| \sum_{l\in I} \lambda_l g_0 \circ h^{-l} - \sum_{l\in I} \lambda_l g_0 \circ h^{-l-1} \right\|.$$

In the second sum in the last term, we change variables to get $\sum_{l+1\in I} \lambda_{l-1} g_0 \circ h^{-l}$. Use $\lambda_{-N-N_0} = \lambda_{N+N_0} = 0$ and combine terms to get

$$\|gu - ug\| = \left\| \sum_{l=-N-N_0+1}^{N+N_0} (\lambda_l - \lambda_{l-1}) g_0 \circ h^{-l} \right\|.$$

The expressions $g_0 \circ h^{-l}$ are orthogonal and have norm 1, so

$$\|gu - ug\| = \max_{-N-N_0+1\leq l\leq N+N_0} |\lambda_l - \lambda_{l-1}| = \frac{1}{N_0} < \varepsilon.$$

It remains to verify condition (3). Since $1-g$ vanishes on the sets

$$h^{-N}(Y),\, h^{-N+1}(Y),\, \ldots,\, h^{N-2}(Y),\, h^{N-1}(Y),$$

Proposition 17.6 implies that $(1-g)u^l \in B$ for $l = -N,\, -N+1,\, \ldots,\, N-1,\, N$. For $j = 1, 2, \ldots, m$, since $c_{j,l} \in C(X) \subset B$ for $l = -N,\, -N+1,\, \ldots,\, N-1,\, N$, we get

$$(1-g)c_j = \sum_{l=-N}^{N} c_{j,l} \cdot (1-g)u^l \in B.$$

This completes the verification of condition (3), and the proof of the theorem. ☐

23. Application to the Radius of Comparison of Crossed Products by Minimal Homeomorphisms

The purpose of this section is to describe some of the ideas involved in Theorem 19.15 and its proof. We describe the mean dimension of a homeomorphism, and we give proofs of simple special cases or related statements for some of the steps in its proof.

We will need simplicial complexes. See Section 2.6 of [197] for a presentation of the basics. Following a common abuse of terminology, we say here that a topological space is a simplicial complex when, formally, we mean that it is homeomorphic to the geometric realization of a simplicial complex.

An explanation of mean dimension starts with dimension theory; see the discussion after Corollary 16.2. The mean dimension of a homeomorphism $h\colon X \to X$ was introduced in [164] For best behavior, h should not have "too many" periodic points. It is designed so that if K is a sufficiently nice compact metric space (in particular, $\dim(K^n)$ should equal $n \cdot \dim(K)$ for all $n \in \mathbb{Z}_{>0}$), then the shift on $X = K^{\mathbb{Z}}$ should have mean dimension equal to $\dim(K)$. Given this heuristic, it should not be surprising that if $\dim(X) < \infty$ then $\mathrm{mdim}(h) = 0$.

We recall that $\mathrm{Cov}(X)$ is the set of finite open covers of X (Notation 16.3), that $\mathcal{V} \prec \mathcal{U}$ means that $\mathcal{V}$ refines $\mathcal{U}$ (Definition 16.5), the order $\mathrm{ord}(\mathcal{U})$ of a finite open cover $\mathcal{U}$ (Definition 16.4), and that $\mathcal{D}(\mathcal{U})$ is the least order of a refinement of $\mathcal{U}$.

Definition 23.1. Let X be a compact Hausdorff space, and let $\mathcal{U}$ and $\mathcal{V}$ be two finite open covers of X. Then the *join* $\mathcal{U} \vee \mathcal{V}$ of $\mathcal{U}$ and $\mathcal{V}$ is

$$\mathcal{U} \vee \mathcal{V} = \big\{ U \cap V \colon U \in \mathcal{U} \text{ and } V \in \mathcal{V} \big\}.$$

Definition 23.2. Let X be a compact Hausdorff space, let $\mathcal{U}$ be a finite open cover of X, and let $h\colon X \to X$ be a homeomorphism. We define

$$h(\mathcal{U}) = \big\{ h(U) \colon U \in \mathcal{U} \big\}.$$

Definition 23.3 (Definition 2.6 of [164])**.** Let X be a compact metric space and let $h\colon X \to X$ be a homeomorphism. Then the *mean dimension* of h is (see Corollary 23.6 below for existence of the limit)

$$\operatorname{mdim}(h) = \sup_{\mathcal{U} \in \operatorname{Cov}(X)} \lim_{n \to \infty} \frac{\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big)}{n}.$$

The expression in the definition uses the join of n covers.

Existence of the limit depends on the following result.

Proposition 23.4 (Corollary 2.5 of [164])**.** Let X be a compact metric space, and let $\mathcal{U}$ and $\mathcal{V}$ be two finite open covers of X. Then $\mathcal{D}(\mathcal{U} \vee \mathcal{V}) \leq \mathcal{D}(\mathcal{U}) + \mathcal{D}(\mathcal{V})$.

We omit the proof, but the idea is similar to that of the proof of Proposition 3.2.6 of [197] ($\dim(X \times Y) \leq \dim(X) + \dim(Y)$ for nonempty compact Hausdorff spaces X and Y). The point is that an open cover $\mathcal{U}$ has $\operatorname{ord}(\mathcal{U}) \leq m$ if and only if there is a finite simplicial complex K of dimension at most m which approximates X "as seen by $\mathcal{U}$", and if K and L are finite simplicial complexes which approximate X as seen by $\mathcal{U}$ and by $\mathcal{V}$, then $K \times L$ is a finite simplicial complex with dimension $\dim(K) + \dim(L)$ which approximates X as seen by $\mathcal{U} \vee \mathcal{V}$.

Lemma 23.5. Let $(\alpha_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $[0, \infty)$ which is subadditive, that is, $\alpha_{m+n} \leq \alpha_m + \alpha_n$ for all $m, n \in \mathbb{Z}_{>0}$. Then $\lim_{n \to \infty} n^{-1} \alpha_n$ exists and is equal to $\inf_{n \in \mathbb{Z}_{>0}} n^{-1} \alpha_n$.

Proof. We follow part of the proof of Theorem 6.1 of [164]. Define $\beta = \inf_{n \in \mathbb{Z}_{>0}} n^{-1} \alpha_n$. Let $\varepsilon > 0$. Choose $N_0 \in \mathbb{Z}_{>0}$ such that $N_0^{-1} \alpha_{N_0} < \beta + \frac{\varepsilon}{2}$. Choose $N \in \mathbb{Z}_{>0}$ so large that

$$N \geq N_0 \qquad \text{and} \qquad \frac{N_0 \alpha_1}{N} < \frac{\varepsilon}{2}.$$

Let $n \geq N$. Since $N \geq N_0$, there are $r \in \mathbb{Z}_{>0}$ and $s \in \{0, 1, \ldots, N_0 - 1\}$ such that $n = r N_0 + s$. Then, using subadditivity at the first step,

$$\frac{\alpha_n}{n} \leq \frac{r \alpha_{N_0} + s \alpha_1}{n} = \frac{r \alpha_{N_0}}{r N_0 + s} + \frac{s \alpha_1}{n} < \frac{\alpha_{N_0}}{N_0} + \frac{N_0 \alpha_1}{n} < \beta + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \beta + \varepsilon.$$

This completes the proof. ∎

Corollary 23.6. Let X be a compact metric space let $\mathcal{U}$ be a finite open cover of X, and let $h\colon X \to X$ be a homeomorphism. Then the limit

$$\lim_{n \to \infty} \frac{\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big)}{n}$$

in Definition 23.3 exists.

Proof. Combine Lemma 23.5 and Proposition 23.4. ∎

The following result is immediate.

Proposition 23.7. Let X be a compact metric space with finite covering dimension, and let $h\colon X \to X$ be a homeomorphism. Then $\mathrm{mdim}(h) = 0$.

Proof. Let $\mathcal{U}$ be a finite open cover of X. Then, by definition,

$$\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big) \leq \dim(X),$$

so

$$\mathrm{mdim}(h) \leq \lim_{n\to\infty} \frac{\dim(X)}{n} = 0.$$

This completes the proof. □

The following result is less obvious, but not difficult (although we refer to [164] for the proof). In particular, it shows that every uniquely ergodic minimal homeomorphism has mean dimension zero.

Proposition 23.8. Let X be a compact metric space, let $h\colon X \to X$ be a homeomorphism, and assume that h has at most countably many ergodic invariant Borel probability measures. Then $\mathrm{mdim}(h) = 0$.

Proof. In [164], see Theorem 5.4 and the discussion after Definition 5.2. □

Proposition 23.7 covers most of the common examples of minimal homeomorphisms. However, not all minimal homeomorphisms have mean dimension zero. We start with the standard nonminimal example. the shift, as in Example 2.20.

Definition 23.9. Let K be a set. The *shift* $h_K\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ is the bijection given by $h_K(x)_k = x_{k+1}$ for $x = (x_k)_{k\in\mathbb{Z}} \in K^{\mathbb{Z}}$ and $k \in \mathbb{Z}$.

Theorem 23.10 (Proposition 3.1 of [164])**.** Let K be a compact metric space, and let h_K be as in Definition 23.9. Then $\mathrm{mdim}(h_K) \leq \dim(K)$.

Theorem 23.11 (Proposition 3.3 of [164])**.** Let $d \in \mathbb{Z}_{>0}$, set $K = [0,1]^d$, and let h_K be as in Definition 23.9. Then $\mathrm{mdim}(h_K) = d$.

We omit the proofs. To understand the result heuristically, in Definition 23.3 consider a finite open cover $\mathcal{U}_0$ of K, for $n \in \mathbb{Z}$ let $p_n\colon K^{\mathbb{Z}} \to K$ be the projection on the nth coordinate, and consider the finite open cover

$$\mathcal{U} = \big\{p_0^{-1}(U)\colon U \in \mathcal{U}_0\big\}.$$

Then the cover $\mathcal{U} \vee h_K^{-1}(\mathcal{U}) \vee \cdots \vee h_K^{-n+1}(\mathcal{U})$ sees only n of the coordinates in $K^{\mathbb{Z}}$, so that

$$\mathcal{D}\big(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\big) \leq \dim(K^n) \leq n\dim(K).$$

The proof of Theorem 23.10 requires only one modification of this idea, namely that the original cover $\mathcal{U}$ must be allowed to depend on an arbitrary finite number of coordinates rather than just one. The proof of Theorem 23.11 requires more work.

One does not expect $\mathrm{mdim}(h_K) = \dim(K)$ in general, because of the possibility of having $\dim(K^n) < n\dim(K)$. (This is the possibility of having strict inequality in (16.1); see the discussion after (16.1).) When $\dim(K) < \infty$, by combining Theorem 1.4 of [61] and Theorem 3.16(b) of [61] and the discussion afterwards, one sees that $\dim(K^n)$ is either always $n\dim(K)$ or always $n\dim(K) - n + 1$. In the first case,

$$\lim_{n\to\infty} \frac{\dim(K^n)}{n} = \dim(K),$$

while in the second case,

$$\lim_{n\to\infty}\frac{\dim(K^n)}{n}=\dim(K)-1.$$

Moreover, the second case actually occurs. (For example, combine [170] and [147].) A modification of the proof of Theorem 23.10 should easily give the upper bound $\operatorname{mdim}(h_K)\leq\dim(K)-1$ in the second case. This suggests the following question, which, as far as we know, has not been addressed.

Question 23.12. Let K be a compact metric space, and let h_K be as in Definition 23.9. Does it follow that $\operatorname{mdim}(h_K)=\dim(K)$ or $\operatorname{mdim}(h_K)=\dim(K)-1$?

Shifts are not minimal (unless K has at most one point), but one can construct minimal subshifts with large mean dimension. A basic construction of this type is given in [164].

Theorem 23.13 (Proposition 3.5 of [164])**.** There exists a minimal invariant subset $X\subset([0,1]^2)^{\mathbb{Z}}$ such that $\operatorname{mdim}(h_{[0,1]^2}|_X)>1$.

A related construction is used in [93] to produce many more examples, including ones with arbitrarily large mean dimension.

We now recall the statement of Theorem 19.15.

Theorem 23.14 ([110])**.** Let X be a compact metric space. Assume that there is a continuous surjective map from X to the Cantor set. Let $h\colon X\to X$ be a minimal homeomorphism. Then $\operatorname{rc}(C^*(\mathbb{Z},X,h))\leq\frac{1}{2}\operatorname{mdim}(h)$.

It is hoped that $\operatorname{rc}(C^*(\mathbb{Z},X,h))=\frac{1}{2}\operatorname{mdim}(h)$ for any minimal homeomorphism of an infinite compact metric space X. This has been proved in [110] for some special systems covered by Theorem 23.14, slightly generalizing the construction of [93].

The hypothesis on existence of a surjective map to the Cantor set has other equivalent formulations, one of which is the existence of an equivariant surjective map to the Cantor set.

Proposition 23.15. Let X be a compact metric space, and let $h\colon X\to X$ be a minimal homeomorphism. Then the following are equivalent:

(1) There exists a decreasing sequence $Y_0\supset Y_1\supset Y_2\supset\cdots$ of nonempty compact open subsets of X such that the subset $Y=\bigcap_{n=0}^{\infty}Y_n$ satisfies $h^r(Y)\cap Y=\varnothing$ for all $r\in\mathbb{Z}\setminus\{0\}$.
(2) There is a minimal homeomorphism of the Cantor set which is a factor of (X,h) (Definition 2.27).
(3) There is a continuous surjective map from X to the Cantor set.
(4) For every $n\in\mathbb{Z}_{>0}$ there is a partition $\mathcal{P}$ of X into at least n nonempty compact open subsets.

We omit the proof.

Assume h is minimal and $h^n(Y)\cap Y=\varnothing$ for $n\in\mathbb{Z}\setminus\{0\}$. As in Remark 17.20, write $Y=\bigcap_{n=0}^{\infty}Y_n$ with $Y_0\supset Y_1\supset\cdots$ and $\operatorname{int}(Y_n)\neq\varnothing$ for all $n\in\mathbb{Z}_{\geq0}$, getting

$$C^*(\mathbb{Z},X,h)_Y=\varinjlim_n C^*(\mathbb{Z},X,h)_{Y_n}.$$

The algebras $C^*(\mathbb{Z},X,h)_{Y_n}$ are recursive subhomogeneous C*-algebras whose base spaces are closed subsets of X. (See Theorem 17.14.) The effect of requiring a

Cantor system factor is that one can choose Y and $(Y_n)_{n\in\mathbb{Z}_{\geq 0}}$ so that Y_n is both closed and open for all $n \in \mathbb{Z}_{\geq 0}$. Doing so ensures that $C^*(\mathbb{Z}, X, h)_{Y_n}$ is a homogeneous C*-algebra whose base spaces are closed subsets of X. Thus $C^*(\mathbb{Z}, X, h)_Y$ is a simple AH algebra. We get such a set Y by taking the inverse image of a point in the Cantor set.

To keep things simple, in these notes we will assume that h has a particular minimal homeomorphism of the Cantor set as a factor, namely an odometer system (Definition 2.22). The further simplification of assuming an odometer factor is that one can arrange $C^*(\mathbb{Z}, X, h)_{Y_n} \cong M_{p_n}(C(Y_n))$, that is, there is only one summand. This simplifies the notation but otherwise makes little difference.

We omit the proof of the following lemma. Some work is required, most of which consists of keeping notation straight. A more general version (assuming an arbitrary minimal homeomorphism of the Cantor set as a factor) is in [110].

Lemma 23.16. Let X be a compact metric space, and let $h\colon X \to X$ be a minimal homeomorphism. We assume that (X, h) has as a factor system the odometer on $X_d = \prod_{n=1}^{\infty}\{0, 1, 2, \ldots, d_n - 1\}$ (Definition 2.22) for a sequence $d = (d_n)_{n\in\mathbb{Z}_{>0}}$ of integers with $d_n \geq 2$ for all $n \in \mathbb{Z}_{>0}$. Let Y be the inverse image of $(0, 0, \ldots)$ under the factor map, and let Y_n be the inverse image of

$$\{0\}^n \times \prod_{k=n+1}^{\infty}\{0, 1, 2, \ldots, d_k - 1\}.$$

For $n \in \mathbb{Z}_{>0}$ set $p_n = \prod_{k=1}^{n} d_k$. For $m, n \in \mathbb{Z}_{\geq 0}$ with $n \geq m$, define

$$\psi_{n,m}\colon C(Y_m, M_{p_m}) \to C(Y_n, M_{p_n})$$

by

$$\psi_{n,m}(f) = \operatorname{diag}\big(f|_{Y_n},\, f\circ h^{p_m}|_{Y_n},\, f\circ h^{2p_m}|_{Y_n},\, \ldots,\, f\circ h^{(p_n/p_m-1)p_m}|_{Y_n}\big)$$

for $f \in C(Y_m, M_{p_m})$. Then

$$C^*(\mathbb{Z}, X, h)_Y \cong \varinjlim_n C(Y_n, M_{p_n}).$$

The map $\psi_{n,0}$ in the statement of the lemma has the particularly suggestive formula

$$\psi_{n,0}(f) = \operatorname{diag}\big(f|_{Y_n},\, f\circ h|_{Y_n},\, f\circ h^2|_{Y_n},\, \ldots,\, f\circ h^{p_n-1}|_{Y_n}\big).$$

The problem is now reduced to showing that if $A = \varinjlim_n C(Y_n, M_{p_n})$, with maps

$$\psi_{n,m}(f) = \operatorname{diag}\big(f|_{Y_n},\, f\circ h^{p_m}|_{Y_n},\, f\circ h^{2p_m}|_{Y_n},\, \ldots,\, f\circ h^{(p_n/p_m-1)p_m}|_{Y_n}\big),$$

then $\operatorname{rc}(A) \leq \frac{1}{2}\operatorname{mdim}(h)$.

We will make a further simplification, and prove instead the following theorem, also from [110].

Theorem 23.17 ([110])**.** Let X be an infinite compact metric space. Let $d = (d_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence of integers with $d_n \geq 2$ for all $n \in \mathbb{Z}_{>0}$. For $n \in \mathbb{Z}_{>0}$ set $p_n = \prod_{k=1}^{n} d_k$. Let $h\colon X \to X$ be a homeomorphism, and suppose that h^{p_n} is minimal for all $n \in N$. For $m, n \in \mathbb{Z}_{>0}$ with $m \leq n$, define $\psi_{n,m}\colon C(X, M_{p_m}) \to C(X, M_{p_n})$ by

$$\psi_{n,m}(f) = \operatorname{diag}\big(f,\, f\circ h^{p_m},\, f\circ h^{2p_m},\, \ldots,\, f\circ h^{(p_n/p_m-1)p_m}\big)$$

for $f \in C(X, M_{p_m})$. Using these maps, define

$$B = \varinjlim_{n} C(X, M_{p_n}).$$

Then $\mathrm{rc}(B) \leq \frac{1}{2}\mathrm{mdim}(h)$.

The following lemma (whose easy proof is left as an exercise) ensures that the direct system in Theorem 23.17 actually makes sense.

Lemma 23.18 ([110])**.** Let X, h, d, and $\psi_{n,m}$ for $m, n \in \mathbb{Z}_{>0}$ with $m \leq n$, be as in Theorem 23.17, but without any minimality assumptions on h. Then for $k, m, n \in \mathbb{Z}_{>0}$ with $k \leq m \leq n$, we have $\psi_{n,m} \circ \psi_{m,k} = \psi_{n,k}$.

Exercise 23.19. Prove Lemma 23.18.

The algebra B in Theorem 23.17 is a kind of AH model for the crossed product $C^*(\mathbb{Z}, X, h)$. In particular, it is always an AH algebra, while we needed the assumption of a Cantor set factor system to find a large subalgebra of $C^*(\mathbb{Z}, X, h)$ which is an AH algebra. This model has the defect that we must now assume that h^{p_n} is minimal for all $n \in \mathbb{Z}_{>0}$. Otherwise, it turns out that the direct limit isn't simple. (This minimality condition on the powers actually excludes systems with odometer factors.) The proof of the following lemma is a fairly direct consequence of the simplicity criterion in Proposition 2.1(iii) of [50].

Lemma 23.20 ([110])**.** Let X, h, d, and $\psi_{n,m}$ for $m, n \in \mathbb{Z}_{>0}$ with $m \leq n$, be as in Theorem 23.17, but without any minimality assumptions on h. Set $B = \varinjlim_n C(X, M_{p_n})$. Then B is simple if and only if h^{p_n} is minimal for all $n \in N$.

Exercise 23.21. Prove Lemma 23.20.

The main effect of passing to the situation of Theorem 23.17 is to further simplify the notation. For minimal homeomorphisms without Cantor set factor systems, the replacement of a direct limit of recursive subhomogeneous algebras with an AH algebra of the sort appearing in Theorem 23.17 is a much more substantial simplification. There are difficulties (presumably technical) in the more general context which we don't (yet) know how to solve.

We would like to use Theorem 6.2 of [176] to prove Theorem 23.17 (and also Theorem 23.14). Unfortunately, the definition there of the mean dimension of an AH direct system requires that the base spaces be connected, or at least have only finitely many connected components. If (X, h) has a Cantor set factor system, the base spaces in the AH model (and also in the direct system in Lemma 23.16) have surjective maps to the Cantor set. So we proceed more directly, although the arguments are closely related.

Lemma 23.22. Let X be a compact metric space and let $h \colon X \to X$ be a homeomorphism with no periodic points. Then for every $\varepsilon > 0$ and every finite subset $F \subset C(X)$ there exists $N \in \mathbb{Z}_{>0}$ such that for all $n \geq N$ there is a compact metric space K and a surjective map $i \colon X \to K$ satisfying:

(1) $\dim(K) < n[\mathrm{mdim}(h) + \varepsilon]$.
(2) For $m = 0, 1, \ldots, n-1$ and $f \in F$ there is $g \in C(K)$ such that $\|f \circ h^m - g \circ i\| < \varepsilon$.

The argument depends on nerves of covers and their geometric realizations. See Section 2.6 of [197], especially Definition 2.6.1, Definition 2.6.2, Definition 2.6.7, and the proof of Proposition 2.6.8, for more details of the theory than are presented here.

Definition 23.23. Let X be a topological space, and let $\mathcal{V}$ be a finite open cover of X, with $\varnothing \not\in \mathcal{V}$. The *nerve* $K(\mathcal{V})$ is the finite simplicial complex with vertices $[V]$ for $V \in \mathcal{V}$, and in which there is a simplex in $K(\mathcal{V})$ with vertices $[V_0], [V_1], \ldots, [V_n]$ if and only if $V_0 \cap V_1 \cap \cdots \cap V_n \neq \varnothing$.

The points $z \in K(\mathcal{V})$ (really, points z in its geometric realization) are thus exactly the formal convex combinations

$$z = \sum_{V \in \mathcal{V}} \alpha_V [V] \tag{23.1}$$

in which $\alpha_V \geq 0$ for all $V \in \mathcal{V}$, $\sum_{V \in \mathcal{V}} \alpha_V = 1$, and $\big\{ [V] \colon \alpha_V \neq 0 \big\}$ is a simplex in $K(\mathcal{V})$, that is,

$$\bigcap \big\{ V \in \mathcal{V} \colon \alpha_V \neq 0 \big\} \neq \varnothing.$$

Lemma 23.24. Let X be a topological space, and let $\mathcal{V}$ be a finite open cover of X, with $\varnothing \not\in \mathcal{V}$. Then $\dim(K(\mathcal{V})) = \operatorname{ord}(\mathcal{V})$.

Proof. It is immediate that $\operatorname{ord}(\mathcal{V})$ is the largest (combinatorial) dimension of a simplex occurring in $K(\mathcal{V})$. It follows from standard results in dimension theory (in [197], see Proposition 3.1.5, Theorem 3.2.5, and Theorem 3.2.7) that this dimension is equal to $\dim(K(\mathcal{V}))$. $\square$

Lemma 23.25. Let X be a topological space, and let $\mathcal{V}$ be a finite open cover of X, with $\varnothing \not\in \mathcal{V}$. Let $(g_V)_{V \in \mathcal{V}}$ be a partition of unity on X such that $\operatorname{supp}(g_V) \subset V$ for all $V \in \mathcal{V}$. Then there is a continuous map $i \colon X \to K(\mathcal{V})$ determined, using (23.1), by

$$i(x) = \sum_{V \in \mathcal{V}} g_V(x)[V]$$

for $x \in X$.

Exercise 23.26. Prove Lemma 23.25.

This exercise is straightforward.

At this point, we leave traditional topology.

Lemma 23.27. Let X be a compact Hausdorff space, and let $\mathcal{V}$, $(g_V)_{V \in \mathcal{V}}$, and $i \colon X \to K(\mathcal{V})$ be as in Lemma 23.25. Let $(x_V)_{V \in \mathcal{V}}$ be a collection of points in X such that $x_V \in V$ for $V \in \mathcal{V}$. Then there is a linear map $P \colon C(X) \to C(K(\mathcal{V}))$ (not a homomorphism) defined, following (23.1), by

$$P(f)\left(\sum_{V \in \mathcal{V}} \alpha_V [V] \right) = \sum_{V \in \mathcal{V}} \alpha_V f(x_V)$$

for $f \in C(X)$. Moreover:

(1) $\|P\| \leq 1$.
(2) For all $f \in C(X)$, we have

$$\|P(f) \circ i - f\| \leq \sup_{V \in \mathcal{V}} \sup_{x, y \in V} |f(x) - f(y)|.$$

The key point is part (2): if $f \in C(X)$ varies by at most $\delta > 0$ over each set $V \in \mathcal{V}$, then $P(f)$ is a function on $K(\mathcal{V})$ whose pullback to X is close to f. That is, if $\mathcal{V}$ is sufficiently fine, then we can approximate a finite set of functions on X by functions on the finite (in particular, finite dimensional) simplicial complex $K(\mathcal{V})$. Moreover, the dimension of $K(\mathcal{V})$ is controlled by the order of $\mathcal{V}$.

Proof of Lemma 23.27. It is easy to check that $P(f)$ is continuous, that P is linear, and that $\|P\| \leq 1$.

For (2), let $r > 0$ and suppose that for all $V \in \mathcal{V}$ and $x, y \in V$ we have $|f(x) - f(y)| \leq r$. Let $x \in X$ and estimate:

$$\begin{aligned} \bigl|P(f)(i(x)) - f(x)\bigr| &= \left|P(f)\left(\sum_{V \in \mathcal{V}} g_V(x)[V]\right) - \sum_{V \in \mathcal{V}} g_V(x) f(x)\right| \\ &\leq \sum_{V \in \mathcal{V}} g_V(x) |f(x_V) - f(x)| \leq \sum_{V \in \mathcal{V}} g_V(x) r = r. \end{aligned}$$

This completes the proof. ☐

Proof of Lemma 23.22. Choose a finite open cover $\mathcal{U}$ of X such that for all $U \in \mathcal{U}$, $x, y \in U$, and $f \in F$, we have $|f(x) - f(y)| < \frac{\varepsilon}{2}$. By definition, we have

$$\lim_{n \to \infty} \frac{\mathcal{D}\bigl(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\bigr)}{n} \leq \mathrm{mdim}(h).$$

Therefore there exists $N \in \mathbb{Z}_{>0}$ such that for all $n \geq N$ we have

$$\frac{\mathcal{D}\bigl(\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})\bigr)}{n} < \mathrm{mdim}(h) + \varepsilon.$$

Let $n \geq N$. Then there is a finite open cover $\mathcal{V}$ of X which refines

$$\mathcal{U} \vee h^{-1}(\mathcal{U}) \vee \cdots \vee h^{-n+1}(\mathcal{U})$$

and such that

$$\mathrm{ord}(\mathcal{V}) < n[\mathrm{mdim}(h) + \varepsilon]. \tag{23.2}$$

Since X is a compact metric space, we can choose a partition of unity $(g_V)_{V \in \mathcal{V}}$ on X such that $\mathrm{supp}(g_V) \subset V$ for all $V \in \mathcal{V}$. Apply Lemma 23.25, getting $i \colon X \to K(\mathcal{V})$, and let $P \colon C(X) \to C(K(\mathcal{V}))$ be as in Lemma 23.27.

Let $f \in F$ and let $m \in \{0, 1, \ldots, n-1\}$. Since $\mathcal{V}$ refines $h^{-m}(\mathcal{U})$, it follows that for all $V \in \mathcal{V}$ and $x, y \in V$ we have $\bigl|(f \circ h^m)(x) - (f \circ h^m)(y)\bigr| < \frac{\varepsilon}{2}$. So

$$\bigl\|P(f \circ h^m) \circ i - f \circ h^m\bigr\| \leq \frac{\varepsilon}{2} < \varepsilon.$$

We are done with the proof except for the fact that i might not be surjective. So define $K = i(X) \subset K(\mathcal{V})$. Since the dimension of a subspace can't be larger than the dimension of the whole space (see Proposition 3.1 5 of [197]),

$$\dim(K) \leq \dim(K(\mathcal{V})) = \mathrm{ord}(\mathcal{V}) < n[\mathrm{mdim}(h) + \varepsilon].$$

In place of $P(f \circ h^m)$ we use $P(f \circ h^m)|_K$. This completes the proof. ☐

The proof of Theorem 23.17 requires two further results. For both proofs, we refer to the original sources. The first is a special case of Theorem 5.1 of [285].

Theorem 23.28 (see Theorem 5.1 of [285])**.** Let X be a compact metric space and let $n \in \mathbb{Z}_{>0}$. Then

$$\operatorname{rc}(M_n \otimes C(X)) \leq \frac{\dim(X) - 1}{2n}.$$

Lemma 23.29 (Lemma 6.1 of [176])**.** Let B be a simple unital exact C*-algebra and let $r \in [0, \infty)$. Suppose:

(1) For every finite subset $S \subset B$ and every $\varepsilon > 0$, there is a unital C*-algebra D such that $\operatorname{rc}(D) < r + \varepsilon$ and an injective unital homomorphism $\rho \colon D \to B$ such that $\operatorname{dist}(a, \rho(D)) < \varepsilon$ for all $a \in S$.

(2) For every $s \in [0, 1]$ and every $\varepsilon > 0$, there exists a projection $p \in B$ with $|\tau(p) - s| < \varepsilon$ for all $\tau \in \mathrm{T}(B)$.

Then $\operatorname{rc}(B) \leq r$.

Proof of Theorem 23.17. We use Lemma 23.29. Certainly B is simple, unital, and exact. Since $C(X, M_{p_n}) \hookrightarrow B$ and $C(X, M_{p_n})$ has projections of constant rank k for any $k \in \{0, 1, \ldots, p_n\}$, condition (2) in Lemma 23.29 is satisfied.

We need to show that for every finite subset $S \subset B$ and every $\varepsilon > 0$, there is a unital C*-algebra D such that $\operatorname{rc}(D) < \frac{1}{2}\operatorname{mdim}(h) + \varepsilon$ and an injective unital homomorphism $\rho \colon D \to B$ such that $\operatorname{dist}(a, \rho(D)) < \varepsilon$ for all $a \in S$.

For $n \in \mathbb{Z}_{\geq 0}$ let $\psi_n \colon C(X, M_{p_n}) \to B$ be the map obtained from the direct limit description of B. Let $S \subset B$ be finite and let $\varepsilon > 0$. Choose $m \in \mathbb{Z}_{>0}$ and a finite set

$$S_0 \subset C(X, M_{p_m}) = M_{p_m}(C(X))$$

such that for every $a \in S$ there is $b \in S_0$ with $\|\psi_m(b) - a\| < \frac{1}{2}\varepsilon$. Let $F \subset C(X)$ be the set of all matrix entries of elements of S_0. Use Lemma 23.22 to find $N \in \mathbb{Z}_{\geq 0}$ such that for all $l \geq N$ there are a compact metric space K and a surjective map $i \colon X \to K$ such that $\dim(K) < l[\operatorname{mdim}(h) + \varepsilon]$ and for $r = 0, 1, \ldots, l - 1$ and $f \in F$ there is $g \in C(K)$ with

$$\|f \circ h^r - g \circ i\| < \frac{\varepsilon}{2p_m^2}.$$

Choose $n \geq m$ such that $p_n \geq N$. Choose K and i for $l = p_n$, so that

$$\dim(K) < p_n[\operatorname{mdim}(h) + \varepsilon]$$

and for $r = 0, 1, \ldots, p_n - 1$ and $f \in F$ there is $g \in C(K)$ with

$$\|f \circ h^r - g \circ i\| < \frac{\varepsilon}{2p_m^2}.$$

Define an injective homomorphism $\rho_0 \colon C(K) \to C(X)$ by $\rho_0(f) = f \circ i$ for $f \in C(K)$. Set $D = M_{p_n}(C(K))$ and define

$$\rho = \psi_n \circ (\mathrm{id}_{M_{p_n}} \otimes \rho_0) \colon D \to B.$$

Then ρ is also injective.

By Theorem 23.28,

$$\operatorname{rc}(D) \leq \frac{\dim(K) - 1}{2p_n} < \frac{\operatorname{mdim}(h) + \varepsilon}{2} < \frac{\operatorname{mdim}(h)}{2} + \varepsilon.$$

It remains to prove that $\operatorname{dist}(a, \rho(D)) < \varepsilon$ for all $a \in S$.

Let $a \in S$. Choose $b \in S_0$ such that $\|\psi_m(b) - a\| < \frac{\varepsilon}{2}$. For $j, k \in \{0, 1, \ldots, p_m - 1\}$, we let $e_{j,k} \in M_{p_m}$ be the standard matrix unit (except that we start the indexing at 0 rather than 1). Then there are $b_{j,k} \in F$ for $j, k \in \{0, 1, \ldots, p_m - 1\}$ such that

$b = \sum_{j,k=0}^{p_m - 1} e_{j,k} \otimes b_{j,k}$. By construction, for $r = 0, 1, \ldots, p_n - 1$ there is $g_{j,k,r} \in C(K)$ such that

$$\left\| g_{j,k,r} \circ i - b_{j,k} \circ h^r \right\| < \frac{\varepsilon}{2p_m^2}.$$

For $t = 0, 1, \ldots, p_n/p_m - 1$, define

$$c_t = \sum_{j,k=0}^{p_m - 1} e_{j,k} \otimes g_{j,k,tp_m} \in M_{p_m}(C(K)).$$

Then define

$$c = \operatorname{diag}\bigl(c_0, c_1, \ldots, c_{p_n/p_m - 1}\bigr) \in M_{p_n}(C(K)).$$

We claim that $\|\rho(c) - a\| < \varepsilon$, which will finish the proof. We have, using the definition of $\psi_{n,m}$ at the third step,

$$\begin{aligned}
\|\rho(c) - a\| &\leq \|a - \psi_m(b)\| + \|\psi_m(b) - \rho(c)\| \\
&< \frac{\varepsilon}{2} + \|\psi_{n,m}(b) - c\| \\
&< \frac{\varepsilon}{2} + \Bigl\| \operatorname{diag}\bigl(f,\ f \circ h^{p_m},\ f \circ h^{2p_m},\ \ldots,\ f \circ h^{(p_n/p_m - 1)p_m}\bigr) \\
&\qquad\qquad\qquad - \operatorname{diag}\bigl(c_0, c_1, c_2, \ldots, c_{p_n/p_m - 1}\bigr) \Bigr\| \\
&\leq \frac{\varepsilon}{2} + \max_{0 \leq t \leq p_n/p_m - 1} \|f \circ h^{p_m r} - c_t\| \\
&\leq \frac{\varepsilon}{2} + \max_{0 \leq t \leq p_n/p_m - 1} \sum_{j,k=1}^{p_m} \left\| g_{j,k,r} \circ i - b_{j,k} \circ h^r \right\| \\
&\leq \frac{\varepsilon}{2} + p_m^2 \left(\frac{\varepsilon}{2p_m^2} \right) = \varepsilon.
\end{aligned}$$

This completes the proof. ∎

24. Open Problems on Large Subalgebras and their Applications to Crossed Products

We discuss some open problems related to large subalgebras, some (but not all) of which have some connection with dynamical systems. We start with some which are motivated by particular applications, and then give some which are suggested by results already proved but for which we don't have immediate applications.

Not all the problems in [215] appear here. In particular, the ones about L^p operator crossed products have been omitted.

The first question is motivated by the hope that large subalgebras can be used to get more information about crossed products than we now know how to get. In most parts, we expect that positive answers would require special hypotheses, if they can be gotten at all. We omit definitions of most of the terms.

Question 24.1. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a large (or centrally large) subalgebra.

(1) Suppose that B has tracial rank zero (Definition 11.35). Does it follow that A has tracial rank zero?
(2) Suppose that B is quasidiagonal. Does it follow that A is quasidiagonal?
(3) Suppose that B has finite decomposition rank. Does it follow that A has finite decomposition rank?

(4) Suppose that B has finite nuclear dimension. Does it follow that A has finite nuclear dimension?

It seems likely that "tracial" versions of these properties pass from a large subalgebra to the containing algebra, at least if the tracial versions are defined using cutdowns by positive elements rather than by projections. But we don't know how useful such properties are. As far as we know, they have not been studied.

Next, we ask whether being stably large is automatic.

Question 24.2. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a large (or centrally large) subalgebra. Does it follow that $M_n(B)$ is large (or centrally large) in $M_n(A)$ for $n \in \mathbb{Z}_{>0}$?

We know that this is true if A is stably finite, by Proposition 20.11. Not having the general statement is a technical annoyance. This result would be helpful when dealing with large subalgebras of $C^*(\mathbb{Z}, C(X, D), \alpha)$ when D is simple unital, X is compact metric, and the homeomorphism of $\mathrm{Prim}(C(X, D)) \cong X$ induced by α is minimal. Some results on large subalgebras of such crossed products can be found in [7]; also see Theorem 22.4.

More generally, does Proposition 19.4 still hold without the finiteness assumption?

Question 24.3. Let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha\colon \mathbb{Z} \to \mathrm{Aut}(A)$ have the tracial Rokhlin property. Is there a useful large or centrally large subalgebra of $C^*(\mathbb{Z}, A, \alpha)$?

We want a centrally large subalgebra of $C^*(\mathbb{Z}, A, \alpha)$ which "locally looks like matrices over corners of A". The paper [185] proves that crossed products by automorphisms with the tracial Rokhlin property preserve the combination of real rank zero, stable rank one, and order on projections determined by traces. The methods were inspired by those of [202], which used large subalgebras (without the name). The proof in [185] does not, however, construct a single large subalgebra. Instead, it constructs a suitable subalgebra (analogous to $C^*(\mathbb{Z}, X, h)_Y$ for a small closed subset $Y \subset X$ with $\mathrm{int}(Y) \neq \varnothing$) for every choice of finite set $F \subset C^*(\mathbb{Z}, A, \alpha)$ and every choice of $\varepsilon > 0$. It is far from clear how to choose these subalgebras to form an increasing sequence so that a direct limit can be built. Similar ideas, under weaker hypotheses (without projections), are used in [182], and there it is also far from clear how to choose the subalgebras to form an increasing sequence.

The first intended application is simplification of [185].

Problem 24.4. Let X be a compact metric space, and let G be a countable amenable group which acts minimally and essentially freely on X. Construct a (centrally) large subalgebra $B \subset C^*(G, X)$ which is a direct limit of recursive subhomogeneous C*-algebras as in [203] whose base spaces are closed subsets of X, and which is the (reduced) C*-algebra of an open subgroupoid of the transformation group groupoid obtained from the action of G on X.

In a precursor to the theory of large subalgebras, this is in effect done in [202] when $G = \mathbb{Z}^d$ and X is the Cantor set, following ideas of [88]. The resulting centrally large subalgebra is used in [202] to prove that $C^*(\mathbb{Z}^d, X)$ has stable rank one, real rank zero, and order on projections determined by traces. (More is now known.) We also know how to construct a centrally large subalgebra of this kind when $G = \mathbb{Z}^d$ and X is finite dimensional (unpublished). This gave the first proof that, in this case, $C^*(\mathbb{Z}^d, X)$ has stable rank one and strict comparison of positive elements. (Again, more is now known.)

Unlike for actions of $\mathbb{Z}$, there are no known explicit formulas like that in Theorem 19.5; instead, centrally large subalgebras must be proved to exist via constructions involving many choices. They are direct limits of C*-algebras of open subgroupoids of the transformation group groupoid as in Problem 24.4. In each open subgroupoid, there is a finite upper bound on the size of the orbits; this is why they are recursive subhomogeneous C*-algebras (homogeneous when X is the Cantor set, as in [204]). In fact, the original motivation for the definition of a large subalgebras was to describe the essential properties of these subalgebras, as a substitute for an explicit description.

We presume, as suggested in Problem 24.4, that the construction can be done in much greater generality.

Problem 24.5. Develop the theory of large subalgebras of not necessarily simple C*-algebras.

One can't just copy Definition 19.1. Suppose B is a nontrivial large subalgebra of A. We surely want $B \oplus B$ to be a large subalgebra of $A \oplus A$. Take $x_0 \in A_+ \setminus \{0\}$, and take the element $x \in A \oplus A$ in Definition 19.1 to be $x = (x_0, 0)$. Writing $g = (g_1, g_2)$, we have forced $g_2 = 0$. Thus, not only would $B \oplus B$ not be large in $A \oplus A$, but even $A \oplus B$ would not be large in $A \oplus A$.

In this particular case, the solution is to require that x and y be full elements in A and B. What to do is much less clear if, for example, A is a unital extension of the form

$$0 \longrightarrow K \otimes D \longrightarrow A \longrightarrow E \longrightarrow 0,$$

even if D and E are simple, to say nothing of the general case.

The following problem goes just a small step away from the simple case, and just asking that x and y be full might possibly work for it, although stronger hypotheses may be necessary.

Question 24.6. Let X be an infinite compact metric space and let $h\colon X \to X$ be a homeomorphism which has a factor system which is a minimal homeomorphism of an infinite compact metric space (or, stronger, a minimal homeomorphism of the Cantor set). Can one use large subalgebra methods to relate the mean dimension of h to the radius of comparison of $C^*(\mathbb{Z}, X, h)$?

We point out that Lindenstrauss's embedding result for systems of finite mean dimension in shifts built from finite dimensional spaces (Theorem 5.1 of [163]) is proved for homeomorphisms having a factor system which is a minimal homeomorphism of an infinite compact metric space.

Problem 24.7. Develop the theory of large subalgebras of simple but not necessarily unital C*-algebras.

One intended application is to crossed products $C^*\big(\mathbb{Z},\, C(X, D),\, \alpha\big)$ when X is an infinite compact metric space, D is simple but not unital, and the induced action on X is given by a minimal homeomorphism. (Compare with Theorem 22.4.) Another possible application is to the structure of crossed products $C^*(\mathbb{Z}, X, h)$ when h is a minimal homeomorphism of a noncompact version of the Cantor set. Minimal homeomorphisms of noncompact Cantor sets have been studied in [166] and [167], but, as far as we know, almost nothing is known about their transformation group C*-algebras.

For a large subalgebra $B \subset A$, the proofs of most of the relations between A and B do not need B to be centrally large. The exceptions so far are for stable rank one and Z-stability. Do we really need centrally large for these results?

Question 24.8. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a large subalgebra (not necessarily centrally large). If B has stable rank one, does it follow that A has stable rank one?

That is, can Theorem 19.12 be generalized from centrally large subalgebras to large subalgebras?

Question 24.9. Let A be an infinite dimensional simple separable nuclear unital C*-algebra, and let $B \subset A$ be a large subalgebra (not necessarily centrally large). If B is Z-stable, does it follow that A is Z-stable?

That is, can Theorem 19.13 be generalized from centrally large subalgebras to large subalgebras?

It is not clear how important these questions are. In all applications so far, with the single exception of [73] (on the extended irrational rotation algebras), the large subalgebras used are known to be centrally large. In particular, all known useful large subalgebras of crossed products are already known to be centrally large.

Question 24.10. Does there exist a large subalgebra which is not centrally large? Are there natural examples?

The results of [73] depend on large subalgebras which are not proved there to be centrally large, but it isn't known that they are not centrally large.

Question 24.11. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a large subalgebra. If $\mathrm{RR}(B) = 0$, does it follow that $\mathrm{RR}(A) = 0$? What about the converse? Does it help to assume that B is centrally large in the sense of Definition 19.2?

If B has both stable rank one and real rank zero, and is centrally large in A, then A has real rank zero (as well as stable rank one) by Theorem 19.12. The main point of Question 24.11 is to ask what happens if B is not assumed to have stable rank one. The proof in [202] of real rank zero for the crossed product $C^*(\mathbb{Z}^d, X)$ of a free minimal action of $\mathbb{Z}^d$ on the Cantor set X (see Theorem 6.11(2) of [202]; the main part is Theorem 4.6 of [202]) gives reason to hope that if B is large in A and $\mathrm{RR}(B) = 0$, then one does indeed get $\mathrm{RR}(A) = 0$. Proposition 19.11 could also be considered evidence in favor. Nothing at all is known about conditions under which $\mathrm{RR}(A) = 0$ implies $\mathrm{RR}(B) = 0$.

Applications to crossed products may be unlikely. It seems possible that $C^*(G, X)$ has stable rank one for every minimal essentially free action of a countable amenable group G on a compact metric space X.

Question 24.12. Let A be an infinite dimensional simple separable unital C*-algebra. Let $B \subset A$ be centrally large in the sense of Definition 19.2. Does it follow that $K_0(B) \to K_0(A)$ is an isomorphism mod infinitesimals?

In other places where this issue occurs (in connection with tracial approximate innerness; see Proposition 6.2 and Theorem 6.4 of [208]), it seems that everything in K_1 should be considered to be infinitesimal.

A six term exact sequence for the K-theory of some orbit breaking subalgebras is given in Example 2.6 of [232]. Related computations for some special more complicated orbit breaking subalgebras can be found in [233]. See Theorem 17.31 and the discussion afterwards. Theorem 17.25, according to which the inclusion of $C^*(\mathbb{Z}, X, h)_{\{y\}}$ in $C^*(\mathbb{Z}, X, h)$ is an isomorphism on K_0, is an important consequence.

A positive answer to Question 24.12 would shed some light on both directions in Question 24.11.

Question 24.13. Let A be an infinite dimensional stably finite simple separable unital C*-algebra. Let $B \subset A$ be centrally large in the sense of Definition 19.2. If A has stable rank one, does it follow that B has stable rank one?

That is, does Theorem 19.12 have a converse? In many other results in Section 19, B has an interesting property if and only if A does.

Question 24.14. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a centrally large subalgebra. Let $n \in \mathbb{Z}_{>0}$. If $\mathrm{tsr}(B) \leq n$, does it follow that $\mathrm{tsr}(A) \leq n$? If $\mathrm{tsr}(B)$ is finite, does it follow that $\mathrm{tsr}(A)$ is finite?

That is, can Theorem 19.12 be generalized to other values of the stable rank? The proof of Theorem 19.12 uses $\mathrm{tsr}(B) = 1$ in two different places, one of which is not directly related to $\mathrm{tsr}(A)$, so an obvious approach seems unlikely to succeed.

As with Question 24.11, applications to crossed products seem unlikely.

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