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ASYMPTOTIC MORPHISMS AND E-THEORY

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Introduction

The purpose of these notes is to introduce the reader with the basic facts about asymptotic morphisms and E-theory, and show some of the recent applications of the theory in the literature. No new results are shown here. However, the notes bring together many of the basic facts of E-theory developed so far. The bulk of sections 1 to 5 are from Chapter II, Appendix B of Connes' book [Co]. We have added more detail to the proofs of the theorems stated there. Sections 3.1 and 6 come from a paper of Dadarlat and Loring [DL1]. Section 7 comes from journal articles by various authors, and should get the reader acquainted with the literature related to particular applications.

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1. Deformations & Asymptotic Morphisms

1.1. **Introduction.** The concrete realization of Higson's abstract category **E** [Hig2], was discovered jointly by A. Connes and N. Higson [CH], and is given by homotopy classes asymptotic morphisms, which can be thought of as "approximate *-homomorphisms" of C*-algebras. One place these maps arise is in certain continuous fields of C*-algebras. We will investigate the nature of such functions, and define composition between them. This composition yields a bilinear map

$$E(A,B) \times E(B,C) \to E(A,C)$$

which extends the Kasparov product [JT].

Definition 1.1. Let A, B be C*-algebras. An asymptotic morphism is a family $(\phi_t)_{t\in[1,\infty)}$ of maps from A to B such that the following conditions hold.

1. For any $a \in A$ the map $t \mapsto \phi_t(a)$ is norm continuous.
2. For any $a, b \in A$, $\lambda \in \mathbb{C}$ the following norm limits vanish
   (a) $\lim_{t\to\infty} (\phi_t(a) + \lambda\phi_t(b) - \phi_t(a + \lambda b))$
   (b) $\lim_{t\to\infty} (\phi_t(ab) - \phi_t(a)\phi_t(b))$
   (c) $\lim_{t\to\infty} (\phi_t(a^*) - \phi_t(a)^*)$

Nothing else is assumed about the ϕ_t's. Because of the weakness of the axioms, one must be careful. For instance, it does not follow directly from the axioms that for $a \in A$, the norm limit $\phi_t(aa^*) - \phi_t(a)\phi_t(a)^*$ vanishes as t goes to infinity. For this to hold (and we want it to hold) we first need norm boundedness for $\phi_t(a)$. Luckily, this can be proven directly from the axioms.

Lemma 1.2. *For any asymptotic morphism $\phi_t : A \to B$,*

$$\limsup_{t\to\infty} ||\phi_t(a)|| \leq ||a||.$$

Proof. We prove the condition first for projections. Let p be a projection in A. Then one has for any $\varepsilon > 0$ a $T_\varepsilon \in [1,\infty)$ such that for $t \geq T_\varepsilon$

$$\begin{aligned} ||\phi_t(p) - \phi_t(p)\phi_t(p)|| &< \varepsilon \text{ and} \\ ||\phi_t(p) - \phi_t(p)^*|| &< \varepsilon \end{aligned}$$

One concludes from the above inequalities that for small enough ε there is projection $q_\varepsilon \in B$ such that

$$||\phi_t(p) - q_\varepsilon|| < 4\varepsilon$$

for every $t \geq T_\varepsilon$. From this we must have that $\limsup ||\phi_t(p)|| \leq 1$. The result will follow for any $0 \leq a \leq 1$ since one can form a projection in $M_2(A)$

$$p = \begin{bmatrix} a & -\sqrt{a-a^2} \\ -\sqrt{a-a^2} & -a \end{bmatrix},$$

and then form the extension $(\phi_t^2) : M_2(A) \to M_2(B)$ (allow ϕ_t to act on each matrix entry). One checks this extension is an asymptotic morphism, and from this we conclude

$$\limsup ||\phi_t(a)|| \leq \limsup ||\phi_t^2(p)|| \leq 1.$$

The result for arbitrary $a \in A$ with $||a|| \leq 1$ follows from the fact that a can be written as a linear combination of four positive elements. So $\limsup ||\phi_t(a)|| < \infty$ for every $a \in A$. That this bound is actually $||a||$ will be explained in section 1.2 ■

Remark 1.3. The argument shown above for the projections is the same argument one uses to define a map on K-theory from an asymptotic morphism (cf. Section 1.4). One actually defines $\phi_*([p]) = [q_\varepsilon]$ for $[p] \in K_0(A)$.

Remark 1.4. Asymptotic morphisms can be defined on any C*-algebra; however in order to make composition work, we will need to assume our C*-algebras are separable. We will assume this without mentioning it from now on.

The exact motivation for an asymptotic morphism comes from a continuous field of C*-algebras defined in Dixmier (see [Dix, Chapter 10]).

Definition 1.5. Let T be a compact Hausdorff space and $(A_t)_{t\in T}$ a family of C*-algebras parameterized by T. Suppose $\Gamma \subset \Pi_{t\in T} A_t$ satisfies the following:

1. Γ is a subspace of $\Pi_{t\in T} A_t$ closed under the multiplication and *-operation (the operations are taken pointwise).
2. The set $\Gamma(t) = \{x(t) \mid x \in \Gamma\}$ is dense in A_t for all $x \in \Gamma$.
3. The map $t \mapsto \|x(t)\|$ is norm continuous for every $x \in \Gamma$.
4. Suppose $y \in \Pi_{t\in T} A_t$ is such that the map
$$t \mapsto \|y(t) - x(t)\|$$
is continuous for every $x \in \Gamma$. Then $y \in \Gamma$.

Then $\mathcal{A} = (A_T, \Gamma)$ is a continuous field of C*-algebras over T.

The continuous fields we are interested in are the *deformations* from A to B. Such is a continuous field of C*-algebras over the interval $[0,1]$ whose fiber at 0 is A, and whose fiber over $(0,1]$ is the constant fiber B. Suppose $(A_{[0,1]}, \Gamma)$ is such a field. Given any $a \in A$ there is a section $\alpha_a \in \Gamma$ satisfying $\alpha_a(0) = a$, $\alpha_a(t) \in B$ for $t \in (0,1]$. Thus we can associate an asymptotic morphism $\phi_t : A \to B$ via
$$\phi_t(a) = \alpha_a(1/t).$$

One may ask if such a process be reversed; that is, given any asymptotic morphism $\phi_t : A \to B$, can one associate a deformation from A to B to it. The answer is a partial yes, but we will need more machinery to see when and how this process can be done.

1.2. **Asymptotic vs Ordinary Morphisms.** In order to further develop the theory of asymptotic morphisms we need to make several observations. Two asymptotic morphisms, $(\phi_t), (\phi'_t) : A \to B$, are said to be *asymptotically equal* if and only if for all $a \in A$
$$\lim_{t\to\infty} (\phi_t(a) - \phi'_t(a)) = 0$$
Given an asymptotic morphism (ϕ_t) from A to B we can define an ordinary morphism $\tilde{\phi} : A \to B_\infty$ by $\tilde{\phi}(a)(t) = \phi_t(a)$, where B_∞ is the quotient algebra
$$\frac{C_b([1,\infty[, B)}{C_0([1,\infty[, B)}$$
of bounded, B-valued, continuous functions modulo functions that vanish at infinity. If (ϕ_t) and (ψ_t) are asymptotic morphisms from A to B, then $\tilde{\phi} = \tilde{\psi}$ as *-homomorphisms if and only if (ϕ_t) and (ψ_t) are asymptotically equal. This correspondence between asymptotic and ordinary morphisms, along with Choi-Effros lifting theorem allows one to view an asymptotic morphism as a family of *-linear maps, at least when A is nuclear.

Lemma 1.6. *Let A be nuclear and $\phi : A \to B_\infty$ a *-homomorphism. Then ϕ lifts to a completely positive linear map $\bar{\phi} : A \to C_b([1,\infty), B)$. In other words, if A is nuclear, then any asymptotic morphism $(\phi_t) : A \to B$ is asymptotically equal to a family of completely positive linear maps $(\psi_t) : A \to B$, parameterized by $t \in [1,\infty)$, satisfying (2 b) of definition 1.1.*

Another place where the correspondence between asymptotic and ordinary morphisms can be exploited completes the proof of lemma 1.2. As $\tilde{\phi}$ is a *-homomorphism, it is always a contraction, hence $\limsup \|\phi_t(a)\| \leq \|a\|$ for every $a \in A$. Much of the theory of asymptotic morphisms relies on this correspondence.

1.3. **Asymptotic Morphisms Associated with Deformations.** As mentioned in the introduction, one would like to know if given an arbitrary asymptotic morphism $(\phi_t) : A \to B$, when is (ϕ_t) associated to a deformation from A to B. The answer mainly lies in a property of "injectivity".

Definition 1.7. Let $(\phi_t) : A \to B$ be an asymptotic morphism.

1. [CH] We say that ϕ is *weakly injective* if the corresponding morphism $\tilde{\phi} : A \to B_\infty$ is injective.
2. [Lor1] We say that ϕ is *injective* if $\liminf_{t\to\infty} \|\phi_t(a)\| > 0$ for every $a \neq 0$.

Note what we call weakly injective is what [CH] call injective. We have adopted the convention of [Lor1]. Using the ordinary morphism associated with an asymptotic morphism, one sees that any asymptotic morphism is equivalent to the composition of a quotient map and a weakly injective asymptotic morphism. Also, injectivity implies weak injectivity, but not vice-versa. To see this, let A be a contractible C*-algebra, and define ϕ_t to be a continuous family of homomorphisms parameterized by $[1, \infty)$ which is the identity homomorphism when t is an odd integer and the zero homomorphism when t is an even integer [Lor1].

Lemma 1.8. [Lor1] *If ϕ is an injective asymptotic morphism, then $\lim_{t\to\infty} \|\phi_t(a)\| = \|a\|$ for every $a \in A$.*

Proof. The proof is achieved by reducing to a discrete parameter. Let $t_n \to \infty$ be a sequence in $[1, \infty)$. Associate to (ϕ_t) the ordinary morphism $\hat{\phi} : A \to \hat{B}$ where $\hat{B}$ is the quotient

$$\prod_{i=1}^{\infty} B \mod \bigoplus_{i=1}^{\infty} B,$$

and $\hat{\phi}(a) = \{\phi_{t_n}(a)\}_{n\in\mathbb{N}}$. If $\liminf \|\phi_t(a)\| < \|a\|$ then t_n can be chosen so that $\|\hat{\phi}(a)\| < \|a\|$. As $\hat{\phi}$ is a *-homomorphism, it must have a non trivial kernel. From this, we get for a non-zero $x \in \ker(\hat{\phi})$

$$0 < \liminf_{t\to\infty} \|\phi_t(x)\| \leq \liminf_{n} \|\phi_{t_n}(x)\| = 0,$$

a contradiction. Hence $\liminf \|\phi_t(a)\| \geq \|a\|$. However by lemma 1.2 we know the $\limsup \|\phi_t(a)\| \leq \|a\|$, so the lemma is proved. ∎

The strategy for constructing a deformation from an asymptotic morphism $(\phi_t) : A \to B$ is as follows [Co]. Regard $(B_t)_{t\in[1,\infty)}$ as the trivial fiber B and for each $a \in A$, $\phi(a)$ the sections of this bundle. Basically, we want to define the fiber at ∞ as A, and in order to have any sort of continuity at infinity, we better have ϕ injective. Regard

$[1, \infty]$ as $[0, 1]$, and place the fiber A at 0 and B everywhere else. If ϕ is injective, then it is weakly injective, and $\tilde{\phi}$ is an isomorphism onto its range. Thus there is a *-subalgebra, $C \subset C_b((0,1], B)$ with $A \cong C/C_0((0,1], B)$. Using the notation of definition 1.5 one lets $\Gamma = \{(\phi_{1/t}(a) + f(t))_{t \in (0,1]} \,|\, a \in A,\ f \in C_0((0,1], B)\}$, and checks that Γ satisfies the axioms of a continuous field over $[0, 1]$ with fiber A when $t = 0$ and B elsewhere.

In light of this, we have the following.

Proposition 1.9. [Lor1] *The following are equivalent:*

1. *ϕ is injective.*
2. *$(\phi_t) : A \to B$ is associated to a deformation from A to B.*
3. *$\lim \|\phi_t(a)\| = \|a\|$ for every $a \in A$.*

Proof. (2)⇒(3) and (3)⇒(1) are obvious. That (1)⇒(2) is true follows from the above discussion and lemma 1.8. ∎

1.4. **Asymptotic Morphisms and K-Theory.** Given an asymptotic morphism $(\phi_t) : A \to B$ one can extend to the unitizations $(\phi_t^+) : A^+ \to B^+$ by $\phi_t(a + \lambda) = \phi_t(a) + \lambda$. We saw how to extend (ϕ_t) to matrices in the proof of lemma 1.2. In order to see how asymptotic morphisms act on K-theory, we are going to need to know when two asymptotic morphisms are homotopic. The definition is the obvious one, stated below.

Definition 1.10. Two asymptotic morphisms, $(\phi_t^i) : A \to B$, $i \in \{0, 1\}$ are said to be *homotopic* if and only if there exists an asymptotic morphism $(\Phi_t) : A \to B[0,1] = C[0,1] \otimes B$ whose evaluation at 0 and 1 yields (ϕ_t^0) and (ϕ_t^1) respectively. We shall denote the homotopy classes of asymptotic morphisms from A to B as $[[A,\ B]]$.

Remark 1.11. It should be noted that if two asymptotic morphisms are asymptotically equal, they are homotopic via the straight line connecting them.

Note that a homotopy of asymptotic morphisms does NOT coincide with a homotopy between the corresponding *-homomorphisms because $B_\infty \otimes C[0,1] \ncong (B \otimes C[0,1])_\infty$ in general.

Given $(\phi_t) : A \to B$ one gets an induced map $\phi_* : K_*(A) \to K_*(B)$ as follows. Suppose $p \in M_n(A)$ is a projection. Then for large enough t, $\phi_t(p)$ approximately satisfies the projection identities (see the proof of lemma 1.2). So there is a projection $q_t \in M_n(B)$ nearby. Continuity in t ensures us that all such q_t's are homotopic. If one chooses a different projection in the range, say r_t, the closeness of $\phi_t(p)$ to both q_t and r_t shows the map is well defined. Using the same argument for unitaries shows the map is independent of the choice of projection from $[p]$, and also gives us a map between the K_1 groups. The fact that ϕ_* is homotopy invariant is clear from the definition of homotopy.

We close this section by stating a theorem due to T. Loring which provides a test for whenever an asymptotic morphism is associated to a deformation. This theorem is useful when constructing deformations of topological spaces (cf. Section 7.4).

Theorem 1.12. [Lor1] *Suppose A is a C^*-algebra such that for any non-zero ideal $I \lhd A$ the quotient map induces a non injective map on K-theory. Then for any C^*-algebra B and asymptotic morphism $(\phi_t) : A \to B$, if $\phi_* : K_*(A) \to K_*(B)$ is injective then ϕ is an asymptotic morphism associated to a deformation from A to B.*

2. Operations on Asymptotic Morphisms

2.1. **Composition of Asymptotic Morphisms.** Given $(\phi_t) : A \to B$ and $(\psi_t) : B \to C$, one would like to define a composition $(\psi \circ \phi)_t : A \to C$. Unfortunately, a direct composition $\theta_t(a) = \psi_t(\phi_t(a))$ will not work because of a lack of uniform continuity on an arbitrary asymptotic morphism. We illustrate with a simple counter-example. Let $f_t \in C_0(\mathbb{R})$, $t \in [1, \infty)$ be an approximate unit that is 1 on $[-t, t]$ and 0 on $(-\infty, -t-1] \cup [t+1, \infty)$. One defines an asymptotic morphism $\phi : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ by $\phi_t(g) = g \circ h_t$ where $h_t(x) = x - t$, $x \in \mathbb{R}$, and an asymptotic morphism $\psi_t : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ by $\psi_t(g) = gf_t$. Define a map $\psi_t \circ \phi_t : C_0(\mathbb{R}) \to C_0(\mathbb{R})$. One checks directly that $\lim_{t\to\infty} ||\psi_t \circ \phi_t(f_1^2) - \psi_t \circ \phi_t(f_1)\psi_t \circ \phi_t(f_1)||_\infty \neq 0$. Uh oh!

The problem in our counter-example above can be corrected by replacing ψ with a suitable homotopic asymptotic morphism. For instance, a change in parameter $t \mapsto r(t)$ where $r(t) : [1, \infty) \to [1, \infty)$ is a continuous, increasing function always yields a homotopic asymptotic morphism. In the example above, just changing ψ_t to $\psi_{r(t)}$ where $r(t) = 2t$ suffices. In order to do this in a more general situation, we'll need some notion of "uniformity" on an asymptotic morphism to see exactly what $r(t)$ should be.

Definition 2.1. Let $\phi_t : A \to B$ be an asymptotic morphism and K a subset of A. Then ϕ_t is *uniform* on K if and only if the following conditions hold.

1. $(t, a) \mapsto \phi_t(a)$ is a continuous map, from $[1, \infty) \times K$ to B.
2. For every $\varepsilon > 0$ there is a $T < \infty$ such that for every $t \geq T$ the following inequalities hold for every a, $b \in K$.
   (a) $||\phi_t(a) + \lambda\phi_t(b) - \phi_t(a + \lambda b)|| < \varepsilon$, $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$.
   (b) $||\phi_t(a)\phi_t(b) - \phi_t(ab)|| < \varepsilon$
   (c) $||\phi_t(a)^* - \phi_t(a^*)|| < \varepsilon$
   (d) $||\phi_t(a)|| < ||a|| + \varepsilon$

An arbitrary asymptotic morphism may not be uniform on compact subsets. However, we shall show that it is asymptotically equal to an asymptotic morphism that is. To do so, we shall need the Bartle-Graves selection theorem. For a proof consult [BP].

Theorem 2.2. *(Bartle-Graves) Let $u : E \to X$ be a surjective continuous linear operator from a Fréchet space E to a Fréchet space X. Then there exists a continuous function $f : X \to E$ such that $uf = id_X$ and $f(0) = 0$.*

It should be noted that f might not be linear.

Proposition 2.3. *Let $\phi_t : A \to B$ be an asymptotic morphism. Then (ϕ_t) is equivalent to an asymptotic morphism, (ψ_t), which is uniform on compact subsets of A.*

Proof. Given a $\phi_t : A \to B$, pass to $\tilde{\phi} : A \to B_\infty$. Apply the Bartle-Graves selection theorem to the quotient map $\pi : C_b([1,\infty[,B) \to B_\infty$ to get a continuous function $\sigma : B_\infty \to C_b([1,\infty[,B)$. Let $\psi_t(a) = \sigma \circ \tilde{\phi}(a)(t)$. Then ψ is clearly equivalent to ϕ. Condition (1) of definition 2.1 follows from the fact that $\tilde{\phi}$ and σ are continuous functions. For condition (2), we will show multiplicativity; the others are similar. So suppose condition (2 b) of definition 2.1 does not hold. This means that there exists a $\varepsilon_0 > 0$, and sequences $\{a_n\}, \{b_n\}$, $n \in \mathbb{N}$ such that

$$(\dagger)\ \varepsilon_0 \le \|\psi_n(a_n b_n) - \psi_n(a_n)\psi_n(b_n)\|$$

As K is compact, we can assume that $a_n \to a$, $b_n \to b$ for some $a, b \in K$. Also we have $a_n b_n \to ab$. Continuity of ψ gives us $\psi(a_n) \to \psi(a)$, $\psi(b_n) \to \psi(b)$ and $\psi(a_n b_n) \to \psi(ab)$ in $C_b([1,\infty),B)$. Thus there is an $N_1, N_2 \in \mathbb{N}$ such that

$$\|\psi(a_n b_n) - \psi(ab)\|_\infty < \frac{\varepsilon_0}{3}, \quad \forall n \ge N_1$$

$$\|\psi(a_n)\psi(b_n) - \psi(a)\psi(b)\|_\infty < \frac{\varepsilon_0}{3}, \quad \forall n \ge N_2$$

(here $\|\ \|_\infty$ is the supremum norm on $C_b([1,\infty),B)$). We also have an $N_3 \in \mathbb{N}$ such that for $t \ge N_3$

$$\|\psi_t(ab) - \psi_t(a)\psi_t(b)\| < \frac{\varepsilon_0}{3}$$

So choose an $n \in \mathbb{N}$ with $n \ge \max\{N_1, N_2, N_3\}$ then

$$\begin{aligned}(\dagger) &\le \|\psi_n(a_n b_n) - \psi_n(ab)\| + \|\psi_n(ab) - \psi_n(a)\psi_n(b)\| + \|\psi_n(a)\psi_n(b) - \psi_n(a_n)\psi_n(b_n)\| \\ &\le \|\psi(a_n b_n) - \psi(ab)\|_\infty + \|\psi_n(ab) - \psi_n(a)\psi_n(b)\| + \|\psi(a)\psi(b) - \psi(a_n)\psi(b_n)\|_\infty \\ &< \frac{\varepsilon_0}{3} + \frac{\varepsilon_0}{3} + \frac{\varepsilon_0}{3} = \varepsilon_0\end{aligned}$$

a contradiction. ■

To define composition, we define it on a dense subalgebra and then extend. In particular, let A be a separable C*-algebra and $(a_n)_{n=1}^\infty$ a countable , dense subset of A. One defines K_n to be the set of polynomials, $p(a_1, \cdots, a_n, a_1^*, \cdots, a_n^*)$, of degree $2n$ with rational coefficients no bigger in absolute value than 2^n. With this one gets a sequence, $(K_n)_{n=1}^\infty$ of compact sets in A. Let

$$\mathcal{A} = \bigcup_{n=1}^{\infty} K_n.$$

Then $\mathcal{A}$ is a dense *-sub-algebra of A. Moreover, we have for every $n \in \mathbb{N}$, $K_n \subseteq K_{n+1}$, $K_n + K_n \subseteq K_{n+1}$, $K_n K_n \subseteq K_{n+1}$, and $\lambda K_n \subseteq K_n$, $\forall \lambda \in \mathbb{C}$, $|\lambda| \leq 1$. We can now define composition of asymptotic morphisms on $\mathcal{A}$ as given by the following proposition.

Proposition 2.4. *Let $\mathcal{A} \subset A$ be as above, and let $\phi_t : A \to B$ and $\psi_t : B \to C$ be asymptotic morphisms with ϕ_t uniform on compact sets of $\mathcal{A}$ and ψ_t uniform on compact sets of B. Then there exists a continuous increasing function, $r : [1, \infty[\to [1, \infty[$ such that the composition $\theta_t = \psi_{s(t)} \circ \phi_t : \mathcal{A} \to C$ is uniform on compact sets for every continuous increasing function $s : [1, \infty[\to [1, \infty[$, with $s \geq r$.*

Proof. Let $\mathcal{A} = \cup K_n$ as above. We shall construct $r(t)$ as follows. Choose t_n such that ϕ_t satisfies condition (2) of definition 2.1 on K_n, for $\varepsilon = \frac{1}{n}$, $t \geq t_n$. One can choose the t_n's to be a non-decreasing, divergent sequence. Define

$$K'_n = \{\phi_t(a) \,|\, a \in K_{n+1},\ t \leq t_{n+1}\}$$

Let $r_n = r(t_n)$ be such that ψ_t satisfies condition (2) of definition 2.1 for K'_n, (which is clearly compact) for $\varepsilon = \frac{1}{n}$ (once again, the r_n are increasing). One then "connects the dots" to get $r(t)$. We must show for any $s \geq r$ the composition $\psi_{s(t)}\phi_t$ is uniform on compact sets.

Let $\varepsilon > 0$ be arbitrary and $K \subset \mathcal{A}$ compact. There exists an $N \in \mathbb{N}$ large enough so that $K \subseteq K_N$, $\frac{1}{N} \leq \varepsilon$. Pick any $t \geq t_N$. Then there exists $m \in \mathbb{N}$, $m \geq N$ with $t_{m+1} > t \geq t_m \geq t_N$. Let $a, b \in K \subseteq K_N$. Then a, b, ab, $a + \lambda b \in K_{N+1} \subseteq K_{m+1}$, $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$. Thus, $\phi_t(a)$, $\phi_t(b)$, $\phi_t(ab)$, $\phi_t(a + \lambda b) \in K'_m$. Moreover $s(t) \geq r(t) \geq r(t_m)$ so as ψ is uniform on K'_m

$$\|\psi_{s(t)}\phi_t(a + \lambda b) - \psi_{s(t)}\phi_t(a) - \lambda\psi_{s(t)}\phi_t(b)\| < \frac{1}{m} \leq \frac{1}{N} \leq \varepsilon$$

as are all the other identities from condition (2) of definition 2.1. Condition (1) is obvious. ■

Given $\phi_t : A \to B$, $\psi_t : B \to C$, one defines $\theta_t : \mathcal{A} \to C$ by $\theta_t = \psi_{s(t)} \circ \phi_t$. Then $\tilde{\theta} : \mathcal{A} \to C_\infty$ is a bounded morphism (since all C*-subalgebra morphisms are contractions). Thus one has a continuous extension $\tilde{\theta} : A \to C_\infty$.

Definition 2.5. With the notation from above, let θ_t the associated asymptotic morphism given by $\theta_t(a) = \tilde{\theta}(a)(t)$ $\forall a \in A$, $t \in [1, \infty)$. θ_t is a well defined (up to asymptotic equality) extension of the composition $\psi \circ \phi$.

Proposition 2.6. *(1) The homotopy class $[\theta] \in [[A, C]]$ is independent of the choice of $\mathcal{A}$ and $s(t)$. It only depends on the homotopy class of ϕ and ψ.*
(2) The composition of asymptotic morphisms is associative.

Proof. [Co] Part (1) follows from the fact that an involutive subalgebra of A generated by two σ-compact sets is still σ-compact. Part (2) follows using the involutive subalgebra $\mathcal{B} \subseteq B$ generated by $\phi_t(\mathcal{A})$ where (ϕ_t) is uniform on compact subsets of A. Since $\mathcal{B}$ is σ-compact, the conclusion follows. ∎

2.2. **Tensor Products of Asymptotic Morphisms.** It is quite straightforward to construct tensor products of asymptotic morphisms by using the corresponding *-homomorphism associated to it. Recall the definition of the maximal tensor product of C*-algebras, A, B, denoted $A \otimes_\vee B$.

Lemma 2.7. *Let $\phi_t : A \to C$ and $\psi_t : B \to C$ be asymptotic morphisms such that the commutator $[\phi_t(a), \psi_t(b)] \to 0$ as $t \to \infty$. Then there exists a unique (up to asymptotic equality) asymptotic morphism $\theta_t : A \otimes_\vee B \to C$ such that*

$$\theta_t(a \otimes b) - \phi_t(a)\psi_t(b) \to 0, \quad \forall a \in A, \ \forall b \in B.$$

Proof. Use the corresponding morphisms, $\tilde{\phi}$ and $\tilde{\psi}$ into C_∞ and apply the universal property of the maximal tensor product. ∎

Corollary 2.8. *If $\phi_t : A \to C$ and $\psi_t : B \to D$ are asymptotic morphisms, then there exists a unique (up to asymptotic equality) asymptotic morphism $(\phi \otimes_\vee \psi)_t$: $A \otimes_\vee B \to C \otimes_\vee D$ with $(\phi \otimes_\vee \psi)_t(a \otimes b) - \phi_t(a) \otimes \psi_t(b) \to 0$ as $t \to \infty$.*

Proof. Unitize C_∞ and D_∞. Now use the lemma on the asymptotic morphisms $\bar{\phi}_t(a) = \phi_t(a) \otimes 1_{D_\infty}$ and $\bar{\psi}_t(b) = 1_{C_\infty} \otimes \psi_t(b)$, $a \in A$, $b \in B$. ∎

Remark 2.9. It should be noted that the construction of $\phi \otimes \psi$ in the above proof may destroy some 'nice' properties that either (ϕ_t) or (ψ_t) already have. For instance, if both $(\phi_t), (\psi_t)$ are linear for each t, or continuous (in A and B), then it may not be the case that $(\phi \otimes \psi)_t$ inherits the same properties. If A and B are both nuclear, however, lemma 1.6 still applies.

We will only be working with tensor products of nuclear C*-algebras, so we can use the above results without explicitly stating the norm.

3. The Category Asym

We will denote by **Sep** the category of separable C*-algebras and *-homomorphisms, **HSep** will be category of separable C*-algebras and homotopy classes of *-homomorphisms, and **Asym** the category of separable C*-algebras and homotopy classes of asymptotic morphisms. There is an obvious functor that takes an ordinary *-homomorphism, $\phi : A \to B$, to the asymptotic morphism $\phi_t = \phi$ for every $t \in [1, \infty)$.

3.1. **Properties of $[[-,-]]$.** One can view $[[A, B]]$ as a pointed set with base point the zero asymptotic morphism, and using composition it is clear how $[[A, -]]$ and $[[-, B]]$ can be made into covariant and contravariant functors respectively from **Asym** to **Asym**. The set $[[A, B]]$ behaves very similarly in terms of the non-commutative homotopy theory for $[A, B]$, the morphism set for **HSep**, set out in [Ros].

First, one can stabilize $[[A, B]]$ to $[[A, B \otimes \mathcal{K}]]$ in order to get an additive structure, which is accomplished as follows. If $\phi, \psi \in [[A, B \otimes \mathcal{K}]]$, then one defines the sum

$$(\phi \oplus \psi)_t(a) = \kappa\left(\begin{bmatrix} \phi_t(a) & 0 \\ 0 & \psi_t(a) \end{bmatrix}\right)$$

where $\kappa : M_2(B \otimes \mathcal{K}) \to \mathcal{K}$ is any *-isomorphism (unique up to homotopy). With this structure, one can only conclude $[[A, B \otimes \mathcal{K}]]$ is an abelian moniod with identity the 0 asymptotic morphism. We can also use the topological suspension functor S, $SB = B \otimes C_0((0,1))$ to make $[[A, SB]]$ a (not necessarily abelian) group. By stabilizing both ways, and forming $[[A, SB \otimes \mathcal{K}]]$ one gets an abelian group with bilinear composition $[[A, SB \otimes \mathcal{K}]] \times [[SB \otimes \mathcal{K}, SC \otimes \mathcal{K}]] \to [[A, SC \otimes \mathcal{K}]]$. The additive inverse is obtained using reflection:$C_0(\mathbb{R}) \to C_0(\mathbb{R}) : f \mapsto -f$. The proofs of these are the same as the proofs for $[A, B]$. See [Ros, Theorem 3.1], or [CH].

The covariant functor $[[C, -]]$ also satisfies the long exact sequence induced by an ordinary morphism $p : A \to B$. To construct this, suppose $p : A \to B$ is a *-homomorphism. One forms the mapping cone, C_p as the pullback of p and evaluation at $0 : C_0([0,1), B) \to B$. Explicitly

$$C_p = \{(a, f) \in A \oplus CB \,|\, p(a) = f(0)\}.$$

Where $CB = C_0([0,1), B) \cong C_0([0,1)) \otimes B$ is the cone of B.

There are *-homomorphisms $k : SB \to C_p$ and $\alpha : C_p \to A$ given by

$$k(f) = (0, f) \text{ and } \alpha(\mathrm{a}, \mathrm{f}) = \mathrm{a}.$$

With these at our disposal, we have the following:

Theorem 3.1. *Let A, B be C^*-algebras, and $p : A \to B$ a *-homomorphism. Then for any C-*-algebra, C the following sequence is exact*

$$\cdots \overset{Sk_*}{\to} [[C, SC_p]] \overset{S\alpha_*}{\to} [[C, SA]] \overset{Sp_*}{\to} [[C, SB]] \overset{k_*}{\to} [[C, C_p]] \overset{\alpha_*}{\to} [[C, A]] \overset{p_*}{\to} [[C, B]].$$

The higher order sets in the $\cdots$ are just suspensions. The contravariant sequence does not have such a property, and exactness for it can only be achieved by forming a suspended stable homotopy category, in which case you get the so called Puppe exact sequences [Ros, Section 3]. The proof of the above theorem is verbatim to [Ros, Theorem 3.8] with $[[-, -]]$ replacing $[-, -]$ as the pointed sets.

If one has a short exact sequence $0 \to J \to A \overset{p}{\to} B \to 0$, then in order to get exactness at A for the functor $[[C, -]]$ there should be an isomorphism between the mapping cone C_p and the ideal J. In the next section, we set out to prove this isomorphism. There is a small price in order to achieve such a goal; we can only get an isomorphism between SJ and SC_p. So when we set out to define the category $\mathbf{E}$, you can bet there will be a suspension in it.

3.2. **Extensions and Asymptotic Morphisms.** Let $0 \to J \to A \to B \to 0$ be a short exact sequence of C*-algebras. There exists, by [Vo], in the ideal J of A, $(u_t)_{t\in[1,\infty)}$ $0 \le u_t \le 1$ satisfying the following:

1. $\|u_t x - x\| \to 0$ whenever $t \to \infty$ $\forall x \in J$
2. $\|x u_t - x\| \to 0$ whenever $t \to \infty$ $\forall x \in J$
3. $\|[u_t, y]\| = 0$ whenever $t \to \infty$ $\forall y \in A$
4. $t \to u_t$ is norm continuous

We call (u_t) a *quasi-central continuous approximate unit* for J in A.

Since $0 \le u_t \le 1$, $f(u_t)$ makes sense for all $f \in C_0(0,1)$. Moreover, $\forall a \in A$, one has $\lim_{t\to\infty}[f(u_t), a] = 0$. To see this, note that f is the uniform limit of a sequence of polynomials, p_n, in $C[0,1]$. Thus $[f(u_t), a] = \lim_{n\to\infty}[p_n(u_t), a]$. But $[p_n(u_t), a] \to 0$ as $t \to \infty$ by (3) above, since every term of p_n has a factor u_t. This allows us to define a "connecting map" on arbitrary short exact sequences which will give us important asymptotic morphisms later.

Proposition 3.2. *Let $0 \to J \to A \xrightarrow{p} B \to 0$ be a short exact sequence of C*-algebras. Let $m : B \to A$ be a continuous section of p (see Theorem 2.2), and let (u_t) be a quasi-central approximate unit for J in A, indexed by $t \in [1, \infty)$. Then there exists an asymptotic morphism $\phi_t : SB \to J$ given by*

$$\phi_t(f \otimes b) = f(u_t)m(b).$$

Moreover the homotopy class ϵ_p of (ϕ_t) is independent of the choice of m and (u_t).

To prove this Proposition we need the following fact.

Lemma 3.3. *Let $J \lhd A$, and u_t a quasi-central continuous approximate unit for J in A. Let $f \in C_0(0,1)$ then $\lim_{t\to\infty} f(u_t)j = 0$ for every $j \in J$.*

Proof. It suffices to prove the result for polynomials h. The result for arbitrary f will follow by the Stone-Weierstrass theorem. So suppose $h \in C_0((0,1))$ and $h = \sum_{i=1}^n \lambda_i x^i$ for $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$. We know that $\sum_{i=1}^n \lambda_i = 0$. Thus

$$\begin{aligned} \|\sum_{i=1}^n \lambda_i u_t^i j\| &= \|\sum_{i=1}^n \lambda_i u_t^i j - \sum_{i=1}^n \lambda_i j\| \\ \|\sum_{i=1}^n [\lambda_i (u_t^i j - j)]\| &\le \sum_{i=1}^n |\lambda_i| \|u_t^i j - j\|. \end{aligned} \tag{1}$$

Let $\varepsilon > 0$. For each i, $1 \le i \le n$, choose t_i such that $\|u_t j - j\| < \frac{\varepsilon}{i \sum |\lambda_k|}$ $\forall t \ge t_i$. Then

$$\begin{aligned} \|u_t^i j - j\| &= \|u_t^i j - u_t^{i-1} j + u_t^{i-1} j - u_t^{i-2} j + \cdots + u_t j - j\| \\ &\le \|u_t^{i-1}\| \|u_t j - j\| + \cdots + \|u_t j - j\| \end{aligned} \tag{2}$$

and since $\|u_t\| \le 1$ we have (2) less than $\frac{\varepsilon}{\sum |\lambda_k|}$ thus (1) is less than ε for every $t \ge \max\{t_1, \cdots, t_n\}$. ∎

The proof of Proposition 3.2: With the notation from the proposition, we must show that the above equation actually defines an asymptotic morphism. To do so, we will show that the map $(f,b) \mapsto f(u_t)m(b)$ defines an ordinary bilinear map from $C_0(0,1) \times B$ to J_∞ that respects multiplication and the *-operation. This will extend to an ordinary morphism:$C_0(0,1) \otimes B \to J_\infty$ by the universal property of the maximal (=minimal) tensor product.

It is clear that linearity, multiplicativity, and the *-operation are preserved in the first variable, since it is just evaluation at u_t. To show linearity in the second variable we must show for every $a, b \in A,\ \lambda \in \mathbb{C},\ f \in C_0(0,1)$

$$||f(u_t)m(a+\lambda b) - f(u_t)m(a) - \lambda f(u_t)m(b)|| \to 0, \text{ as t} \to \infty$$

However, we note that the above is equal to

$$||f(u_t)(m(a+\lambda b) - m(a) - \lambda m(b))||$$

and $m(a+\lambda b) - m(a) - \lambda m(b) \in J$. So the result follows from lemma 3.3. The other identities are similar (multiplicativity relies on the fact that u_t is quasi-central in A). So the map extends to $\phi : SB \to J_\infty$. This results in an asymptotic morphism $\phi_t : SB \to J$.

The part about homotopy classes follows, since sections are unique up to homotopy and quasi-central approximate units are convex. ■

Remark: If the short exact sequence of proposition 3.2 splits, the proof is an immediate application of lemma 2.7.

With proposition 3.2 we can associate to any class of extensions a class of asymptotic morphisms. In order to get a KK-class one needs to have a completely positive lifting of p. This occurs in particular if J is nuclear. Conversely, given an asymptotic morphism associated to a deformation from A to B, one can get an extension of SB by A as follows: Denote by E the C*-algebra generated by the restriction of the continuous field to $[0,1)$ (see [Dix, Chapter 10]), and note the following is exact

$$0 \to SB \to E \to A \to 0.$$

Here the map $SB \to E$ is the inclusion and $E \to A$ is evaluation at 0 [CH, Co].

Proposition 3.2 is the key for achieving the isomorphism between the ideal in a short exact sequence and the mapping cone of the surjective morphism.

3.3. **The Ideal-Mapping Cone Isomorphism.** Let $0 \to J \xrightarrow{j} A \xrightarrow{p} B \to 0$ be an exact sequence of C*-algebras. Let C_p be the mapping cone of p, and α, k as in section 3.1. One also has the natural inclusion, $i : J \to C_p$ given by $i(y) = (y, 0)$. Clearly we have the identity $\alpha \circ i = j$. Although we cannot invert the morphism i in **Asym** we can invert its suspension Si. The inverse will be the asymptotic morphism $\epsilon_\sigma \in [[SC_p, SJ]]$ associated to the short exact sequence

$$0 \to SJ \to CA \xrightarrow{\sigma} C_p \to 0$$

where $\sigma(f) = (f(0), p \circ f)$. It is a simple exercise to check exactness. Let $\epsilon_\sigma \in [[SC_p, SJ]]$ be as in Proposition 3.2 applied to the above sequence.

Lemma 3.4. *The map $Sj \circ \epsilon_\sigma : SC_p \to SA$ is homotopic to $S\alpha$ in the class of asymptotic morphisms from SC_p to SA.*

Proof. Let u_t, $t \in [1, \infty)$ be a quasi-central approximate unit for J in A. We will choose h_t, $t \in [2, \infty)$ as in Figure 1 for our quasi-central approximate unit for $C_0(0,1)$ in $C[0,1]$ (let $h_t = h_2$ for $1 \le t < 2$). We let $\epsilon_\sigma \in [[SC_p, SJ]]$ be as in proposition 3.2 represented as

$$\phi_t(f \otimes x) = f(h_t \otimes u_t)\tilde{x}$$

where $\tilde{x} \in CA$ is such that $\sigma(\tilde{x}) = x$. Evaluation at a point $s \in (0,1)$ gives

$$f(h_t(s)u_t)\tilde{x}(s).$$

We note that Sj is just the inclusion of SJ in SA and in the algebra SA, ϕ_t equivalent to ψ_t where

$$\psi_t(f \otimes x)(s) = f(h_t(s))\tilde{x}(s).$$

since $u_t \to 1$ in $\mathcal{M}(J)$. We now show that ψ_t is equivalent to θ_t, where

$$\theta_t(f \otimes x)(s) = f(j_t(s))\tilde{x}(s)$$

where $j_t : [0,1] \to [0,1]$ is as in Figure 2. So we must show that

$$\lim_{t \to \infty} \sup_{s \in [0,1)} \|[f(h_t(s)) - f(j_t(s))]\tilde{x}(s)\| \to 0$$

One notes that

$$\begin{aligned} \sup_{0 \le s \le 1} \|(f(h_t(s)) - f(j_t(s)))\tilde{x}(s)\| &= \sup_{1-\frac{1}{t} \le 1} \|(f(h_t(s)) - f(j_t(s)))\tilde{x}(s)\| \\ &\le \|f\|_\infty \sup_{1-\frac{1}{t} \le 1} \|\tilde{x}(s)\| \to 0 \end{aligned}$$

as $t \to \infty$ since $\tilde{x}(s) \to 0$ as $s \to 1$. One can use a similar argument to show that θ_t is equivalent to

$$\eta_t(f \otimes x)(s) = (f(j_t(s)))\tilde{x}(0)$$

Only one more homotopy to go! For each t define a homotopy $H_t(\lambda) = j_{\lambda(t-1)+1}$ (we are just deforming $j_t(s)$ to $j_1(s) = s$). Then η_t is homotopic to $S\alpha$ via the homotopy

$$\Phi_t(f \otimes x)(\lambda, s) = f(H_t(\lambda)(s))\tilde{x}(0)$$

since $\tilde{x}(0) = \alpha(x)$. ∎

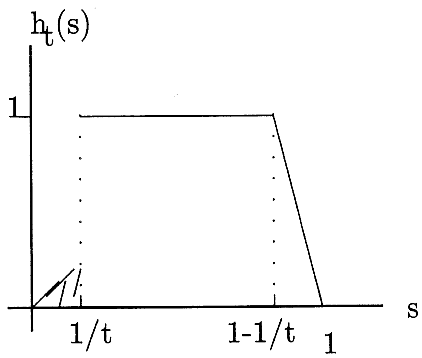

Figure 1

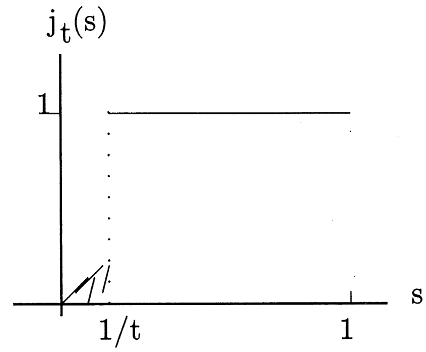

Figure 2

Corollary 3.5. *The map* $Si \in [[SJ, SC_p]]$ *is an isomorphism with inverse* $\epsilon_\sigma \in [[SC_p, SJ]]$.

Proof. After Lemma 3.4 we have (up to homotopy)
$Sj \circ \epsilon_\sigma = S\alpha$ and $Sj \circ \epsilon_\sigma = S(\alpha \circ i) \circ \epsilon_\sigma$. Combining these two gives $Si \circ \epsilon_\sigma = id$.
Conversely $Sj \circ \epsilon_\sigma \circ Si = S\alpha \circ Si = S(\alpha \circ i) = Sj$. So $\epsilon_\sigma \circ Si = id$. ∎

4. E-Theory

4.1. **The Additive Category E.** We define the category **E** as follows. The objects of **E** are separable C*-algebras, and the arrows $E(A, B) = [[SA \otimes \mathcal{K}, SB \otimes \mathcal{K}]]$ with $SA = A \otimes C_0(\mathbb{R})$. In light of section 3.1 $E(A, B)$ is an abelian group with identity the zero asymptotic morphism. Using the bilinear map $E(A, B) \times E(B, C) \to E(A, C)$ given by composition of asymptotic morphisms makes **E** into an additive category.

It is clear how one can define a functor F from **Sep** to **E** that takes an object A to $SA \otimes \mathcal{K}$ and takes an ordinary *-homomorphism p to the homotopy class of the constant asymptotic morphism, $\phi_t = p \otimes id$ for each $t \in [1, \infty)$. We will denote the image of a morphism p from **Sep** in **E** as $\hat{p}$. We can also define covariant and contravariant functors $E(A, -)$ and $E(-, A)$: **Sep** $\to$ **E** respectively, by defining $\hat{p}_* = E(A, p)$ and $\hat{p}^* = E(p, A)$ as composition of $\hat{p}$ on the left and right respectively, for any $p : B \to C$. Both these functors are homotopy invariant and stable by the definition of **E**.

Remark 4.1. By taking suspensions in the definition of **E**, we ensure half-exactness of the functors $E(A, -)$ and $E(-, A)$. Stabilizing with the compacts ensures matrix stability. The above definition is "symmetric" and makes proofs for half exactness and Bott periodicity simpler. However, for calculations, it is sometimes easier to define $E(A, B) = [[SA, SB \otimes \mathcal{K}]]$. Both of these groups are isomorphic, as the next proposition shows. We state it in its full generality: that of moniods.

Proposition 4.2. *For every (separable) C*-algebra* A, B $[[A, B \otimes \mathcal{K}]] \cong [[A \otimes \mathcal{K}, B \otimes \mathcal{K}]]$ *as abelian monoids (or groups).*

Proof. Define a map $F : [[A \otimes \mathcal{K}, B \otimes \mathcal{K}]] \to [[A, B \otimes \mathcal{K}]]$ by $F(\phi) = \phi \circ \alpha$. Where $\alpha : A \to A \otimes \mathcal{K} : a \mapsto a \otimes e$, e a rank one projection in $\mathcal{K}$. This is clearly an abelian monoid (group) homomorphism. It is also injective. For surjectivity, suppose $\psi \in [[A, B \otimes \mathcal{K}]]$. Define $\phi \in [[A \otimes \mathcal{K}, B \otimes \mathcal{K}]]$ by $\phi = \beta \circ (\psi \otimes id_{\mathcal{K}})$, where $\beta : \mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$ is any isomorphism (all such β are homotopic). Then $\phi \circ \alpha(a) = \phi(a \otimes e) = \beta(\psi \otimes id)(a \otimes e) = \beta(\psi(a) \otimes e) \sim_h \psi(a)$. The last homotopy follows because if B is stable, the map $b \mapsto b \otimes e$ is homotopic to a *-isomorphism. ■

4.2. **Bott Periodicity and Half Exactness of E(A, B).** In this section, we prove the following:

Theorem 4.3. *Let $0 \to J \xrightarrow{j} A \xrightarrow{p} B \to 0$ be exact in* **Sep**. *Then for any separable C^*-algebra D the following are exact:*

$$(a)\ E(D, J) \xrightarrow{\hat{j}_*} E(D, A) \xrightarrow{\hat{p}_*} E(D, B)$$
$$(b)\ E(B, D) \xrightarrow{\hat{p}^*} E(A, D) \xrightarrow{\hat{j}^*} E(J, D).$$

That the covariant sequence, (a), is exact follows immediately from Theorem 3.1 and Corollary 3.5. We have to sweat it out a bit to prove (b). Functoriality gives Image$(\hat{p}^*) \subseteq \ker(\hat{j}^*)$. The best we can get right now regarding the reverse inclusion is the following:

Proposition 4.4. *Let $0 \to J \xrightarrow{j} A \xrightarrow{p} B \to 0$ be exact. Let $x \in [[A, D]]$ be such that $x \circ j \sim_h 0$. Then there is a $k \in [[S^2B, S^2D]]$ with $S^2x = k \circ S^2p$.*

Proof. Using the notation of lemma 3.4 $x \circ j \sim_h 0$ implies that $Sx \circ S\alpha = Sx \circ Sj \circ \epsilon_\sigma \sim_h 0$. Thus $S(x \circ j) \circ \epsilon_\sigma \sim_h 0$, which means that $S(h \circ \alpha) \sim_h 0$. Let $(\Phi_t) : SC_p \to SD \otimes C([0, 1])$ be the homotopy connecting $S(x \circ \alpha)$ and 0. By restricting Φ_t to $SB \lhd C_p$ we get an asymptotic morphism $(k_t) : S^2B \to SD \otimes C([0, 1])$. We finally note that $ev_1(k_t) = ev_0(k_t) = 0$, were ev_i is evaluation at $i \in [0, 1]$. So $(k_t) : S^2B \to SD \otimes C_0(]0, 1[) = S^2D$, is the required map. ■

With the above proposition, it is clear what we must set out to do; prove Bott Periodicity in both variables. As a result of Cuntz's Bott Periodicity theorem, we already have $E(A, S^2D) \cong E(A, D)$ [Cu]. Using this, we can get Bott periodicity in the first variable.'

Theorem 4.5. Bott Periodicity *There are natural isomorphisms $E(S^2A, B) \cong E(A, B) \cong E(A, S^2B)$.*

This gives us:

Corollary 4.6. *The suspension functor $S : E(A, B) \to E(SA, SB)$ is an abelian group isomorphism for every A and B.*

Corollary 4.6 and proposition 4.4 give us the rest of the proof of theorem 4.3 (b).

4.3. **The Proof of Bott Periodicity.** We already mentioned we have Bott periodicity in the functor $E(D,-)$. We will exploit this property to get Bott periodicity for the contravariant case. To do so, we will use the "reduced Toeplitz sequence"

$$0 \to \mathcal{K} \to \mathcal{T}_0 \xrightarrow{p} C_0(\mathbb{R}) \to 0$$

constructed by taking the classical Toeplitz extension

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow{q} C(\mathbb{T}) \to 0$$

and denoting by $\mathcal{T}_0$ the kernel of the map q composed with evaluation at a point $* \in \mathbb{T}$ (see [Cu] for more on this). Using a result from [Cu] we have that $E(D, \mathcal{T}_0)$ and $E(D, S\mathcal{T}_0)$ are zero for every C*-algebra D. So by the six-term cyclic exact sequence we have the following.

$$\begin{array}{ccccc} E(D,\mathcal{K}) & \rightarrow & 0 & \rightarrow & E(D,C_0(\mathbb{R})) \\ \delta \uparrow & & & & \downarrow \\ E(D,C_0(\mathbb{R}^2)) & \leftarrow & 0 & \leftarrow & E(D,S\mathcal{K}) \end{array}$$

Whence the connecting map δ is an isomorphism. We wish to investigate exactly what the connecting map is. It is constructed as follows. One checks that

$$0 \to C_0(\mathbb{R}^2) \xrightarrow{k} C_p \xrightarrow{\alpha} \mathcal{T}_0 \to 0$$

is exact. We regard $C_0(\mathbb{R}^2)$ as $C_0(]0,1[, C_0(]0,1[))$, and define $k(f) = (0, f)$. The map α is the usual projection. Thus

$$E(D, C_0(\mathbb{R}^2)) \xrightarrow{\hat{k}_*} E(D, C_p) \xrightarrow{\hat{\alpha}_*} E(D, \mathcal{T}_0) = \{0\}$$

is exact. Hence by the six-term cyclic exact sequence associated with the above exact sequence, $\hat{k}_*$ is an isomorphism of abelian groups for every D.

Since δ is unique, it must be the composition

$$E(D, C_0(\mathbb{R}^2)) \xrightarrow{\hat{k}_*} E(D, C_p) \xrightarrow{\hat{i}_*^{-1}} E(D, \mathcal{K}) \tag{3}$$

where $\hat{i}^{-1} : SC_p \otimes \mathcal{K} \to S\mathcal{K} \otimes \mathcal{K}$ is the asymptotic morphism inverse to the map $\hat{i} : S\mathcal{K} \otimes \mathcal{K} \to SC_p \otimes \mathcal{K}$ (lemma 3.4 & corollary 3.5), and the '$_*$' denotes composition on the left.

Lemma 4.7. *The map $k : C_0(\mathbb{R}^2) \to C_p$ is an isomorphism in the category* **E**. *i.e. There is an asymptotic morphism $\phi \in E(C_p, C_0(\mathbb{R}^2))$ with $\phi \circ \hat{k} \sim_h id$ and $\hat{k} \circ \phi \sim_h id$.*

Proof. Replace D by C_p in (3) and observe that the map $\hat{k}_* : E(C_p, C_0(\mathbb{R}^2)) \to E(C_p, C_p)$ is an isomorphism. So there is a $\phi \in E(C_p, C_0(\mathbb{R}^2))$ with $k_*(\phi)$ equal to the identity map, $id : SC_p \otimes \mathcal{K} \to SC_p \otimes \mathcal{K}$. But $\hat{k}_*(\phi) = \hat{k} \circ \phi = id$, so ϕ is the left inverse to k in **E**. To show that $\phi \circ \hat{k} = id$ replace D by $C_0(\mathbb{R}^2)$ in (3) and note that $\hat{k}_*(\phi \circ \hat{k}) = \hat{k}_*(id)$. Since $\hat{k}_*$ is injective, we must have $\phi \circ \hat{k} \sim_h id$. ■

Theorem 4.8. *For every C^*-algebra A there is a natural isomorphism of abelian groups $\Sigma : E(A, B) \to E(S^2A, B)$.*

Proof. Define $\Sigma : E(A, B) \to E(S^2A, B)$ by $\Sigma(\psi) = \psi \circ (\hat{k} \circ \hat{\imath})^{-1}$. It is checked that this defines the isomorphism needed.

To show it is natural one checks for any morphism $f : A \to A'$ in **E** the following diagram commutes for any C*-algebra B:

$$\begin{array}{ccc} E(A', B) & \stackrel{\Sigma}{\to} & E(S^2A', B) \\ f^* \downarrow & & \downarrow (S^2f)^* \\ E(A, B) & \stackrel{\Sigma}{\to} & E(S^2A, B) \end{array}$$

where f^* is just composition on the right. ■

Thus, theorem 4.5 is true.

Remark 4.9. One can give an alternative proof for Bott Periodicity in **E** by using the so called Heisenberg deformation [ENN]. Roughly speaking, the irreducible representations of $C^*(H_3)$, the group algebra of the Heisenberg group, form a continuous field over $\mathbb{R}$ with fiber at 0 equal to $C_0(\mathbb{R})$ and whose fiber elsewhere is $\mathcal{K}$. One can restrict to the interval $[0, 1]$ and form a deformation from $C_0(\mathbb{R})$ to $\mathcal{K}$. The asymptotic morphism associated to this deformation yields an isomorphism (in **E**) between $\mathcal{K}$ and $C_0(\mathbb{R}^2)$.

Knowing how asymptotic morphisms act on K-theory, one has a map

$$K_0(C_0(\mathbb{R}^2)) \times E(C_0(\mathbb{R}^2), \mathcal{K}) \to K_0(\mathcal{K}).$$

By applying the asymptotic morphism associated with the Heisenberg deformation to the generator of $K_0(C_0(\mathbb{R}^2)) \cong \mathbb{Z}$, one gets Bott Periodicity for the functor K_0. This was proven in [ENN].

5. The Universal Property of E

What good is a category without a universal property? In this section we will define the universal property of the category **E**. Once this is defined, it links our category to the abstract one defined in [Hig2], and also to KK-theory if the C*-algebras are (K-)nuclear (see [Sk] for the definition of K-nuclearity).

5.1. **Passing Functors to E.** We begin by letting $F : \mathbf{Sep} \to \mathbf{Ab}$ be a homotopy invariant, half-exact, stable functor. Cuntz's Bott Periodicity theorem [Cu] states F has Bott Periodicity. Thus we have a natural isomorphism, $F(A) \to F(S^2A \otimes \mathcal{K})$.

We would like to pass F to a functor on **E**. We recall that every asymptotic morphism $(\phi_t) : A \to B$ corresponds to an ordinary morphism $\tilde{\phi} : A \to B_\infty$, where B_∞ is the quotient $\frac{C_b([1,\infty),B)}{C_0([1,\infty),B)}$. We wish to exploit the short exact sequence

$$C_0([1, \infty), B) \to C_b([1, \infty), B) \stackrel{\pi}{\to} B_\infty$$

since the first algebra is isomorphic to the cone of B, and hence is contractible. However, neither $C_b([1,\infty), B)$ or B_∞ are separable, so we'll have to work with separable subalgebras. Let $\phi_t(A) = \tilde{B}_\phi \subset B_\infty$, $B_\phi = \pi^{-1}(\tilde{B})$. With these, we get a short exact sequence of separable C*-algebras

$$0 \to C_0([1,\infty), B) \to B_\phi \xrightarrow{\pi} \tilde{B}_\phi \to 0.$$

The first algebra is contractible. So $F(\pi)$ is an isomorphism by the six-term cyclic exact sequence.

Using this, we can show how to pass F to a functor on $\mathbf{E}$. To do so, let $(\phi_t) : SA \otimes \mathcal{K} \to SB \otimes \mathcal{K}$ be an asymptotic morphism uniform on compact sets. Denote by $\phi_{F*} : F(A) \to F(B)$ the following composition

$$\begin{aligned} F(S^2A \otimes \mathcal{K}) &\xrightarrow{F(id\otimes\tilde{\phi})} F(S(\widetilde{SB \otimes \mathcal{K}})_\phi) \xrightarrow{F(id\otimes\pi)^{-1}} \\ &F(S(SB \otimes \mathcal{K})_\phi) \xrightarrow{F(id\otimes ev_1)} F(S^2B \otimes \mathcal{K}) \end{aligned} \tag{4}$$

where ev_1 is evaluation at 1. We finish off the construction by applying the Bott isomorphism and the stability isomorphism on either end to get $F(-) \cong F(S^2 - \otimes\mathcal{K})$. One checks the following:

Proposition 5.1. *The above composition maps $1_A \in E(A, A)$ to $1_{F(A)}$.*

By letting the functor $E(A, -)$ play the role of F we get:

Proposition 5.2. (Compare [Hig1, Theorem 3.5]) *The homomorphism $\phi_{E(A,-)*} = \phi_{E*}$ maps 1_A to $[[\phi]]$.*

The map ϕ_{E*} is a long-winded way of composing ϕ on the left.

Theorem 5.3. (Compare [Hig1, Theorem 3.7]) *For every C^*-algebra A and $x \in F(A)$ there exists a unique natural transformation $\alpha : E(A, -) \to F(-)$ with $\alpha_A : E(A, A) \to F(A)$ satisfying $\alpha_A(1_A) = x$.*

Proof. If such $\alpha : E(A, -) \to F(-)$ exists, one must have $\alpha_B : E(A, B) \to F(B)$ satisfying

$$\alpha_B([[\phi]]) = \alpha_B(\phi_{E*}(1_A)) = \phi_{F*}(\alpha_A(1_A)).$$

So α_B is determined by α_A, hence it is unique. To show existence, one defines $\alpha_B(\phi) = \phi_{F*}(x)$, where $x \in F(A)$. One has $\alpha_A(1_A) = x$ by proposition 5.1, and must show that $\alpha_B(\phi)$ is independent of representative chosen in the homotopy class. To do this, suppose $\phi \sim_h \psi$. By doing a similar construction in (4) to the homotopy $\Phi_t : SA \otimes \mathcal{K} \to SB \otimes \mathcal{K} \otimes C[0,1]$ connecting ϕ_t and ψ_t, one checks that $\phi_{F*} = \psi_{F*}$. Naturality of α is straightforward, and is left as an exercise. ■

Remark 5.4. It should be noted that the natural transformations α_A are necessarily additive, thus abelian group homomorphisms. It is a straightforward exercise to check this using functoriality and additivity [Cu, Proposition 4.1 (c)] of F.

Of course, all this can be applied to homotopy invariant, half-exact, stable *contravariant* functors by reversing the arrows in (4). This gives us a contravariant analogue of proposition 5.2.

Proposition 5.5. *The homomorphism* $\phi^{E(-,B)*}$ *maps* 1_B *to* $[[\phi]]$.

We can now state the universal property of $\mathbf{E}$ (compare [Hig1, Theorem 4.5] and [Hig2, Theorem 3.6]).

Theorem 5.6. The universal property of E: *Let* $\mathbf{A}$ *be an additive category and* $F : \mathbf{Sep} \to \mathbf{A}$ *a functor such that the bifunctor* $\mathbf{A}(F(-), F(-))$ *is homotopy invariant, half-exact and stable in each variable. Then there exists a unique additive functor* $\hat{F} : \mathbf{E} \to \mathbf{A}$ *with* $F = \hat{F} \circ E$.

Proof. We define a functor $\hat{F}$ by $\hat{F}(A) = F(A)$ on objects. On arrows we define $\hat{F} : E(A, -) \to \mathbf{A}(F(A), F(-))$ to be the natural transformation that takes $1_A \in E(A, A)$ to $1_{F(A)}$. Such a functor is unique by theorem 5.3. One checks that $\hat{F}$ respects compositions, and is thus a functor. ■

The above propositions' proofs are almost verbatim to the proofs Higson used to define the category **KK** in [Hig1]; the difference being we use the machinery we built for **E** rather than **KK**. As a result of the universal property, the category **E** defined is the same as the abstract category defined in [Hig2] by N. Higson. Furthermore, as the abstract category coincides with the category **KK** whenever A is (K-)nuclear [Hig2, Theorem 3.5], $E(A, B) \cong KK(A, B)$.

In light of the isomorphism between **E** and **KK**, one sees that $E(\mathbb{C}, A) \cong K_0(A)$ as $\mathbb{C}$ is nuclear. Thus by Proposition 4.2 and [Ros, Theorem 4.1, Corollary 4.2] we have

$$[[C_0(\mathbb{R}), SA \otimes \mathcal{K}]] = E(\mathbb{C}, A) \cong K_0(A) \cong \pi_1(SA \otimes \mathcal{K}) \cong [C_0(\mathbb{R}), SA \otimes K].$$

Here π_1 denotes the first homotopy functor. In particular, if A is stable, every asymptotic morphism from $C_0(\mathbb{R})$ into SA is homotopic to a *-homomorphism. We also have $E(\mathbb{C}, SA) \cong K_1(A)$. The isomorphism takes a unitary, u, in the unitization of $A \otimes \mathcal{K}$ and associates an ordinary morphism $C_0(\mathbb{R}) \to A \otimes \mathcal{K} : f \mapsto f(u)$ via the functional calculus [Co].

For K-nuclear A, the homology theory $E(-, B)$ links up with the K-homology of Kasparov [Kas] and the Brown-Douglas-Fillmore description [BDF]. In particular, for a locally compact Hausdorff space, X

$$E(C_0(X), \mathbb{C}) = [[SC_0(X), S\mathcal{K}]] \cong K^0(C_0(X)) \cong K_0(X).$$

We shall see in next section that for many X, the suspension is superfluous.

6. Unsuspended E-theory

In the general definition of E-theory, one is inevitably stuck with a suspension. The reason is twofold: first the functor $[[A, -]]$ is not half exact, and secondly, $[[A, B \otimes \mathcal{K}]]$ in general is only an abelian monoid. One way of remedying the latter case is to ask for the map $A \to A \otimes \mathcal{K} : a \mapsto a \otimes e$, where $e \in \mathcal{K}$ is a rank 1 projection, to have an additive inverse in $[[A, A \otimes \mathcal{K}]]$. That is, assume the existence of an asymptotic morphism $(\eta_t) : A \to A \otimes \mathcal{K}$ such that the map

$$a \mapsto \begin{bmatrix} a \otimes e & 0 \\ 0 & \eta_t(a) \end{bmatrix}$$

is homotopic to the zero map. This certainly makes $[[A, B \otimes \mathcal{K}]]$ into an abelian group; the remarkable result of Dadarlat and Loring is that this is a sufficient condition to "unsuspend" E-theory.

Theorem 6.1. [DL1] *If the map $a \to a \otimes e$ has an additive inverse in $[[A, A \otimes \mathcal{K}]]$ then for every separable C^*-algebra B*

$$E(A, B) \cong [[A, B \otimes \mathcal{K}]].$$

Moreover, the suspension map $S : [[A, B \otimes \mathcal{K}]] \to [[SA, SB \otimes \mathcal{K}]]$ induces the isomorphism.

The proof exploits the additive inverse and the split exactness of the functor $[[A, -]]$ to produce isomorphisms between $[[A, B \otimes \mathcal{K}]]$ and $[[S^2A, S^2B \otimes \mathcal{K}]] \cong E(A, B)$. For the details, see [DL1].

We will call, as in [DL1], a C*-algebra, A, *homotopy symmetric* if the hypothesis of theorem 6.1 holds. There are many cases where this is true. In particular, it holds for $A = C_0(\mathbb{R})$. In fact, for any locally compact metrizable topological space X, $C_0(X)$ is homotopy symmetric [DL1].

Up to homotopy, there are no interesting *-homomorphisms from $C_0(X)$ to $\mathcal{K}$ where X is a locally compact, finite CW-complex. As we have seen, the E-theory groups give us another description of the K-homology of X in the sense of Brown-Douglas-Fillmore [BDF]. As $C_0(X)$ satisfies the hypothesis of theorem 6.1 [DL1], we have

$$E(C_0(X), \mathcal{K}) = [[SC_0(X), S\mathcal{K}]] \cong [[C_0(X), \mathcal{K}]].$$

Yielding a description of the K-homology of X in terms of homotopy classes of asymptotic morphisms. We close off this section with a lemma whose proof is straightforward.

Lemma 6.2. [DL1] *If A is homotopy symmetric then for any C^*-algebra B, $A \otimes B$ is homotopy symmetric.*

7. Applications of E-theory

The final section of these notes gives a quick overview of some of the areas where E-theory has been used.

7.1. **The Classification of AH-algebras.** One of the recent classification problems has been to determine an invariant for the so called approximately homogeneous (AH) algebras. These algebras are inductive limits $A = \varinjlim A_n$ where

$$A_n = \bigoplus_{i=1}^{k_n} M_{n,i}(C_0(X_{n,i})).$$

Here $X_{n,i}$ is a finite CW-complex. In the case $X_{n,i} = \{pt\}$ for every n and i, one gets an AF-algebra. If the $X_{n,i}$ are intervals, or circles, one gets AI and A$\mathbb{T}$-algebras respectively. In all these cases, K-theory has been the invariant for classification; it stands to reason that K-theory could be the invariant for the AH-algebras.

Elliott and Gong [EG] classified the AH-algebras whose $\dim(X_{n,i}) \leq 3$, and either (a) the inductive limit algebras were simple, or (b) $K^*(X_{n,i})$ are torsion free (here $K^*(X_{n,i}) = K_*(C_0(X_{n,i}))$). The invariant was ordered K-theory. Elliott also conjectured that this was the complete invariant for all such AH-algebras. In [D1], Dadarlat showed the conjecture was true for simple AH-algebras with slow dimension growth (so $\sup(\dim(X_{n,i})) < \infty$), and whose $X_{n,i}$ had torsion free K-theory. However Gong [G] showed the conjecture was false if the $X_{n,i}$'s had torsion in their K-groups.

In [G], Gong constructed two AH-algebras using the space $X = \mathbb{RP}^2 \vee S^2$, the wedge of the real projective plane and the 2-sphere, as the $X_{n,i}$'s. One has $\dim(X) = 2$, and $K^0(X) = \mathbb{Z}/2$ has torsion. These two AH-algebras had isomorphic ordered K-theory, but they were not unsuspended E-equivalent, hence could not be isomorphic. Gong showed this was the only restriction. In other words, if two unital real rank zero AH-algebras were unsuspended E-equivalent, then they were isomorphic. Since finite CW-complexes are homotopy symmetric, this amounts to a KK-equivalence (or E-equivalence).

Theorem 7.1. [G] *Two unital real rank zero AH-algebras, A, B, are isomorphic if and only if A is KK-equivalent (or E-equivalent) to B.*

7.2. **Dimension Drop Algebras and Mod-p K-theory.** Another useful application of the unsuspended E-theory has been on the dimension drop algebras. The non unital one being defined as

$$A_n = \{f \in C_0(0,1] \otimes M_n(\mathbb{C}) \mid f(1) \text{ is scalar}\}.$$

The unitized A_n takes a scalar value at 0 also. The unitized A_n has been used by Elliott as building blocks in A$\mathbb{T}$-algebras with torsion in their K_1 group [Ell]. A result from [DL1] is that A_n is homotopy symmetric, and moreover

$$[[A_n, B]] \cong [A_n, B].$$

So there are no 'real' asymptotic morphisms from a the non-unital dimension drop into any C*-algebra. The proof uses a property called semiprojectivity, which can be used to show that any *-homomorphism $A \to B_\infty$ (hence asymptotic morphism) can be lifted, for some m, to a *-homomorphism $A \to C_b([1,\infty), B)/C_0([1,m), B)$.

This shows that any asymptotic morphism from A_n to B is equivalent to a path of *-homomorphisms, so the result follows.

The importance of such a result is it gives us a definition of the Mod-p K-theory introduced by Cuntz and Schochet, in terms of homotopy classes of ordinary *-homomorphisms. One defines mod-p K-theory as

$$K_*(A;\mathbb{Z}/p) = K_*(A \otimes D)$$

where D is any C*-algebra KK-equivalent to a commutative one and $K_0(D) \cong \mathbb{Z}/p$, $K_1(D) = 0$. One can use the algebra A_p since $K_0(A_p) = 0$ and $K_1(A_p) \cong \mathbb{Z}/p$, as long as one does a degree shift. Extending the concept to KK-theory [Bl]

$$KK(A, B;\mathbb{Z}/p) = KK(A, B \otimes D) \cong KK_1(A \otimes D, B).$$

In particular, as $\mathbb{C}$ is nuclear [DL1]

$$\begin{aligned} K_0(B;\mathbb{Z}/p) &\cong KK(\mathbb{C}, B;\mathbb{Z}/p) \cong E_1(\mathbb{C}, B \otimes A_p) \cong E(\mathbb{C} \otimes A_p, B) \\ &= [[SA_p, SB \otimes \mathcal{K}]] \cong [[A_p, B \otimes \mathcal{K}]] \cong [A_p, B \otimes \mathcal{K}]; \end{aligned}$$

giving a definition of mod-p K-theory in terms of homotopy classes of ordinary morphisms. This definition of mod-p K-theory has played an important role in classifying certain C*-algebras of real rank zero [DL2].

7.3. **Classification of Purely Infinite, Simple, Unital C*-algebras.** As we have seen in the previous sections, for nuclear C*-algebras, E-theory can be used in place of KK-theory in order to get classification results. Sometimes, it is more convenient to use the theory of asymptotic morphisms rather than KK-elements. Nowhere is this more apparent than in N.C. Phillips paper [Ph] on the classification of separable, purely infinite, simple, unital C*-algebras.

In [Ph], Phillips shows that a subset of full asymptotic morphisms $[[A, B \otimes \mathcal{K} \otimes \mathcal{O}_\infty]]$, denoted $\tilde{E}_A(B)$, is actually a group isomorphic to $KK(A, B)$ (see [Ph] for the definition of "full"). Here $\mathcal{O}_\infty$ is denotes the Cuntz algebra: the universal C*-algebra generated by countably many isometries satisfying $u_i^* u_i = 1$, $u_i u_j = 0$ for $i \neq j$ and $\sum u_i u_i^* = 1$. The heart of this proof is two results of Kirchberg.

Lemma 7.2. [Kir] *(1) If A is separable, unital, and simple then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$.*
(2) If A is separable, unital, nuclear and simple then $\mathcal{O}_\infty \otimes A \cong A$.

With the above isomorphisms, one is able to show that $\tilde{E}_A(-)$ is a homotopy invariant, half-exact, stable functor from separable, purely infinite, unital, nuclear, simple C*-algebras into abelian groups, and $\tilde{E}_A(B) \cong KK(A, B)$ [Ph]. From this isomorphism, one is able to construct from a KK-equivalence of A and B, a *-isomorphism.

Theorem 7.3. [Ph] *Let A and B be separable, purely infinite, simple, unital C*-algebras. Suppose there is an invertible element $\eta \in KK(A, B)$ satisfying $\eta \times [1_A] = [1_B]$. Then $A \cong B$.*

Here $[1_A], [1_B]$ denote the class of the identity in $KK(A, A)$ and $KK(B, B)$ respectively (see for example [Cu, Hig1]). We will show the highlights of the proof. The key is the notion of asymptotic unitary equivalence.

Definition 7.4. Two asymptotic morphisms $(\phi_t), (\psi_t)$ are said to be *asymptotically unitary equivalent* if there exists a family of unitaries $(u_t)_{t\in[1,\infty)}$ satisfying

1. $t \mapsto u_t$ is norm continuous.
2. $\lim_{t\to\infty} u_t\phi_t(a)u_t^* - \psi_t(a) = 0$

Unlike approximate unitary equivalence (see [D1, Rø]), asymptotic unitary equivalence determines the same class in E-theory (or KK-theory for the nuclear case). The drawback to it is that in general, an asymptotic morphism may not be equivalent to a reparameterization with a continuous increasing function. So the product is not well defined on this equivalence class. The strategy used by Phillips as an attempt to correct it was to ask for a "decoupled version". That is, a continuous family of unitaries $u_{s,t}$ $s, t \in [1, \infty)$ such that

$$\lim_{s,t\to\infty} \|u_{s,t}\phi_t(a)u_{s,t}^* - \psi_s(a)\| = 0, \ \forall a \in A.$$

The decoupled version above cures this problem, and introduces a new one. The asymptotic morphism $\phi : C(S^1) \to \mathbb{C}$ defined by $\phi_t(f) = f(\exp(\imath t))$ is no longer equivalent to itself. In fact, if an asymptotic morphism is asymptotically unitarily equivalent to itself, then it is asymptotically unitarily equivalent to a *-homomorphism [Ph].

However, in spite of the problem, this is the condition that "works". At least for the case we need. The next two remarkable facts are the key ingredients to the proof of theorem 7.3.

Lemma 7.5. [Ph] *Let A be separable, nuclear, unital and simple, and let D_0 be unital and $D = \mathcal{O}_\infty \otimes D_0$.*

1. *Two full asymptotic morphisms from A to $\mathcal{K} \otimes D$ are asymptotically unitarily equivalent iff they are homotopic.*
2. *Any full asymptotic morphism $\phi : A \to \mathcal{K} \otimes D$ is asymptotically unitarily equivalent to a *-homomorphism.*

With these at our disposal, we can supply the ingredients of the proof of theorem 7.3. As $\tilde{E}_A(D) \cong KK(A, B)$ there is a full asymptotic morphism $\phi_0 : A \to \mathcal{K}\otimes\mathcal{O}_\infty\otimes B$ whose KK-class is η. Using the above lemma, we see that ϕ_0 is homotopic to a *-homomorphism. By exploiting the isomorphism between B and $B\otimes\mathcal{O}_\infty$, and the fact that $\mathcal{O}_\infty$ is purely infinite, one can construct a *-homomorphism $\phi : A \to B$ whose KK-class is η. A similar construction on η^{-1} gives us a $\psi : B \to A$ whose KK-class is η^{-1}.

In order to show that A is isomorphic to B, one shows that $\phi \circ \psi$ is approximately unitary equivalent to id_A and $\psi \circ \phi$ is approximately unitarily equivalent to id_B. The result follows from Theorem 5.1 of [Rø]. See [Ph] for the details.

It should be noted that E. Kirchberg independently proved the same result using different techniques [Kir].

7.4. **Deformations of Topological Spaces.** In this section, we return to the relationship between asymptotic morphisms and C*-algebras deformations. In particular, we look at deformations from $C_0(X)$ to $B \otimes \mathcal{K}$, where X is a locally compact space. Such deformations are quite common, as the next theorem shows.

Theorem 7.6. [DL3] *Suppose $X \cup \{pt\}$ is a compact orientable manifold, and B a C^*-algebra. If $\eta : K^*(X) \to K_*(B)$ is an isomorphism then there is a deformation from $C_0(X)$ to $B \otimes \mathcal{K}$ which induces η.*

The proof is fairly straightforward, and we include it. Using the universal coefficient theorem (see [RS]), the isomorphism η is induced by a KK-equivalence in $KK(C_0(X), B)$. As $C_0(X)$ is homotopy symmetric and nuclear, $KK(C_0(X), B) \cong [[C_0(X), B \otimes \mathcal{K}]]$. Let ϕ be the asymptotic morphism inducing η. As ϕ induces an isomorphism on K-theory, the result follows from theorem 1.12.

A result of this theorem is that one can 'custom make' topological spaces whose K-theory agrees with that of a well known C*-algebra, and form deformations. In particular, deformations of topological spaces to non-unital dimension drop algebras show how topological torsion can be transformed into matrical torsion. A specific example for the dimension drop algebra A_2 is worked out in [DL3].

Specific cases for X being non-orientable were worked out in [Lor2]. In particular, one can form deformations of real projective space and the Klein bottle with algebras which are very similar to the Toeplitz algebra. Such deformations are candidates for quantum deformations. See [Lor2] for more.

7.5. **Other Areas of Interest.** We have by no means covered all the areas where E-theory and the theory asymptotic morphisms are used. In particular, E-theory and deformations have been used by Connes and others to gain better insights into the Baum-Connes conjecture [Co]. There are other applications of E-theory to 'non commutative geometry' in Connes' book [Co].

E-theory has also been linked up with the shape theory of C*-algebras introduced by Effros and Kaminker and further developed by Blackadar. Much of the theory was only developed at a formal level, and there were few explicit examples. In [D2] M. Dadarlat showed that shape theory and E-theory actually coincided, and shape equivalence amounted to E-equivalence. This gave a new perspective to many concepts in shape theory. See [D2] for more on this topic.

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