MATH 617 (WINTER 2024, PHILLIPS): HOMEWORK 1

This homework assignment is due Wednesday 17 January 2024.

Problem 1 (Problem A)**.** Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces, let $h \colon X \to Y$ be a measurable function (that is, if $F \in \mathcal{N}$ then $h^{-1}(F) \in \mathcal{M}$), and let μ be a measure on $(X, \mathcal{M})$. Define $\nu \colon \mathcal{N} \to [0, \infty]$ by $\nu(F) = \mu(h^{-1}(F))$ for $F \in \mathcal{N}$. Prove the following:

(1) ν is a measure on $(X, \mathcal{M})$.
(2) If $f \colon Y \to [0, \infty]$ is measurable, then $f \circ h \colon X \to [0, \infty]$ is measurable, and $\int_X (f \circ h) \, d\mu = \int_Y f \, d\nu$.
(3) If $f \colon Y \to \mathbb{C}$ is integrable, then $f \circ h \colon X \to \mathbb{C}$ is integrable, and $\int_X (f \circ h) \, d\mu = \int_Y f \, d\nu$.

The measure ν is called the push forward of μ under h, and written $h_*(\mu)$.

Problem 2 (Problem 13 in Chapter 4 of Rudin)**.** Let $f \colon \mathbb{R} \to \mathbb{C}$ be a 1-periodic continuous function. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Prove that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t) \, dt.$$

Hint. First consider the case $f(t) = e^{2\pi i k t}$ for some $k \in \mathbb{Z}$.

Problem 3 (Problem 4 in Chapter 5 of Rudin)**.** Define $M \subset C([0,1])$ by

$$M = \left\{ f \in C([0,1]) \colon \int_0^{1/2} f(t) \, dt - \int_{1/2}^1 f(t) \, dt = 1 \right\}.$$

Prove that M is a closed convex set which contains no element of minimal norm.

Why does this not contradict Theorem 4.10 of Rudin's book?

Problem 4 (Problem 5 in Chapter 5 of Rudin)**.** Define $M \subset L^1([0,1])$ (using Lebesgue measure) by

$$M = \left\{ \xi \in L^1([0,1]) \colon \int_0^1 \xi(t) \, dt = 1 \right\}.$$

Prove that M is a closed convex set which contains infinitely many element of minimal norm.

Why does this not contradict Theorem 4.10 of Rudin's book?

Problem 5 (Problem 14 in Chapter 5 of Rudin)**.** For $n \in \mathbb{Z}_{>0}$ define a subset $X_n \subset C([0,1])$ by

$$X_n = \big\{ f \in C([0,1]) \colon \text{there is } t \in [0,1] \text{ such that } |f(s-t)| \leq n|s-t| \text{ for all } s \in [0,1] \big\}.$$

(1) Let $n \in \mathbb{Z}_{>0}$ and let $U \subset C([0,1])$ be a nonempty open set. Prove that there is a nonempty open set $V \subset U$ such that $V \cap X_n = \varnothing$.

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(2) Prove that there is a dense G_δ-set $G \subset C([0,1])$ such that every function $f \in G$ is nowhere differentiable.

Hint. For (1), every function $f \in C([0,1])$ can be uniformly approximated by a zigzag function g with very large slopes. If these slopes are large enough, and $\|g - h\|_\infty$ is small enough, then $h \notin X_n$.