MATH 617 (WINTER 2024, PHILLIPS): SOLUTIONS TO HOMEWORK 3

This homework assignment is due Wednesday 31 January 2024.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solutions files.

Problem 1 (Problem 6 in Chapter 5 of Rudin)**.** Let H be a Hilbert space, let $M \subset H$ be a closed subspace, and let $\omega_0 \in M^*$. Prove that there is a *unique* $\omega \in H^*$ such that $\omega|_M = \omega_0$ and $\|\omega\| = \|\omega_0\|$. Moreover, prove that ω vanishes on $M^{\perp}$.

Solution. Since M is a Hilbert space, there is $\eta_0 \in M$ such that for all $\xi \in M$ we have $\omega_0(\xi) = \langle \xi, \eta_0 \rangle$. Moreover, $\|\eta_0\| = \|\omega_0\|$. Define $\omega \in H^*$ by $\omega(\xi) = \langle \xi, \eta_0 \rangle$ for $\xi \in H$. Obviously $\omega|_M = \omega_0$. We have $\|\omega\| = \|\eta_0\| = \|\omega_0\|$. Also, if $\xi \in M^{\perp}$, then, since $\eta_0 \in M$, we have $\omega(\xi) = \langle \xi, \eta_0 \rangle = 0$.

It remains to prove that if $\rho \in H^*$ satisfies $\rho|_M = \omega_0$ and $\|\rho\| = \|\omega_0\|$, then $\rho = \omega$. Given ρ, there is $\eta \in H$ such that for all $\xi \in M$ we have $\rho(\xi) = \langle \xi, \eta \rangle$. Since $\rho|_M = \omega_0$, for all $\xi \in M$ we have

$$0 = \rho(\xi) - \omega(\xi) = \langle \xi, \eta \rangle - \langle \xi, \eta_0 \rangle = \langle \xi, \eta - \eta_0 \rangle.$$

In particular, $\langle \eta_0, \eta - \eta_0 \rangle = 0$. Therefore

$$\|\eta_0\|^2 + \|\eta - \eta_0\|^2 = \|\eta\|^2 = \|\rho\|^2 = \|\omega_0\|^2 = \|\eta_0\|^2.$$

Thus $\|\eta - \eta_0\| = 0$, whence $\eta = \eta_0$, so $\rho = \omega$. □

Problem 2 (Problem 18 in Chapter 5 of Rudin)**.** Let E be a normed vector space, let F be a Banach space, and let $(a_n)_{n \in \mathbb{Z}_{>0}}$ be a bounded sequence in $L(E, F)$. Suppose that there is a dense set $S \subset E$ such that $\lim_{n \to \infty} a_n \xi$ exists for all $\xi \in S$. Prove that $\lim_{n \to \infty} a_n \xi$ exists for all $\xi \in E$.

Solution. For $\xi \in E$ set $g(\xi) = \lim_{n \to \infty} a_n \xi$. Also set $M = 1 + \sup_{n \in \mathbb{Z}_{>0}} \|a_n\|$.

We claim that for $\xi \in E$, the sequence $(a_n \xi)_{n \in \mathbb{Z}_{>0}}$ is Cauchy. Since E is complete, the result will follow.

To prove the claim, let $\xi \in E$ and let $\varepsilon > 0$. Choose $\eta \in S$ such that $\|\xi - \eta\| < \varepsilon/(4M)$. Choose $N \in \mathbb{Z}_{>0}$ such that for all $n \geq N$ we have $\|a_n \eta - g(\eta)\| < \frac{\varepsilon}{4}$. Let $m, n \geq N$. Then

$$\begin{aligned} \|a_m \xi - a_n \xi\| &\leq \|a_m\| \|\xi - \eta\| + \|a_m \eta - g(\eta)\| + \|g(\eta) - a_n \eta\| + \|a_n\| \|\xi - \eta\| \\ &< M\left(\frac{\varepsilon}{4M}\right) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + M\left(\frac{\varepsilon}{4M}\right) = \varepsilon. \end{aligned}$$

The claim is proved, and therefore the solution is complete. □

We took $M = 1 + \sup_{n \in \mathbb{Z}_{>0}} \|a_n\|$ rather than $M = \sup_{n \in \mathbb{Z}_{>0}} \|a_n\|$ to avoid the possibility of dividing by zero.

The following example shows that completeness of F is necessary, even if E is assumed complete. (It was not asked for in the problem.)

Date: 31 January 2024.

Example 1. Let $E = l^1(\mathbb{Z}_{>0})$ with its usual norm. Let F be the set of sequences in $l^1(\mathbb{Z}_{>0})$ with finite support, with the l^1 norm. For $n \in \mathbb{Z}_{>0}$ define $a_n \in L(E, F)$ by, for $\xi = (\xi_1, \xi_2, \ldots) \in L^1(\mathbb{Z}_{>0})$,

$$a\xi = \big(\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots\big).$$

Then $\|a_n\| \leq 1$. Set $S = F$. Then for all $\xi \in S$ we have $\lim_{n\to\infty} a_n\xi = \xi$. However, if $\xi \in E \setminus S$, for example if $\xi = (n^{-2})_{n\in\mathbb{Z}_{>0}}$, then $\lim_{n\to\infty} a_n\xi$ does not exist in F.

Problem 3 (An expansion of Problem 17 in Chapter 5 of Rudin)**.** (This problem is worth two ordinary problems.) Let μ be a n onzero positive measure on a measurable space X. Let $p \in [1, \infty)$. For $f \in L^\infty(X, \mu)$, let $m(f)\colon L^p(X, \mu) \to L^p(X, \mu)$ be defined by $m(f)(\xi)(x) = f(x)\xi(x)$, that is, $m(f)$ is the multiplication operator by f.

(1) Prove that $\|m(f)\| \leq \|f\|_\infty$ for all $f \in L^\infty(X, \mu)$.
(2) Prove that $\|m(f)\| = \|f\|_\infty$ for all $f \in L^\infty(X, \mu)$ if and only if μ is semifinite.
(3) Assume that μ is semifinite. Give, with proof, a characterization in terms of f of those $f \in L^\infty(X, \mu)$ for which the operator $m(f)$ is surjective.
(4) Assume that μ is semifinite. Let $f \in L^\infty(X, \mu)$, and suppose that $m(f)$ is surjective. Prove that $m(f)$ is injective.
(5) Give, with proof, an example of a finite measure μ on a measurable space X and $f \in L^\infty(X, \mu)$ such that that $m(f)$ is injective but not surjective.

Recall that a measure μ on X is called *semifinite* if for every measurable set $E \subset X$ with $\mu(E) > 0$, there is a measurable set $F \subset E$ with $0 < \mu(F) < \infty$.

Example 2. Here are some examples of measures which are semifinite and some which are not. (This isn't an exercise.)

(1) Every σ-finite measure is semifinite.
(2) Counting measure on $\mathbb{R}$ is semifinite but not σ-finite.
(3) On any set X take the measurable sets to be $\varnothing$ and X, and take $\mu(\varnothing) = 0$ and $\mu(X) = \infty$. Then μ is not semifinite.
(4) On $\mathbb{R}$ take the measurable sets to be the countable sets and their complements. Take $\mu(E) = 0$ if E is countable and $\mu(E) = \infty$ if $\mathbb{R} \setminus E$ is countable. Then μ is not semifinite.

I know of no real use for measures which are not semifinite.

We break the solution into several propositions.

Proposition 3. For every positive measure μ and every $f \in L^\infty(\mu)$, we have $m(f) \in L(L^p(\mu))$ and $\|m(f)\| \leq \|f\|_\infty$.

Proof. We estimate $\|m(f)\xi\|$ for $\xi \in L^p(\mu)$. We have

$$\|m(f)\xi\|^p = \int_X |f(x)\xi(x)|^p \, d\mu(x) \leq \int_X \|f\|_\infty^p \|\xi(x)|^p \, d\mu(x) = \|f\|_\infty^p \|\xi\|^p.$$

It is clear that $m(f)$ is linear, so we conclude that $m(f)$ is a bounded linear map with $\|m(f)\| \leq \|f\|_\infty$. □

Proposition 4. For every semifinite positive measure μ and every $f \in L^\infty(\mu)$, we have $\|m(f)\| = \|f\|_\infty$.

Proof. We need only show $\|m(f)\| \geq \|f\|_\infty$. Let $c < \|f\|_\infty$; we show $\|m(f)\| \geq c$. Set $E = \{x \in X : |f(x)| \geq c\}$. Then $\mu(E) > 0$ by the definition of the essential supremum. By semifiniteness, there is a measurable set $F \subset E$ with $0 < \mu(F) < \infty$. Define $\xi = (\mu(F))^{-1/p}\chi_F$. Then $\xi \in L^p(\mu)$ and $\|\xi\|_p = 1$. We have

$$\|m(f)\xi\|^p = \int_F [(\mu(F))^{-1/2}|f(x)|]^p \, d\mu(x) \geq \left(\frac{1}{\mu(F)}\right) \cdot c^p \cdot \mu(F) = c^p.$$

So $\|m(f)\xi\| \geq c$. Thus $\|m(f)\| \geq c$. □

Proposition 5. For every positive measure μ which is not semifinite, there exists $f \in L^\infty(\mu)$ such that $\|f\|_\infty = 1$ and $m(f) = 0$.

Proof. By definition, there is a measurable set $E \subset X$ such that $\mu(E) > 0$, and with the property that for every measurable $F \subset E$ we have $\mu(F) = 0$ or $\mu(F) = \infty$.

We first claim that every $\xi \in L^p(\mu)$ must vanish almost everywhere on E. The set $\left\{x \in E : |\xi(x)| \geq \frac{1}{n}\right\}$ is a measurable subset of E with finite measure, hence has measure zero. Therefore

$$\mu(\{x \in E : \xi(x) \neq 0\}) \leq \sup_{n \in \mathbb{Z}_{>0}} \mu\left(\left\{x \in E : |\xi(x)| \geq \tfrac{1}{n}\right\}\right) = 0.$$

This proves the claim.

Now set $f = \chi_E$. Since $\mu(E) \neq 0$, we have $\|f\|_\infty = 1$. However, for any $\xi \in L^p(\mu)$, the function $f\xi$ vanishes almost everywhere on E (because ξ does), and vanishes everywhere on $X \setminus E$ (because χ_E does). Thus $m(f)\xi = 0$. So $m(f)$ is the zero operator. □

We now consider when $m(f)$ is surjective.

Lemma 6. Let μ be a positive measure, and let $f \in L^\infty(\mu)$. Suppose $\mu(\{x \in X : f(x) = 0\}) = 0$. Then $m(f)$ is injective.

Proof. Set $E = \{x \in X : f(x) = 0\}$. Let $\xi \in L^p(\mu)$ with $m(f)\xi = 0$. Then $\xi = 0$ almost everywhere on $X \setminus E$. Since $\mu(E) = 0$, we get $\xi = 0$ almost everywhere on X, so ξ is the zero element of $L^p(\mu)$. □

Proposition 7. Let μ be a semifinite positive measure, and let $f \in L^\infty(\mu)$. The following are equivalent:

(1) $m(f)$ is surjective.
(2) $m(f)$ is bijective.
(3) $1/f \in L^\infty(\mu)$.
(4) There exists $\alpha > 0$ such that $\mu(\{x \in X : |f(x)| < \alpha\}) = 0$.

Proof. The equivalence of (3) and (4) is easy and is omitted. That (3) implies (2) is clear, since $m(f)$ has the inverse operator $m(1/f)$. That (2) implies (1) is trivial. We complete the proof by showing that (1) implies (2) and that (2) implies (4).

Assume (1). Set

$$E = \{x \in X : f(x) = 0\}.$$

We claim that $\mu(E) = 0$. Suppose not. By semifiniteness, there is $F \subset E$ with $0 < \mu(F) < \infty$. Then $\chi_F \in L^p(\mu)$, but χ_F is not in the range of $m(f)$ because $f\xi$ vanishes on F for every $\xi \in L^p(\mu)$. This proves the claim. It now follows from Lemma 6 that $m(f)$ is bijective, which is (2).

Now assume (2). The Open Mapping Theorem provides $c > 0$ such that $\|m(f)\xi\| \geq c\|\xi\|$ for all $\xi \in L^p(\mu)$. Set

$$F = \left\{x \in X \colon |f(x)| \leq \tfrac{1}{2}c\right\}.$$

We claim $\mu(F) = 0$, which will prove (4). If not, by semifiniteness, there is $G \subset F$ with $0 < \mu(G) < \infty$. Then $\chi_G \in L^p(\mu)$ and satisfies $\|\chi_G\| = [\mu(g)]^{1/2}$ and

$$\|m(f)\chi_G\|^p = \int_G |f|^p \, d\mu \leq \left(\tfrac{1}{2}c\right)^p \mu(G),$$

so that $\|m(f)\chi_G\| \leq \frac{1}{2}c[\mu(g)]^{1/2}$. Since $\mu(G) \neq 0$, this contradicts the choice of c above. □

Remark 8. It is not really necessary to use the Open Mapping Theorem in the proof of Proposition 7, but it significantly simplifies the proof. We demonstrate with a direct proof of the implication from (1) to (4).

Alternate proof of (1) imples (4) in Proposition 7. Assume that (4) fails. Set $E = \left\{x \in X \colon f(x) = 0\right\}$. There are two cases.

First suppose that $\mu(E) > 0$. By semifiniteness, there is $F \subset E$ with $0 < \mu(F) < \infty$. Then $\chi_F \in L^p(X, \mu)$. However, $\chi_F \notin \operatorname{Ran}(m(f))$, because for all $\xi \in L^p(X, \mu)$ we have $m(f)\xi|_F = 0$ almost everywhere.

So suppose that $\mu(E) = 0$. For $n \in \mathbb{Z}_{>0}$ set

$$S_n = \left\{x \in X \colon \frac{1}{n+1} \leq |f(x)| < \frac{1}{n}\right\}.$$

If there are only finitely many $n \in \mathbb{Z}_{>0}$ such that $\mu(S_n) > 0$, then there is $N \in \mathbb{Z}_{>0}$ such that $\mu(S_n) = 0$ for all $n \geq N$. Therefore

$$\mu\left(\left\{x \in X \colon |f(x)| < \frac{1}{N}\right\}\right) = \mu\left(\coprod_{n=N}^{\infty} S_n\right) = \sum_{n=N}^{\infty} \mu(S_n) = 0,$$

contradicting the failure of (4). So there is a sequence $n(1) < n(2) < \cdots$ in $\mathbb{Z}_{>0}$ such that for all $k \in \mathbb{Z}_{>0}$ we have $\mu(S_{n(k)}) > 0$. By semifiniteness, there is $T_k \subset S_{n(k)}$ such that $0 < \mu(T_k) < \infty$.

For $k \in \mathbb{Z}_{>0}$ set

$$r_k = \frac{1}{k^{1/(2p)} n(k) \mu(T_k)^{1/p}}.$$

The sets $T_1, T_2, \ldots$ are disjoint. Set $T = \bigcup_{k=1}^{\infty} T_k$. Then we can define $\eta \colon X \to [0, \infty)$ by

$$\eta(x) = \begin{cases} r_k & k \in \mathbb{Z}_{>0} \text{ and } x \in T_k \\ 0 & x \in X \setminus T. \end{cases}$$

We have

$$\int_X |\eta|^p \, d\mu = \sum_{k=1}^{\infty} r_k^p \mu(T_k) = \sum_{k=1}^{\infty} \frac{1}{k^{1/2} n(k)^p} \leq \sum_{k=1}^{\infty} \frac{1}{k^{1/2} \cdot k^p} \leq \sum_{k=1}^{\infty} \frac{1}{k^{1/2} \cdot k} < \infty.$$

so $\eta \in L^p(X, \mu)$.

Suppose $\xi \in L^p(X, \mu)$ and $m(f)\xi = \eta$. For $k \in \mathbb{Z}_{>0}$ and almost every $x \in T_k$, we then have

$$|\xi(x)| = \left|\frac{\eta(x)}{f(x)}\right| = \left|\frac{r_k}{f(x)}\right| \geq r_k n(k).$$

Therefore, using the Monotone Convergence Theorem,

$$\int_X |\xi|^p \, d\mu \geq \sum_{k=1}^{\infty} \int_{T_k} |\xi|^p \, d\mu \geq \sum_{k=1}^{\infty} r_k^p n(k)^p \mu(T_k) = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}} = \infty.$$

This contradicts $\xi \in L^p(X, \mu)$. So $m(f)$ is not surjective. □

Example 9. Proposition 7 fails whenever μ is not semifinite. With f as in the proof of Proposition 5, the function $1 - f$ satisfies (1) and (2) but not (3) or (4).

Remark 10. The conditions in Proposition 7 are not equivalent to injectivity of $m(f)$. Example: with μ being Lebesgue measure on $[0, 1]$, take $f(t) = t$ for all t. Then $m(f)$ is injective by Lemma 6. However, $m(f)$ is not surjective, by the criterion in Proposition 7(4).

Problem 4 (Problem 8(c) in Chapter 5 of Rudin)**.** Let E be a normed vector space, and let $(\xi_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in E. Suppose that $\lim_{n \to \infty} \omega(\xi_n)$ exists for all $\omega \in E^*$. Prove that $(\xi_n)_{n \in \mathbb{Z}_{>0}}$ is bounded.

Solution. Let $\Lambda \colon E \to E^{**}$ be the standard map, that is, $\Lambda(\xi)(\omega) = \omega(\xi)$ for $\xi \in E$ and $\omega \in E^*$. We have seen that $\|\Lambda(\xi)\| = \|\xi\|$ for all $\xi \in E$.

For $\omega \in E^*$, we have $\sup_{n \in \mathbb{Z}_{>0}} |\Lambda(\xi_n)(\omega)| = \sup_{n \in \mathbb{Z}_{>0}} |\omega(\xi_n)|$. Since $\lim_{n \to \infty} \omega(\xi_n)$ exists, it follows that $\sup_{n \in \mathbb{Z}_{>0}} |\Lambda(\xi_n)(\omega)| < \infty$. Since E^* is complete, the Uniform Boundedness Principle implies that $\sup_{n \in \mathbb{Z}_{>0}} \|\Lambda(\xi_n)\| < \infty$. So $\sup_{n \in \mathbb{Z}_{>0}} \|\xi_n\| < \infty$. □