MATH 617 (WINTER 2024, PHILLIPS): SOLUTIONS TO HOMEWORK 4

This homework assignment is due Wednesday 7 February 2024.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solution file.

Problem 1 (Problem 1 in Chapter 5 of Rudin)**.** Let $X = \{a, b\}$, let $\alpha, \beta \in (0, \infty)$, and let $\mu_{\alpha,\beta}$ be the measure on X such that that $\mu_{\alpha,\beta}(\{a\}) = \alpha$ and $\mu_{\alpha,\beta}(\{b\}) = \beta$. In this problem, we use the spaces $L^p(X, \mu, \mathbb{R})$ of *real* valued L^p functions on X modulo functions vanishing almost everywhere. (In this problem, the only set of measure zero will be $\varnothing$.)

(1) For $p \in (0, \infty]$ describe the closed unit ball of $L^p(X, \mu_{1,1}, \mathbb{R})$. In particular, show that it is convex if and only if $p \in [1, \infty]$, determine for which values of p it is a circle, and determine for which values of p it is a square. Draw pictures of these unit balls for representative choices of p, such as $p = 1, 2, \infty$ and some value of p in each of the intervals $(0, 1)$, $(1, 2)$, and $(2, \infty)$.

(2) Describe what happens to your solution to part (1) for $\alpha \neq \beta$, say for $\mu_{1,1/2}$ in place of $\mu_{1,1}$.

Solution for (1). For $p \in [1, \infty)$, convexity is just the triangle inequality and homogeneity for $\|\cdot\|_p$: for any $\xi, \eta \in L^p(X, \mu_{1,1})$ with $\|\xi\|_p \leq 1$ and $\|\eta\|_p \leq 1$, and any $\lambda \in [0, 1]$, we have

$$\begin{aligned} \|\lambda\xi + (1-\lambda)\eta\|_p &\leq \|\lambda\xi\|_p + \|(1-\lambda)\eta\|_p = |\lambda|\|\xi\|_p + |1-\lambda|\|\eta\|_p \\ &= \lambda\|\xi\|_p + (1-\lambda)\|\eta\|_p \leq \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1. \end{aligned}$$

Now suppose $p \in (0, 1)$. Let $\xi, \eta \in L^p(X, \mu_{1,1}, \mathbb{R})$ be

$$\xi(x) = \begin{cases} 1 & x = a \\ 0 & x = b \end{cases} \qquad \text{and} \qquad \eta(x) = \begin{cases} 0 & x = a \\ 1 & x = b \end{cases}$$

Then $\|\xi\|_p = 1$ and $\|\eta\|_p = 1$. However,

$$\left(\tfrac{1}{2}\xi + \tfrac{1}{2}\eta\right)(x) = \begin{cases} \frac{1}{2} & x = a \\ \frac{1}{2} & x = b, \end{cases}$$

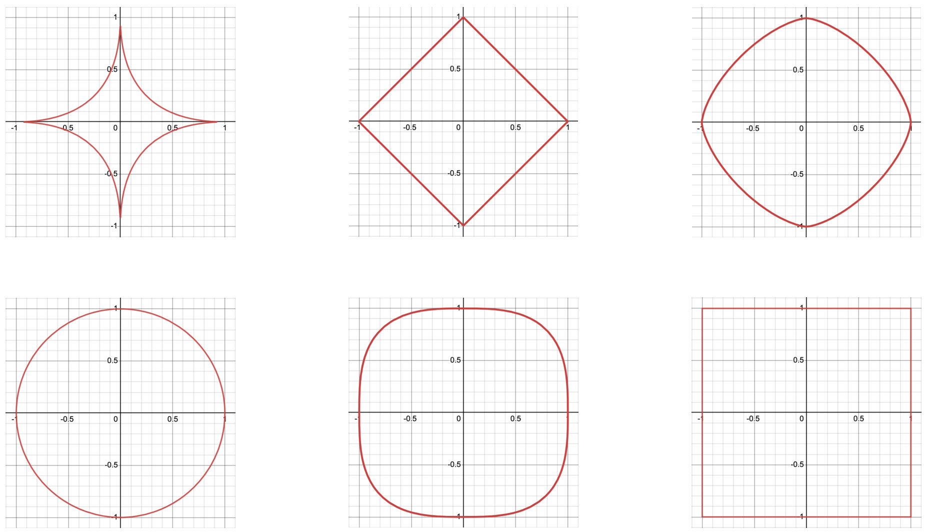
so

$$\left\|\tfrac{1}{2}\xi + \tfrac{1}{2}\eta\right\|_p = \left(\tfrac{1}{2^p} + \tfrac{1}{2^p}\right)^{1/p} = 2^{-1+1/p}.$$

Since $p < 1$, we have $-1 + \frac{1}{p} > 0$. Thus, $2^{-1+1/p} > 1$, and $\frac{1}{2}\xi + \frac{1}{2}\eta$ is not in the closed unit ball. This shows that the closed unit ball is not convex.

Here are pictures for $p = \frac{1}{2}$, $p = 1$ (this one is a square with diagonals on the coordinate axes), $p = \frac{3}{2}$, $p = 2$ (this one is a circle), $p = 3$, and $p = \infty$ (this one is a square with sides parallel to the axes).

Date: 7 February 2024.



Only $p = 2$ gives a circle, and only $p = 1$ and $p = \infty$ give boundaries containing any straight line segments. $\square$

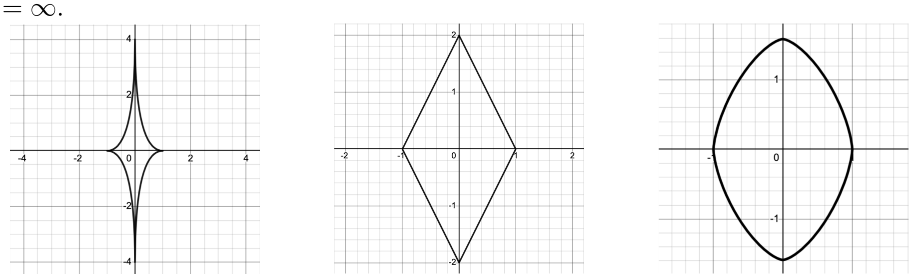
Solution for (2). Identify the set of all functions $f\colon X \to \mathbb{R}$ (which as a vector space is equal to $L^p(X, \mu_{\alpha,\beta})$ for every $p \in)0, \infty]$ and all $\alpha, \beta \in \mathbb{R}$) with $\mathbb{R}^2$ via $f \mapsto (f(a), f(b))$. Write $\| \cdot \|_{p,\alpha,\beta}$ for the corresponding norm on $\mathbb{R}^2$. Thus, for $(x, y) \in \mathbb{R}^2$, we have $\|(x, y)\|_{p,\alpha,\beta} = \big(\alpha|x|^p + \beta|y|^p\big)^{1/p}$ when $p \neq \infty$ and $\|(x, y)\|_{\infty,\alpha,\beta} = \max(|x|, |y|)$. Also, let $B_{p,\alpha,\beta}$ be the closed unit ball for $\| \cdot \|_{p,\alpha,\beta}$.

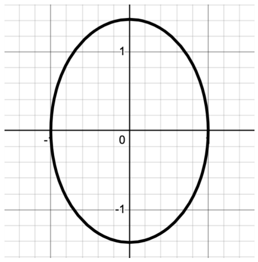
The closed unit ball for $\| \cdot \|_{\infty,\alpha,\beta}$ obviously doesn't depend on α and β. So, from now on, suppose $p \neq \infty$.

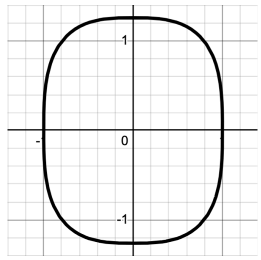
Fix $\alpha, \beta > 0$ and $p \in (0, \infty)$. Define $T_p\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T_p(x, y) = \big(\alpha^{-1/p}x,\ \beta^{-1/p}y\big)$. Then $\|T_p(x, y)\|_{p,\alpha,\beta} = \|(x, y)\|_{p,1,1}$. Therefore $T_p(B_{p,1,1}) = B_{p,\alpha,\beta}$. Since T_p is a linear bijection, this map preserves convexity and nonconvexity. Therefore, by part (1), $B_{p,\alpha,\beta}$ is convex if and only if $p \geq 1$.

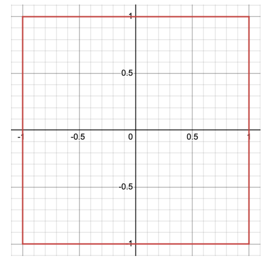
The closed unit ball for $p \neq \infty$ is gotten from the one for $\mu_{p,1,1}$ by expanding or contracting suitably on the x and y directions.

Here are pictures for $\mu_{p,1,1/2}$ with $p = \frac{1}{2}$, $p = 1$ (this one is a rhombus with diagonals on the coordinate axes), $p = \frac{3}{2}$, $p = 2$ (this one is an ellipse), $p = 3$, and $p = \infty$.









The closed unit ball for $p = \infty$ is the same as above. $\square$

Problem 2 (Problems 2 and 3 in Chapter 5 of Rudin)**.** This problem counts as two regular problems.

Three problems on convexity:

(1) Let E be a Banach space. Prove that the closed unit ball B of E is convex, that is, if $\xi, \eta \in B$ and $\alpha \in [0, 1]$ then $(1 - \alpha)\xi + \alpha\eta \in B$.

(2) Let (X, μ) be a measure space, and let $p \in (1, \infty)$. Prove that the closed unit ball B of E is strictly convex, that is, if $\xi, \eta \in B$ are distinct and $\alpha \in (0, 1)$ then $\|(1-\alpha)\xi+\alpha\eta\| < 1$. (The statement means that the surface of the closed unit ball contains no straight line segments. You will need the criterion for equality in the triangle inequality for $\| \cdot \|_p$.)

(3) Let E be any nontrivial space of the form $C(X)$, $L^1(X, \mu)$, or $L^\infty(X, \mu)$. Prove that the closed unit ball B of E is not strictly convex. (Part of the problem is to determine what "trivial" means. If X has only one point then E is certainly trivial for the purposes of this problem, but there are other ways for $L^1(X, \mu)$ and $L^\infty(X, \mu)$ to be trivial.)

Solution to (1). Convexity is just the triangle inequality and homogeneity for the norm: for any $\xi, \eta \in E$ with $\|\xi\| \leq 1$ and $\|\eta\| \leq 1$, and any $\alpha \in [0, 1]$, we have

$$\begin{aligned} \|\alpha\xi + (1 - \alpha)\eta\| &\leq \|\alpha\xi\| + \|(1 - \alpha)\eta\| = |\alpha|\|\xi\| + |1 - \alpha|\|\eta\| \\ &= \alpha\|\xi\| + (1 - \alpha)\|\eta\| \leq \alpha \cdot 1 + (1 - \alpha) \cdot 1 = 1. \end{aligned} \tag{1}$$

This completes the solution. $\square$

Solution to (2). We have to prove that if $\alpha \in (0, 1)$, $\|\xi\| \leq 1$, $\|\eta\| \leq 1$, and $\|\alpha\xi+(1-\alpha)\eta\| = 1$, then $\xi = \eta$. Under these conditions, we must have equality throughout (1) in the solution to part (1). On the second line, this implies that

$$\|\xi\| = \|\eta\| = 1. \tag{2}$$

On the first line, by the condition for equality in the triangle inequality for $\| \cdot \|$, there is $\lambda \geq 0$ such that $\lambda\alpha\xi = (1-\alpha)\eta$ or there is $\lambda \geq 0$ such that $\lambda(1-\alpha)\eta = \alpha\xi$. If $\lambda\alpha\xi = (1 - \alpha)\eta$, then, by (2),

$$\lambda\alpha = \|\lambda\alpha\xi\| = \|(1 - \alpha)\eta\| = 1 - \alpha.$$

Since $1 - \alpha \neq 0$, we deduce that $\xi = \eta$. The case $\lambda(1 - \alpha)\eta = \alpha\xi$ is similar. $\square$

For (3), we give the solution as three theorems, one about each of the cases. Each one contains the appropriate nontriviality condition, and a proof that this condition implies failure of strict convexity of the closed unit ball. It also contains a proof (not asked for in the problem) that the appropriate nontriviality condition is equivalent to the space involved having dimension greater than 1.

Theorem 1. Let X be a compact Hausdorff space. Then the following are equivalent:

(1) $\dim(C(X)) > 1$.
(2) X has at least two points.
(3) The closed unit ball of $C(X)$ is not strictly convex.

Proof. We prove that (2) implies (3). Let $x, y \in X$ be distinct. Choose disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$. Choose continuous functions $f, k \colon X \to [0,1]$ such that $f(x) = 1$, $\mathrm{supp}(f) \subset U$, $k(y) = 1$, and $\mathrm{supp}(k) \subset V$. Set $g = f + h$. Then $g \neq f$, $0 \leq g \leq 1$, and the function $h = \frac{1}{2}f + \frac{1}{2}g$ satisfies $h(x) = 1$ and $0 \leq h \leq 1$. It follows that $\|f\| = \|g\| = \|h\| = 1$. Thus the closed unit ball of $L^1(X, \mu)$ is not strictly convex.

It is immediate that (1) implies (2). Since the closed unit ball of $\mathbb{C}$ is strictly convex, it is obvious that (3) implies (1). $\square$

For the rest of (3), we first give a lemma which helps identify the trivial cases.

Lemma 2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $E \in \mathcal{M}$ be a measurable set such that $\mu(E) > 0$ and such that for every $F \in \mathcal{M}$ with $F \subset E$, either $\mu(F) = 0$ or $\mu(E \setminus F) = 0$. Then for every measurable function $f \colon X \to \mathbb{C}$ there is a measurable function $g \colon X \to \mathbb{C}$ such that $g(x) = f(x)$ for almost all $x \in X$ and g is constant on E.

Sketch of proof. The main difficulty is to identify what the constant value of g on E should be. (It is possible, for example, that every one point subset of E has measure zero.)

To deal with this, first assume that f is real valued. Then let r be the essential supremum of $f|_E$. For all $s > r$ we have $\mu(\{x \in E \colon f(x) > s\}) = 0$. Also, if $s < r$ then $\mu(\{x \in E \colon f(x) > s\}) > 0$. (Otherwise the essential supremum of $f|_E$ would be at most s.) But then $\mu(\{x \in E \colon f(x) \leq s\}) = 0$. By considering sequences $s_n \searrow r$ and $s_n \nearrow r$, it is now easy to see that $f(x) = r$ for almost every $x \in E$. $\square$

Theorem 3. Let (X, μ) be a measure space. Then the following are equivalent:

(1) $\dim(L^1(X, \mu)) > 1$.
(2) There exist disjoint measurable sets $E, F \subset X$ such that $0 < \mu(E) < \infty$ and $0 < \mu(F) < \infty$.
(3) The closed unit ball of $L^1(X, \mu)$ is not strictly convex.

In (2), it isn't enough to require that there be two distinct sets E and F with finite strictly positive measure. Example: $X = \{0, 1, 2\}$, all subsets are measurable, $\mu(\{0\}) = 91$ while $\mu(\{1\}) = \mu(\{2\}) = 0$, and $E = \{0, 1\}$ and $F = \{0, 2\}$. It isn't enough to require that μ be nontrivial and that there be a proper subset E with $0 < \mu(F) < \infty$; the same example works. It isn't enough if even just one of the conditions $\mu(E) < \infty$ and $\mu(F) < \infty$ is omitted. Example: $X = \{0, 1\}$, all subsets are measurable, $\mu(\{0\}) = 91$ while $\mu(\{1\}) = \infty$, and $E = \{0\}$ and $F = \{1\}$. It isn't enough to require that the set of equivalence classes of measurable functions have dimension at least 2; the same example works.

Proof of Theorem 3. We prove that (2) implies (3). Set $f = \mu(E)^{-1}\chi_E$ and $g = \mu(F)^{-1}\chi_F$. Then $f \neq g$ as elements of $L^1(X, \mu)$, and $\|f\|_1 = \|g\|_1 = 1$. Moreover, the function $h = \frac{1}{2}f + \frac{1}{2}g$ satisfies

$$\|h\|_1 = \int_X h\,d\mu = \frac{1}{2}\int_X f\,d\mu + \frac{1}{2}\int_X g\,d\mu = \frac{1}{2} + \frac{1}{2} = 1.$$

This shows that the closed unit ball of $L^1(X,\mu)$ is not strictly convex.

Since the closed unit ball of $\mathbb{C}$ is strictly convex, it is obvious that (3) implies (1).

We prove that (1) implies (2). Let $f, g \in L^1(X,\mu)$ be linearly independent. Since $f \neq 0$, there is $\varepsilon > 0$ such that the set $G = \bigl\{x \in X : |f(x)| > \varepsilon\bigr\}$ satisfies $\mu(G) > 0$. Since $f \in L^1(X,\mu)$, we must have $\mu(G) < \infty$.

If G does not satisfy the hypotheses of Lemma 2, then it is easy to find disjoint measurable sets $E, F \subset G$ such that $0 < \mu(E) < \infty$ and $0 < \mu(F) < \infty$. Otherwise, by Lemma 2, we can assume f and g are constant on G, say with values α and β. Clearly $\alpha \neq 0$.

By linear independence, $\beta f - \alpha g$ is not the zero function. It vanishes on G and is in $L^1(X,\mu)$. Therefore there is $\delta > 0$ such that the set

$$F = \bigl\{x \in X : |\beta f(x) - \alpha g(x)| > \delta\bigr\}$$

satisfies $\mu(F) > 0$. Since $\beta f - \alpha g$ is an L^1 function, we must have $\mu(F) < \infty$. Take $E = G$. Then E and F disjoint measurable sets such that $0 < \mu(E) < \infty$ and $0 < \mu(F) < \infty$. □

Theorem 4. Let (X,μ) be a measure space. Then the following are equivalent:

(1) $\dim(L^\infty(X,\mu)) > 1$.
(2) There exist disjoint measurable sets $E, F \subset X$ such that $\mu(E) > 0$ and $\mu(F) > 0$.
(3) The closed unit ball of $L^\infty(X,\mu)$ is not strictly convex.

Some of the examples used for $L^1(X,\mu)$ show that weaker conditions than (2) are not enough for nontriviality.

Proof of Theorem 4. We prove that (2) implies (3). Set $f = \chi_E$ and $g = \chi_{E\cup F}$. Then $f \neq g$ as elements of $L^1(X,\mu)$, because $\mu(F) > 0$, and $\|f\|_\infty = \|g\|_\infty = 1$, because $\mu(E) > 0$. Moreover, the function $h = \frac{1}{2}f + \frac{1}{2}g$ satisfies Since $\mu(E) > 0$, this implies that $\|h\|_\infty = 1$ Thus the closed unit ball of $L^1(X,\mu)$ is not strictly convex.

Since the closed unit ball of $\mathbb{C}$ is strictly convex, it is obvious that (3) implies (1).

The proof that (1) implies (2) is essentially the same as the proof that (1) implies (2) in Theorem 3: one just omits the parts of the argument used to show that the sets involved have finite meaure. (These parts are in any case not valid here.) □

Problem 3 (Problem C)**.** This problem counts as two regular problems.

Let (X,μ) be a measure space. For a measurable function f on X and $\alpha > 0$, define

$$\lambda_f(\alpha) = \mu\bigl(\bigl\{x \in X : |f(x)| > \alpha\bigr\}\bigr).$$

For $p \in [1,\infty)$ define

$$C_p(f) = \left(\sup_{\alpha>0} \alpha^p \lambda_f(\alpha)\right)^{1/p},$$

and define $L^p_{\mathrm{w}}(\mu)$ ("weak $L^p(\mu)$") to be the set of measurable functions f on X such that $C_p(f) < \infty$ (modulo equality almost everywhere, as usual).

Prove the following:

(1) $C_p(f) \leq \|f\|_p$ and $L^p(\mu) \subset L^p_{\mathrm{w}}(\mu)$.
(2) $L^p_{\mathrm{w}}(\mu)$ is a vector space. (Hint: prove $C_p(f+g) \leq 2\big(C_p(f)^p + C_p(g)^p\big)^{1/p}$ and $C_p(\beta f) = |\beta| C_p(f)$.)
(3) C_1 need not be a norm.
(4) If $\mu(X) < \infty$, then $L^p_{\mathrm{w}}(\mu) \subset L^r(\mu)$ whenever $1 \leq r < p$.

Remark 5. Some remarks on Problem 3:

(1) Most of this problem is taken from Folland's book. (I didn't check the inequality in the hint in part (2) for correctness, but something similar is certainly true.)
(2) The notation is mine, and is probably nonstandard. In another book, for the case of Lebesgue measure on $\mathbb{R}$, $L^p_{\mathrm{w}}(\mu)$ is called $L(p, \infty)$, and there are "L^p type spaces" $L(p, r)$ for $1 \leq r \leq \infty$.
(3) This is from my reading elsewhere (which actually only specifically talked about Lebesgue measure on $\mathbb{R}$). None of the functions C_p is a norm. But if $p > 1$, then there are norms on the spaces $L^p_{\mathrm{w}}(\mu)$ which are equivalent to C_p (in the sense we usually apply to norms). With these norms, the spaces $L^p_{\mathrm{w}}(\mu)$ are Banach spaces. The space $L^1_{\mathrm{w}}(\mu)$ is a topological vector space, but metrizability and completeness were not mentioned.

Solution to (1). For a measurable function $f \colon X \to \mathbb{C}$ and $\alpha \in (0, \infty)$, set $S_f(\alpha) = \{x \in X \colon |f(x)| > \alpha\}$. Thus, $\lambda_f(\alpha) = \mu(S_f(\alpha))$.

For any $\alpha \in (0, \infty)$, we have the pointwise inequality of functions $\alpha \chi_{S_f(\alpha)} \leq |f| \chi_{S_f(\alpha)}$. This inequality still holds when raised to the power p, so

$$\alpha^p \mu(S_f(\alpha)) \leq \int_{S_f(\alpha)} |f|^p \, d\mu \leq \int_X |f|^p \, d\mu = \|f\|_p^p.$$

Therefore

$$\left(\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha) \right)^{1/p} \leq \|f\|_p.$$

This says that $C_p(f) \leq \|f\|_p$. The relation $L^p(\mu) \subset L^p_{\mathrm{w}}(\mu)$ is an immediate consequence. □

Solution to (2). As in the solution to (1), for a measurable function $f \colon X \to \mathbb{C}$ and $\alpha \in (0, \infty)$, set $S_f(\alpha) = \{x \in X \colon |f(x)| > \alpha\}$.

We prove the first part of the hint. If $f, g \colon X \to \mathbb{C}$ are measurable and $\alpha \in (0, \infty)$, then one can have $|(f+g)(x)| > \alpha$ only if $|f(x)| > \frac{\alpha}{2}$ or $|g(x)| > \frac{\alpha}{2}$. In particular,

$$S_{f+g}(\alpha) \subset S_f(\alpha/2) \cup S_g(\alpha/2).$$

It follows that

$$\lambda_{f+g}(\alpha) \leq \lambda_f(\alpha/2) + \lambda_g(\alpha/2).$$

Therefore

$$\alpha^p \lambda_{f+g}(\alpha) \leq 2^p (\alpha/2)^p \lambda_f(\alpha/2) + 2^p (\alpha/2)^p \lambda_g(\alpha/2).$$

Now, since $\sup_{\alpha > 0} h(\alpha/2) = \sup_{\alpha > 0} h(\alpha)$ for any function $h \colon (0, \infty) \to \mathbb{R}$,

$$\begin{aligned} \sup_{\alpha > 0} \alpha^p \lambda_{f+g}(\alpha) &\leq \sup_{\alpha > 0} 2^p (\alpha/2)^p \lambda_f(\alpha/2) + \sup_{\alpha > 0} 2^p (\alpha/2)^p \lambda_g(\alpha/2) \\ &= \sup_{\alpha > 0} 2^p \alpha^p \lambda_f(\alpha) + \sup_{\alpha > 0} 2^p \alpha^p \lambda_g(\alpha). \end{aligned}$$

Since $t \mapsto t^p$ and $t \mapsto t^{1/p}$ are strictly increasing on $(0, \infty)$, this is the same as saying

$$C_p(f+g)^p \leq 2^p \big(C_p(f)^p + C_p(g)^p\big)$$

or

$$C_p(f+g) \leq 2\big(C_p(f)^p + C_p(g)^p\big)^{1/p}.$$

The second part of the hint is trivial if $\beta = 0$. Otherwise, we observe that $|\beta f(x)| > \alpha$ if and only if $|f(x)| > \alpha/|\beta|$, so $S_{\beta f}(\alpha) = S_f(\alpha/|\beta|)$. Therefore

$$\sup_{\alpha>0} \alpha^p \lambda_{\beta f}(\alpha) = \sup_{\alpha>0} \alpha^p S_f(\alpha/|\beta|) = \sup_{\alpha>0} (|\beta|\alpha)^p S_f(\alpha) = |\beta|^p \sup_{\alpha>0} \alpha^p \lambda_f(\alpha).$$

Taking the $1/p$ power and using the fact that $t \mapsto t^{1/p}$ is strictly increasing on $(0, \infty)$, we get $C_p(\beta f) = |\beta| C_p(f)$, as desired.

To prove the result, we simply observe that the first part of the hint implies that if $C_p(f)$ and $C_p(g)$ are finite, then so is $C_p(f+g)$, while the second part of the hint implies that if $C_p(f)$ is finite and $\beta \in \mathbb{C}$ then $C_p(\beta f)$ is finite. $\square$

Solution to (3). Take $X = [0,1]$ and take μ to be Lebesgue measure. Define

$$f(x) = x \qquad \text{and} \qquad g(x) = 1 - x$$

for $x \in [0,1]$. Then for $\alpha > 0$ we have

$$\alpha \lambda_f(\alpha) = \alpha \lambda_g(\alpha) = \begin{cases} \alpha(1-\alpha) & 0 < \alpha \leq 1 \\ 0 & \alpha > 1 \end{cases} \qquad \text{and} \qquad \alpha \lambda_{f+g}(\alpha) = \begin{cases} \alpha & 0 < \alpha \leq 1 \\ 0 & \alpha > 1. \end{cases}$$

It is easily checked that this gives $C_1(f) = C_1(g) = \frac{1}{4}$, and it is immediate that $C_1(f+g) = 1$. So $C_1(f+g) > C_1(f) + C_1(g)$, and the triangle inequality fails. $\square$

Alternate solution to (3). Take $X = (0,1)$ and take μ to be Lebesgue measure. Define

$$f(x) = \frac{1}{x} \qquad \text{and} \qquad g(x) = \frac{1}{1-x}$$

for $x \in [0,1]$. Then for $\alpha > 0$ we have

$$\alpha \lambda_f(\alpha) = \alpha \lambda_g(\alpha) = \begin{cases} \alpha & 0 < \alpha \leq 1 \\ 1 & \alpha > 1. \end{cases}$$

So $C_1(f) = C_1(g) = 1$. However,

$$f(x) + g(x) = \frac{1}{x(1-x)}.$$

The minimum value of this function on $(0,1)$ is 4, occurring at $x = \frac{1}{2}$ and nowhere else in $(0,1)$. (This can be checked by the usual methods of elementary calculus.) Therefore

$$C_1(f+g) \geq 4\lambda_f(4) = 4 > 2 = C_1(f) + C_1(g)$$

(One can show that $C_1(f+g) = 4$, but this is a bit messy and is not necessary for the problem.) $\square$

Second alternate solution to (3). Take $X = (0, 2)$ and take μ to be Lebesgue measure. Define $f = 1 + \chi_{(0,1)}$ and $g = \chi_{[1,2)}$. Then for $\alpha > 0$ we have

$$\alpha \lambda_f(\alpha) = \begin{cases} 2\alpha & 0 < \alpha \leq 1 \\ \alpha & 1 < \alpha \leq 2 \\ 0 & \alpha > 2 \end{cases} \qquad \text{and} \qquad \alpha \lambda_g(\alpha) = \begin{cases} \alpha & 0 < \alpha \leq 1 \\ 0 & \alpha > 1. \end{cases}$$

So $C_1(f) = C_1(g) = 1$. However, $f + g = 2$, so

$$\alpha \lambda_{f+g}(\alpha) = \begin{cases} 2\alpha & 0 < \alpha \leq 2 \\ 0 & \alpha > 2. \end{cases}$$

Therefore $C_1(f + g) = 4 > 2 = C_1(f) + C_1(g)$. $\square$

The usual solution to (4) uses Fubini's Theorem. Here is the intended solution, which does not use Fubini's Theorem, although the version for counting measures (a statement about changing the order of summation) is used.

It uses a lemma. (This is where the order of summation is interchanged.)

Lemma 6. Let $a_0, a_1, \ldots \in [0, \infty)$, let $r, p \in [1, \infty)$ satisfy $r < p$, and assume that

$$\sup_{n \in \mathbb{Z}_{>0}} n^p \sum_{k=n}^{\infty} a_k < \infty.$$

Then $\sum_{k=0}^{\infty} (k + 1)^r a_k < \infty$.

Proof. We have, using nonnegativity of the terms to justify interchanging the order of summation at the third step,

$$\begin{aligned} \sum_{k=1}^{\infty} (k+1)^r a_k &= \sum_{k=0}^{\infty} \sum_{n=0}^{k} [(n+1)^r - n^r] a_k \\ &= \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} [(n+1)^r - n^r] a_k \\ &= \sum_{k=0}^{\infty} a_k + \sum_{n=1}^{\infty} \frac{(n+1)^r - n^r}{n^p} \left(n^p \sum_{k=n}^{\infty} a_k \right) \\ &\leq \sum_{k=0}^{\infty} a_k + \left(\sup_{n \in \mathbb{Z}_{>0}} n^p \sum_{k=n}^{\infty} a_k \right) \sum_{n=1}^{\infty} \frac{(n+1)^r - n^r}{n^p}. \end{aligned}$$

Using the Mean Value Theorem and $r \geq 1$, one checks that

$$(n+1)^r - n^r \leq r(n+1)^{r-1}$$

for $n \in \mathbb{Z}_{>0}$. Therefore

$$\sum_{n=1}^{\infty} \frac{(n+1)^r - n^r}{n^p} \leq r \sum_{n=1}^{\infty} \frac{(n+1)^{r-1}}{n^p}.$$

Since $r < p$, one can easily check that the series on the right converges, as follows. Choose $s \in \mathbb{R}$ such that $r < s < p$. Then

$$\lim_{n \to \infty} \frac{(n+1)^s}{n^p} = 0,$$

so the terms of this sequence are bounded: there is $M \in (0, \infty)$ such that

$$\frac{(n+1)^s}{n^p} < M$$

for all $n \in \mathbb{Z}_{>0}$. Then

$$\sum_{n=1}^{\infty} \frac{(n+1)^{r-1}}{n^p} = \sum_{n=1}^{\infty} \left(\frac{(n+1)^s}{n^p} \right) (n+1)^{r-1-s} \leq M \sum_{n=1}^{\infty} (n+1)^{r-1-s},$$

which is finite since $r - 1 - s < -1$.

Since also $\sum_{k=0}^{\infty} a_k$ converges, it follows that $\sum_{k=0}^{\infty} (k+1)^r a_k$ converges. □

Solution to (4). For $n \in \mathbb{Z}_{\geq 0}$, define

$$X_n = \{ x \in X : n \leq |f(x)| < n+1 \}.$$

Then define $g \colon X \to \mathbb{R}$ by $g = \sum_{n=0}^{\infty} (n+1) \chi_{X_n}$. Then $|f| \geq g$,

so it is enough to prove that g is an L^r function. Now

$$\int_0^1 g^r \, d\mu = \sum_{k=0}^{\infty} (k+1)^r \mu(X_k)$$

and

$$\begin{aligned} \sup_{n \in \mathbb{Z}_{>0}} n^p \sum_{k=n}^{\infty} \mu(X_k) &= \sup_{n \in \mathbb{Z}_{>0}} n^p \mu\left(\bigcup_{k=n}^{\infty} X_k \right) \\ &= \sup_{n \in \mathbb{Z}_{>0}} n^p \mu\big(\{ x \in X : |f(x)| > n \} \\ &\leq \sup_{\alpha \in (0, \infty)} \lambda_f(\alpha). \end{aligned}$$

The desired result therefore follows by applying the lemma with $a_k = \mu(X_k)$ for $k \in \mathbb{Z}_{\geq 0}$. □