MATH 617 (WINTER 2024, PHILLIPS): HOMEWORK 5

Problem 1 (Problem 4 parts (a) and (d) in Chapter 7 of Rudin)**.** Each part of this problem counts as one normal problem.

Let $p \in [1, \infty]$.

(1) Let $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$. Imitate the proof of Theorem 7.14 of Rudin to show that the integral defining $(f * g)(x)$ exists for almost all x, that $f * g \in L^p(\mathbb{R})$, and that $\|f * g\|_p \leq \|f\|_1 \|g\|_p$. (For $p \in (1, \infty)$, you will need to use Hölder's inequality on carefully chosen functions involving powers of the ones you are given.)

(2) Prove that for every $\varepsilon > 0$ there are nonzero $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ such that
$$\|f * g\|_p > (1 - \varepsilon)\|f\|_1 \|g\|_p.$$

Problem 2 (Problem 6 in Chapter 7 of Rudin)**.** This problem counts as 1.5 ordinary problems. Do not use anything about polar coordinates from previous courses.

Let
$$S^{d-1} = \{x \in \mathbb{R}^d \colon \|x\| = 1\}$$
be the unit sphere in $\mathbb{R}^d$. Show that every $x \in \mathbb{R}^d \setminus \{0\}$ has a unique representation $x = rz$ with $r \in (0, \infty)$ and $z \in S^{d-1}$. Thus, $\mathbb{R}^d \setminus \{0\}$ may be regarded as the Cartesian product $(0, \infty) \times S^{d-1}$.

Let m_d be Lebesgue measure on $\mathbb{R}^d$. Define a measure σ_{d-1} on S^{d-1} by
$$\sigma_{d-1}(E) = d \cdot m_d(\{rz \colon z \in E \text{ and } 0 < r < 1\})$$
for every Borel set $E \subset S^{d-1}$. Prove that for every nonnegative Borel function $f \colon \mathbb{R}^d \to [0, \infty]$ we have
$$\int_{\mathbb{R}^d} f \, dm_d = \int_0^\infty r^{d-1} \left(\int_{S^{d-1}} f(rz) \, d\sigma_{d-1}(z) \right) dr. \tag{1}$$
Check that this coincides with familiar results when $d = 2$ and when $d = 3$.

Hint. Check that the formula is true when f is the characteristic function of a set of the form
$$\big\{rz \colon z \in E \text{ and } r_1 < r < r_2\big\}$$
for a Borel set $E \subset S^{d-1}$ and $0 \leq r_1 < r_2 \leq \infty$. Pass from these to characteristic functions of Borel sets in $\mathbb{R}^d$.

Problem 3 (Taken from some edition of Rudin, but not in the one I am working from)**.** This problem counts as 1.5 ordinary problems. (There are a number of estimates to do. Be sure to prove that the hypotheses of the theorems you use are really satisfied.)

Use Fubini's Theorem and the relation
$$\frac{1}{x} = \int_0^\infty e^{-xt} \, dt$$

Date: 7 February 2024.

for $x > 0$ to prove that

$$\lim_{a\to\infty} \int_0^a \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}.$$

Remark 1. The function $f(x) = \frac{\sin(x)}{x}$ is not Lebesgue integrable on $(0, \infty)$, because

$$\int_0^\infty \left| \frac{\sin(x)}{x} \right| dx = \infty.$$

The problem asserts the existence of the improper Riemann integral, not of the Lebesgue integral.