MATH 617 (WINTER 2024, PHILLIPS): SOLUTIONS TO HOMEWORK 5

Problem 1 (Problem 4 parts (a) and (d) in Chapter 7 of Rudin)**.** Each part of this problem counts as one normal problem.

Let $p \in [1, \infty]$.

(1) Let $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$. Imitate the proof of Theorem 7.14 of Rudin to show that the integral defining $(f * g)(x)$ exists for almost all x, that $f * g \in L^p(\mathbb{R})$, and that $\|f * g\|_p \leq \|f\|_1 \|g\|_p$. (For $p \in (1, \infty)$, you will need to use Hölder's inequality on carefully chosen functions involving powers of the ones you are given.)

(2) Prove that for every $\varepsilon > 0$ there are nonzero $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ such that
$$\|f * g\|_p > (1 - \varepsilon)\|f\|_1 \|g\|_p.$$

Solution to part (1). As in the proof of Theorem 7.14 of Rudin, we may assume that f and g are Borel functions, and we consider $|f|$ and $|g|$ first.

Our first objective is to show that if $p < \infty$ then
$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f(x-t)g(t)|\, dt \right)^p dx \leq \|f\|_1^p \|g\|_p^p, \tag{1}$$
in particular, that the outer integral exists. Let q be the conjugate exponent to p. We apply Hölder's inequality (in Theorem 3.5 of Rudin), in the form
$$\int_{-\infty}^{\infty} h_1(x) h_2(x)\, dx \leq \left(\int_{-\infty}^{\infty} h_1(x)^q \, dx \right)^{1/q} \left(\int_{-\infty}^{\infty} h_2(x)^p \, dx \right)^{1/p}$$
for nonnegative measurable functions h_1 and h_2. We take
$$h_1(t) = |f(x-t)|^{1/q} \qquad \text{and} \qquad h_2(t) = |f(x-t)|^{1/p} |g(t)|.$$
This gives
$$\begin{aligned} \int_{-\infty}^{\infty} |f(x-t)g(t)|\, dt &\leq \left(\int_{-\infty}^{\infty} |f(x-t)|\, dt \right)^{1/q} \left(\int_{-\infty}^{\infty} |f(x-t)| \cdot |g(t)|^p \, dt \right)^{1/p} \\ &= \|f\|_1^{1/q} \left(\int_{-\infty}^{\infty} |f(x-t)| \cdot |g(t)|^p \, dt \right)^{1/p}. \end{aligned}$$
So
$$\left(\int_{-\infty}^{\infty} |f(x-t)g(t)|\, dt \right)^p \leq \|f\|_1^{p/q} \int_{-\infty}^{\infty} |f(x-t)| \cdot |g(t)|^p \, dt. \tag{2}$$
Now $(x, t) \mapsto |f(x-t)g(t)|$ is Borel, for the same reason as in the proof of Theorem 7.14 of Rudin, and Lebesgue measure on $\mathbb{R}$ is σ-finite, so Fubini's Theorem for nonnegative functions implies that
$$x \mapsto \int_{-\infty}^{\infty} |f(x-t)g(t)|\, dt$$

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is Borel. Thus the left hand side of (2) is also a Borel function of x. The right hand side of (2) is also a Borel function of x, by Theorem 7.8 of Rudin applied to the L^1 functions $|f|$ and $|g|^p$. Therefore, using the L^1 norm estimate from Theorem 7.14 of Rudin at the second step and $\frac{p}{q}+1=p$ at the third step, we get

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|f(x-t)g(t)|\,dt\right)^p dx &\leq \|f\|_1^{p/q}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|f(x-t)|\cdot|g(t)|^p\,dt\right)dx \\ &\leq \|f\|_1^{p/q}\left(\int_{-\infty}^{\infty}|f|\,dm\right)\left(\int_{-\infty}^{\infty}|g|^p\,dm\right) \\ &\leq \|f\|_1^p\|g\|_p^p.\end{aligned}$$

This proves the claim.

For $p=\infty$ the analog of (1) is that

$$\sup_{x\in\mathbb{R}}\int_{-\infty}^{\infty}|f(x-t)g(t)|\,dt \leq \|f\|_1\|g\|_\infty. \tag{3}$$

This is obvious.

It is now immediate that $\int_{-\infty}^{\infty}|f(x-t)g(t)|\,dt<\infty$ for almost every x (for every x if $p=\infty$). Therefore $\int_{-\infty}^{\infty}f(x-t)g(t)\,dt$ exists for almost every x (for every x if $p=\infty$).

We next claim that the function

$$h(x)=\int_{-\infty}^{\infty}f(x-t)g(t)\,dt$$

is measurable. For $n\in\mathbb{Z}_{>0}$ define

$$g_n(x)=\begin{cases} g(x) & |x|\leq n \text{ and } |g(x)|\leq n \\ 0 & \text{otherwise.}\end{cases}$$

Then $g_n\in L^1(\mathbb{R})$, so Theorem 7.8 of Rudin implies that

$$h_n(x)=\int_{-\infty}^{\infty}f(x-t)g_n(t)\,dt$$

is measurable. For every x for which $\int_{-\infty}^{\infty}|f(x-t)g(t)|\,dt<\infty$, the Dominated Convergence Theorem implies that $\lim_{n\to\infty}h_n(x)=h(x)$. Thus, h is the pointwise almost everywhere limit of measurable functions, hence measurable. This proves the claim.

Now for $p<\infty$ we get, using (1) at the second step,

$$\int_{-\infty}^{\infty}|h|\,dm \leq \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|f(x-t)g(t)|\,dt\right)^p dx \leq \|f\|_1^p\|g\|_p^p.$$

whence

$$\|f*g\|_p=\|h\|_p\leq\|f\|_1\|g\|_p.$$

For $p=\infty$ we get, using (3) instead,

$$\|f*g\|_\infty\leq\sup_{x\in\mathbb{R}}|h(x)|\leq\sup_{x\in\mathbb{R}}\int_{-\infty}^{\infty}|f(x-t)g(t)|\,dt\leq\|f\|_1\|g\|_\infty.$$

This completes the proof. □

The first two solutions to part (2) are written using functions normalized to have $\|f\|_1=\|g\|_p=1$. This is neater in a sense, but not really needed, and not normalizing would simplify the notation a little.

Solution to part (2). We first assume $p < \infty$. Choose $\rho \in (0,1)$ so small that $\rho < 1 - (1-\varepsilon)^p$. Define

$$f = \left(\frac{1}{2\rho}\right)\chi_{[-\rho,\rho]} \qquad \text{and} \qquad g = \left(\frac{1}{2^{1/p}}\right)\chi_{[-1,1]}.$$

Then one checks directly that $\|f\|_1 = \|g\|_p = 1$. Let m be Lebesgue measure on $\mathbb{R}$. We have

$$\begin{aligned}
(f*g)(x) &= \int_{-\infty}^{\infty} f(y)g(x-y)\,dy = \left(\frac{1}{2\rho\cdot 2^{1/p}}\right)\int_{-\rho}^{\rho}\chi_{[-1,1]}(x-y)\,dy \\
&= \left(\frac{1}{2\rho\cdot 2^{1/p}}\right)\int_{-\rho}^{\rho}\chi_{[x-1,\,x+1]}(y)\,dy \\
&= \left(\frac{1}{2\rho\cdot 2^{1/p}}\right) m\big([-\rho,\rho]\cap[x-1,\,x+1]\big) \\
&= \begin{cases} 0 & x \le -1-\rho \\ \frac{x+1+\rho}{2\rho\cdot 2^{1/p}} & -1-\rho < x \le -1+\rho \\ \frac{2\rho}{2\rho\cdot 2^{1/p}} & -1+\rho < x \le 1-\rho \\ \frac{1+\rho-x}{2\rho\cdot 2^{1/p}} & 1-\rho < x \le 1+\rho \\ 0 & 1+\rho < x. \end{cases}
\end{aligned}$$

Considering only the interval $[-1+\rho,\, 1-\rho]$, we get

$$\|f*g\|_p^p \ge [1-\rho-(-1+\rho)](2^{-1/p})^p = 1-\rho > (1-\varepsilon)^p,$$

so $\|f*g\|_p > 1-\varepsilon = (1-\varepsilon)\|f\|_1\|g\|_p$, as desired..

If $p = \infty$, take $f = \chi_{[0,1]}$ and $g = 1$. Then

$$\|f\|_1 = \|g\|_\infty = 1 \qquad \text{and} \qquad f*g = 1,$$

so $\|f*g\|_\infty = 1 > (1-\varepsilon)\|f\|_1\|g\|_\infty$. □

Alternate solution to part (2). We first assume $p < \infty$. Instead of choosing f to be a multiple of the characteristic function of a short interval, as in the first solution, we choose g to be a multiple of the characteristic function of a long interval.

Choose $M \in (0,\infty)$ so large that $\frac{1}{M} < 1 - (1-\varepsilon)^p$. Define

$$f = \left(\frac{1}{2}\right)\chi_{[-1,1]} \qquad \text{and} \qquad g = \left(\frac{1}{(2M)^{1/p}}\right)\chi_{[-M,M]}.$$

Then one checks directly that $\|f\|_1 = \|g\|_p = 1$. Let m be Lebesgue measure on $\mathbb{R}$. We have

$$\begin{aligned}
(f*g)(x) &= \int_{-\infty}^{\infty} f(y)g(x-y)\,dy = \left(\frac{1}{2\cdot(2M)^{1/p}}\right)\int_{-1}^{1}\chi_{[-M,M]}(x-y)\,dy \\
&= \left(\frac{1}{2\cdot(2M)^{1/p}}\right)\int_{-1}^{1}\chi_{[x-M,\,x+M]}(y)\,dy \\
&= \left(\frac{1}{2\cdot(2M)^{1/p}}\right) m\big([-1,1]\cap[x-M,\,x+M]\big) \\
&= \begin{cases} 0 & x \le -M-1 \\ \frac{x+M+1}{2\cdot(2M)^{1/p}} & -M-1 < x \le -M+1 \\ \frac{2}{2\cdot(2M)^{1/p}} & -M+1 < x \le M-1 \\ \frac{M+1-x}{2\cdot(2M)^{1/p}} & M-1 < x \le M+1 \\ 0 & M+1 < x. \end{cases}
\end{aligned}$$

Considering only the interval $[-M+1,\,M-1]$, we get

$$\|f*g\|_p^p \ge [M-1-(-M+1)]((2M)^{-1/p})^p = 1 - \frac{1}{M} > (1-\varepsilon)^p,$$

so $\|f*g\|_p > 1-\varepsilon = (1-\varepsilon)\|f\|_1\|g\|_p$, as desired..

If $p=\infty$, take $f=\chi_{[0,1]}$ and $g=1$. Then

$$\|f\|_1 = \|g\|_\infty = 1 \qquad \text{and} \qquad f*g = 1,$$

so $\|f*g\|_\infty = 1 > (1-\varepsilon)\|f\|_1\|g\|_\infty$. □

Third solution to part (2). In this solution, we prove something stronger, and much more interesting, namely that for *every* $g \in L^p(\mathbb{R})$ (for every $g \in C_0(\mathbb{R})$ if $p=\infty$) there exists a sequence $(f_n)_{n\in\mathbb{Z}_{>0}}$ in $L^1(\mathbb{R})$ such that $\|f_n\|_1 = 1$ for all n, and such that $\lim_{n\to\infty}\|f_n * g\|_p = \|g\|_p$. In fact, the sequence $(f_n)_{n\in\mathbb{Z}_{>0}}$ will depend on neither p nor the choice of the function g, and we will even get

$$\lim_{n\to\infty}\|f_n * g - g\|_p = 0.$$

Choose any nonnegative continuous function $f\colon \mathbb{R}\to[0,\infty)$ such that $\mathrm{supp}(f)\subset[-1,1]$ and $\int_{-\infty}^{\infty} f\,dm = 1$. Define $f_n(x) = nf(x/n)$.

First suppose that $g \in C_{\mathrm{c}}(X)$. Choose M such that $\mathrm{supp}(g)\subset[-M,\,M]$. If $p\neq\infty$, set

$$\varepsilon_0 = \frac{\varepsilon}{2^{1/p}(M+1)^{1/p}}.$$

If $p=\infty$, set $\varepsilon_0 = \frac{1}{2}\varepsilon$. Use uniform continuity of g to choose $\delta>0$ such that whenever $s,t\in\mathbb{R}$ satisfy $|s-t|<\delta$, then $|g(s)-g(t)|<\varepsilon_0$. Without loss of generality $\delta<1$. Let $n>1/\delta$, so that $\mathrm{supp}(f_n)\subset[-\delta,\,\delta]$. A change of variables allows us to rewrite the convolution as

$$(f_n*g)(x) = \int_{-\infty}^{\infty} f_n(t)g(x-t)\,dt.$$

Also, from $\int_{-\infty}^{\infty} f \, dm = 1$ we get $\int_{-\infty}^{\infty} f_n \, dm = 1$, whence

$$\begin{aligned} |(f_n * g)(x) - g(x)| &= \left| \int_{-\infty}^{\infty} f_n(t) g(x-t) \, dt - \int_{-\infty}^{\infty} f_n(t) g(x) \, dt \right| \\ &\leq \int_{-\infty}^{\infty} f_n(t) |g(x-t) - g(x)| \, dt. \end{aligned}$$

The integrand on the right is zero whenever $|t| \geq \delta$, and also whenever $|x| \geq M + \delta$. For $|t| < \delta$ we have $|g(x-t) - g(x)| < \varepsilon_0$. From $\int_{-\infty}^{\infty} f_n \, dm = 1$ we therefore get

$$|(f_n * g)(x) - g(x)| \leq \varepsilon_0$$

for all x. When $p = \infty$, this tells us that $\|f_n * g - g\|_\infty \leq \varepsilon_0 < \varepsilon$. So assume $p \neq \infty$. We have already observed that $(f_n * g)(x) = g(x) = 0$ for $|x| \geq M + \delta$. Therefore

$$\begin{aligned} \|f_n * g - g\|_p &\leq \left(\int_{-\infty}^{\infty} |(f_n * g)(x) - g(x)|^p \, dx \right)^{1/p} \\ &< [2(M+\delta)\varepsilon_0^p]^{1/p} < 2^{1/p}(M+1)^{1/p}\varepsilon_0 = \varepsilon. \end{aligned}$$

Now we consider an arbitrary function $g \in L^p(\mathbb{R})$ ($g \in C_0(\mathbb{R})$ if $p = \infty$). Let $\varepsilon > 0$. Choose $h \in C_{\mathrm{c}}(X)$ such that $\|h - g\|_p < \frac{1}{3}\varepsilon$. Choose N so large that $\|f_n * h - h\|_p < \frac{1}{3}\varepsilon$ for all $n \geq N$. One checks that $f_n * g - f_n * h = f_n * (g - h)$. Using the result of Part (a) and $\|f_n\|_1 = 1$ for all n, we then get

$$\begin{aligned} \|f_n * g - g\|_p &\leq \|f_n * (g-h)\|_p + \|f_n * h - h\|_p + \|h - g\|_p \\ &\leq \|f_n\|_1 \|g - h\|_p + \|f_n * h - h\|_p + \|h - g\|_p \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon \end{aligned}$$

for all $n \geq N$, as desired. □

Problem 2 (Problem 6 in Chapter 7 of Rudin)**.** This problem counts as 1.5 ordinary problems. Do not use anything about polar coordinates from previous courses.

Let

$$S^{d-1} = \{x \in \mathbb{R}^d \colon \|x\| = 1\}$$

be the unit sphere in $\mathbb{R}^d$. Show that every $x \in \mathbb{R}^d \setminus \{0\}$ has a unique representation $x = rz$ with $r \in (0, \infty)$ and $z \in S^{d-1}$. Thus, $\mathbb{R}^d \setminus \{0\}$ may be regarded as the Cartesian product $(0, \infty) \times S^{d-1}$.

Let m_d be Lebesgue measure on $\mathbb{R}^d$. Define a measure σ_{d-1} on S^{d-1} by

$$\sigma_{d-1}(E) = d \cdot m_d(\{rz \colon z \in E \text{ and } 0 < r < 1\})$$

for every Borel set $E \subset S^{d-1}$. Prove that for every nonnegative Borel function $f \colon \mathbb{R}^d \to [0, \infty]$ we have

$$\int_{\mathbb{R}^d} f \, dm_d = \int_0^\infty r^{d-1} \left(\int_{S^{d-1}} f(rz) \, d\sigma_{d-1}(z) \right) dr. \tag{4}$$

Check that this coincides with familiar results when $d = 2$ and when $d = 3$.

Hint. Check that the formula is true when f is the characteristic function of a set of the form

$$\{rz \colon z \in E \text{ and } r_1 < r < r_2\}$$

for a Borel set $E \subset S^{d-1}$ and $0 \leq r_1 < r_2 \leq \infty$. Pass from these to characteristic functions of Borel sets in $\mathbb{R}^d$.

Solution. (This solution is a little sketchy at one point, and omits the check of the last sentence when $d = 3$.)

To do the first paragraph properly, we really want to show that $h(r, z) = rz$ is a homeomorphism $h\colon (0, \infty) \times S^{d-1} \to \mathbb{R}^d \setminus \{0\}$. To prove this, we simply write down the continuous inverse $x \mapsto \big(\|x\|,\, (1/\|x\|)x\big)$.

Next, we verify the formula (4). First, let $E \subset S^{d-1}$ be Borel, suppose $0 \le r_1 < r_2 \le \infty$, and set

$$R = \{rz \colon z \in E \text{ and } r_1 \le r < r_2\} \cap (\mathbb{R}^d \setminus \{0\}).$$

(This set differs slightly from the set in the suggestion, but is easier to work with.) We verify (4) for $f = \chi_R$. Set

$$S = \{rz \colon z \in E \text{ and } 0 < r < 1\}.$$

Using $m_d(rF) = r^d m_d(F)$ for every Borel set F (Theorem 2.20(e) of Rudin, together with the identification of the factor appearing there as $|\det(T)|$), from $r_1 S \subset r_2 S$ and $R = r_2 S \setminus r_1 S$ we get

$$k \cdot m_d(R) = r_2^d \sigma_{d-1}(E) - r_1^d \sigma_{d-1}(E).$$

It is immediate to check that this is consistent with the right hand side of (4).

We next verify that (4) holds for the characteristic function of an open subset $U \subset \mathbb{R}^d \setminus \{0\}$. Choose a sequence of finite partitions $\mathcal{P}_n$ for $n \in \mathbb{Z}_{>0}$ of S^{d-1} into Borel sets such that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for every n, and such that the diameter of each set in $\mathcal{P}_n$ is at most 2^{-n}. The collections

$$\mathcal{Q}_n = \left\{ \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right) \cap (0, \infty) \colon k \in \mathbb{Z} \text{ and } k \ge 0 \right\}$$

are partitions of $(0, \infty)$ with the same properties. Then (some detail omitted) $h^{-1}(V) \subset (0, \infty) \times S^{d-1}$ is the increasing union of the sets

$$F_n = \bigcup \{A \times B \colon A \in \mathcal{P}_n,\, B \in \mathcal{Q}_n, \text{ and } A \times B \subset h^{-1}(V)\}. \tag{5}$$

Thus U is the increasing union of the sets $h(F_n)$. For each $n \in \mathbb{Z}_{>0}$, the set $h(F_n)$ is a disjoint union of sets $h(A \times B)$ with $A \in \mathcal{P}_n$ and $B \in \mathcal{Q}_n$, and we proved above that (4) holds for the characteristic functions of such sets. Therefore (4) holds for χ_{F_n} by the version of the Monotone Convergence Theorem for series of nonnegative functions. Thus (4) holds for χ_U by the usual version of the Monotone Convergence Theorem.

It now follows from the Dominated Convergence Theorem that (4) holds for the characteristic function of a bounded G_δ-set $G \subset \mathbb{R}^d \setminus \{0\}$. Every bounded Borel set $F \subset \mathbb{R}^d \setminus \{0\}$ is contained in a bounded G_δ-set $G \subset \mathbb{R}^d \setminus \{0\}$ such that $m_d(G \setminus F) = 0$. If $m_d(F) = 0$, applying the first sentence to G, we see that both sides of (4) for χ_G are zero. By monotonicity, both sides of (4) for χ_F are zero. Thus, (4) holds for bounded Borel sets of measure zero. Letting F be an arbitrary bounded Borel set and letting G be as before, we find that (4) holds for χ_G and $\chi_{G \setminus F}$. Since everything is finite, we subtract, showing that (4) holds for χ_F.

Now let $F \subset \mathbb{R}^d \setminus \{0\}$ be an arbitrary Borel set. Applying the Monotone Convergence Theorem to the characteristic functions of $F_n = F \cap B_n(0)$, we see that (4) holds for χ_F. Writing an arbitrary nonnegative Borel function as the pointwise increasing limit of simple nonnegative Borel functions and applying the Monotone

Convergence Theorem once again, we see that (4) holds for arbitrary nonnegative Borel functions on $\mathbb{R}^d \setminus \{0\}$. Since $m_d(\{0\}) = 0$, we may replace $\mathbb{R}^d \setminus \{0\}$ by $\mathbb{R}^d$.

It remains to check that this gives the usual results in dimensions 2 and 3. We give the solution only in dimension 2; dimension 3 is handled the same way.

Recall the usual formula:

$$\int_{\mathbb{R}^2} f \, dm_2 = \int_0^\infty r \left(\int_0^{2\pi} f\big(r\cos(\theta),\, r\sin(\theta)\big)\, d\theta \right) dr.$$

We thus want to show that if f is a nonnegative Borel function on $S^1 \subset \mathbb{R}^2$, then

$$\int_{S^1} f \, d\sigma_1 = \int_0^{2\pi} f\big(\cos(\theta),\, \sin(\theta)\big)\, d\theta.$$

Define $g \colon B_1(0) \setminus \{0\} \to [0, \infty]$ by

$$g(x, y) = f\left(\frac{x}{\sqrt{x^2 + y^2}},\, \frac{y}{\sqrt{x^2 + y^2}} \right).$$

That is, $g(w) = f((1/\|w\|)w)$. Using the definition of σ_1 at the first step, and Fubini and the change of variables formula for the function $k \colon (0,1) \times (0, 2\pi) \to B_1(0) \setminus \{0\}$ given by $k(r, \theta) = \big(r\cos(\theta),\, r\sin(\theta)\big)$ at the second step, we get

$$\begin{aligned} \int_{S^1} f \, d\sigma_1 &= 2 \int_{B_1(0)\setminus\{0\}} g \, dm_2 = 2 \int_0^1 r \left(\int_0^{2\pi} f\big(\cos(\theta),\, \sin(\theta)\big)\, d\theta \right) dr \\ &= \int_0^{2\pi} f\big(\cos(\theta),\, \sin(\theta)\big)\, d\theta, \end{aligned}$$

as desired. □

It is essential for the argument that the union in (5) be disjoint, even when only finitely many sets are involved (such as when U is bounded). The previous step doesn't help with the union of two sets of the form used when they are not disjoint.

Alternate solution (sketch). We describe a different method to verify the formula (4) for the characteristic functions of arbitrary open sets. From this point on, the rest of the proof is the same.

First, let $E \subset S^{d-1}$ be Borel, suppose $0 \leq r_1 < r_2 \leq \infty$, and set

$$R(r_1, r_2, E) = \{rz \colon z \in E \text{ and } r_1 \leq r < r_2\} \cap (\mathbb{R}^d \setminus \{0\}).$$

One verifies (4) for $f = \chi_{R(r_1, r_2, E)}$ as in the first solution. Next, again with $0 \leq r_1 < r_2 \leq \infty$, set

$$S(r_1, r_2, E) = \{rz \colon z \in E \text{ and } r_1 < r < r_2\}.$$

Then $S(r_1, r_2, E)$ is an increasing union

$$S(r_1, r_2, E) = \bigcup_{n=1}^\infty R\left(r_1 + \tfrac{1}{n},\, r_2,\, E\right).$$

Applying the Monotone Convergence Theorem on both sides of (4), with $f_n = \chi_{R(r_1 + 1/n,\, r_2,\, E)}$, we see that (4) holds for $f = \chi_{S(r_1, r_2, E)}$. In particular, it holds for the characteristic function of any open set of the form $S(r_1, r_2, E)$ with $E \subset S^{d-1}$ open.

Now we claim that (4) holds for any finite union of sets of the form $S(r_1, r_2, E)$. This is proved by induction on the size of the union. We know it holds for one such set. So assume it holds for any union of n such sets, and let $S_1, S_2, \ldots, S_{n+1}$ be $n+1$ such sets. Set

$$E = \bigcup_{k=1}^{n+1} S_k, \quad F = \bigcup_{k=1}^{n} S_k \qquad \text{and} \qquad G = F \cap S_{n+1} = \bigcup_{k=1}^{n} (S_k \cap S_{n+1}).$$

Each set $S_k \cap S_{n+1}$ is easily checked to be again of the form we are considering. So the formula (4) holds for χ_F and χ_G by the induction hypothesis, and for $\chi_{S_{n+1}}$ by the case $n = 1$. Since $\chi_E = \chi_F + \chi_{S_{n+1}} - \chi_G$ and everything is finite, we conclude that the formula (4) holds for χ_E. This completes the induction.

Now apply the Monotone Convergence Theorem to show that the formula (4) holds for χ_E whenever E is a countable union of sets of the form $S(r_1, r_2, E)$.

Next, one proves using easy point set topology arguments that every open subset of $\mathbb{R}^d \setminus \{0\}$ is a countable union of sets of the form $S(r_1, r_2, E)$ with E open. It follows that (4) holds for χ_U for every open set $U \subset \mathbb{R}^d \setminus \{0\}$. The conclusion for arbitrary open sets $U \subset \mathbb{R}^d$ follows, because $\{0\}$ has measure zero.

The proof is now completed as in the first solution. $\square$

Second alternate solution (brief sketch). We could also verify (4) for characteristic functions of Borel sets by showing that the collection $\mathcal{C}$ of sets E such that it holds for χ_E is a monotone class containing all finite disjoint unions of sets $h(A \times B)$ with $A \subset (0, \infty)$ and $B \subset S^{d-1}$ Borel. Using Theorem 7.3 of Rudin, it is easy to show that every such class contains all Borel subsets of $\mathbb{R}^d \setminus \{0\}$. The proof that $\mathcal{C}$ is a monotone class is essentially the same as the proof of Theorem 7.6 of Rudin. $\square$

Problem 3 (Taken from some edition of Rudin, but not in the one I am working from)**.** This problem counts as 1.5 ordinary problems. (There are a number of estimates to do. Be sure to prove that the hypotheses of the theorems you use are really satisfied.)

Use Fubini's Theorem and the relation

$$\frac{1}{x} = \int_0^\infty e^{-xt}\, dt$$

for $x > 0$ to prove that

$$\lim_{a \to \infty} \int_0^a \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}.$$

Remark 1. The function $f(x) = \frac{\sin(x)}{x}$ is not Lebesgue integrable on $(0, \infty)$, because

$$\int_0^\infty \left| \frac{\sin(x)}{x} \right| dx = \infty.$$

The problem asserts the existence of the improper Riemann integral, not of the Lebesgue integral.

Solution to Problem 3. We first claim that if $a > 0$ then

$$\int_0^a \frac{\sin(x)}{x}\, dx = \int_0^\infty \left(\int_0^a e^{-xt} \sin(x)\, dx \right) dt.$$

Define $f\colon [0,\infty)\times(0,a]\to\mathbb{R}$ by $f(t,x)=e^{-xt}\sin(x)$. Then f is continuous, hence measurable. We have $|f(t,x)|\le xe^{-xt}$ for all $t,x\in[0,\infty)$, and an application of Fubini's Theorem for nonnegative functions shows that

$$\int_{[0,\infty)\times[0,a]} xe^{-xt}d(m\times m)(t,x)=\int_0^a\left(\int_0^\infty xe^{-xt}\,dt\right)dx=\int_0^a 1\,dx<\infty.$$

It follows that f is integrable on $[0,\infty)\times(0,a]$. (Remembering to do this step is a substantial part of the problem.) Now we may apply Fubini's Theorem for integrable functions to get

$$\int_0^\infty\left(\int_0^a e^{-xt}\sin(x)\,dx\right)dt=\int_0^a\left(\int_0^\infty e^{-xt}\sin(x)\,dt\right)dx=\int_0^a x^{-1}\sin(x)\,dx.$$

This proves the claim.

We next claim that

$$\int_0^a e^{-xt}\sin(x)\,dx=\frac{1}{1+t^2}-\frac{e^{-at}(\cos(a)+t\sin(a))}{1+t^2}.$$

This is most easily done by writing $\sin(x)=\frac{1}{2i}(e^{ix}-e^{-ix})$, combining the products of exponentials, and doing some algebra after the integration is done. It can also be done by two integrations by parts: one gets an equation for the integral, which can be solved. Details are omitted.

Now we claim that

$$\lim_{a\to\infty}\int_0^\infty\frac{e^{-at}(\cos(a)+t\sin(a))}{1+t^2}\,dt=0.$$

To prove this, we estimate, for $t\ge 0$:

$$\left|\frac{e^{-at}(\cos(a)+t\sin(a))}{1+t^2}\right|\le\frac{e^{-at}(|\cos(a)|+t|\sin(a)|)}{1+t^2}\le\frac{e^{-at}(1+t)}{1+t^2}.$$

Separate estimates on $[0,1]$ and on $[1,\infty)$ show that

$$\frac{1+t}{1+t^2}\le 2$$

for all $t\ge 0$. Therefore

$$\left|\int_0^\infty\frac{e^{-at}(\cos(a)+t\sin(a))}{1+t^2}\,dt\right|\le\int_0^\infty 2e^{-at}\,dt=\frac{2}{a},$$

from which the claim follows.

Now we prove the result. We have

$$\begin{aligned}\lim_{a\to\infty}\int_0^a\frac{\sin(x)}{x}\,dx&=\lim_{a\to\infty}\int_0^\infty\left(\int_0^a e^{-xt}\sin(x)\,dx\right)dt\\&=\int_0^\infty\frac{1}{1+t^2}\,dt-\lim_{a\to\infty}\int_0^\infty\frac{e^{-at}(\cos(a)+t\sin(a))}{1+t^2}\,dt\\&=\int_0^\infty\frac{1}{1+t^2}\,dt=\frac{\pi}{2}.\end{aligned}$$

This completes the proof. □

Remark 2. One can also use the Dominated Convergence Theorem to prove the third claim, that

$$\lim_{a\to\infty}\int_0^\infty\frac{e^{-at}(\cos(a)+t\sin(a))}{1+t^2}\,dt=0.$$

One must, of course, prove that for every sequence $(a_n)_{n \in \mathbb{Z}_{>0}}$ in $[0, \infty)$ such that $\lim_{n \to \infty} a_n = \infty$, one has

$$\lim_{n \to \infty} \int_0^\infty \frac{e^{-a_n t}(\cos(a_n) + t \sin(a_n))}{1 + t^2} \, dt = 0.$$

One part, pointwise convergence of the integrand to zero almost everywhere, is easy. The other part requires a nonnegative integrable function $g \colon [0, \infty) \to \mathbb{R}$ such that

$$\left| \frac{e^{-at}(\cos(a) + t \sin(a))}{1 + t^2} \right| \leq g(t)$$

for all $a \in [0, \infty)$ and $t \in [0, \infty)$. In fact, one easily sees that it is enough to find g and some $N > 0$ such that this estimate holds for all $a \in [N, \infty)$ and $t \in [0, \infty)$. We sketch two possibilities.

First, the proof above shows that

$$\left| \frac{e^{-at}(\cos(a) + t \sin(a))}{1 + t^2} \right| \leq 2e^{-at}$$

for all $a \in [0, \infty)$ and $t \in [0, \infty)$. Therefore the estimate holds for $a \geq 1$ with $g(t) = 2e^{-t}$.

Second, the proof above shows that

$$\left| \frac{e^{-at}(\cos(a) + t \sin(a))}{1 + t^2} \right| \leq \frac{e^{-at}(1 + t)}{1 + t^2} \leq \frac{1 + te^{-at}}{1 + t^2}$$

for all $a \in [0, \infty)$ and $t \in [0, \infty)$. If $a \geq 1$ then $te^{-at} \leq te^{-t} \leq 1$, so the estimate holds for $a \geq 1$ with $g(t) = \frac{2}{1+t^2}$.