# MATH 617 (WINTER 2024, PHILLIPS): SOLUTIONS TO HOMEWORK 6

Essentially no proofreading has been done, and there are slight gaps.

**Problem 1** (Problem 5 in Chapter 7 of Rudin)**.** This problem counts as five regular problems.

Let $M$ be the Banach space of all complex Borel measures on $\mathbb{R}$. Recall that $\|\mu\| = |\mu|(\mathbb{R})$ for $\mu \in M$.

Let $s\colon \mathbb{R}^2 \to \mathbb{R}$ be the function $s(x, y) = x + y$ for $x, y \in \mathbb{R}$. For $\mu, \nu \in M$, define $\mu * \nu$ to be the set function given by $(\mu * \nu)(E) = (\mu \times \nu)(s^{-1}(E))$ for every Borel set $E \subset \mathbb{R}$. (The function $\mu * \nu$ is called the *convoluton* of $\mu$ and $\nu$.)

(1) Let $\mu, \nu \in M$. Prove that $\mu * \nu$ is a complex Borel measure on $\mathbb{R}$ which satisfies $\|\mu * \nu\| \leq \|\mu\| \|\nu\|$.

(2) Let $\mu, \nu \in M$. Prove that $\mu * \nu$ is the unique measure $\lambda \in M$ which satisfies

$$\int_{\mathbb{R}} f \, d\lambda = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x + y) \, d\mu(x) \right) d\nu(y)$$

for all $f \in C_0(\mathbb{R})$.

(3) Prove that the operation $(\mu, \nu) \mapsto \mu * \nu$, from $M \times M$ to $M$, is commutative, associative, distributes over addition, and satisfies $\alpha(\mu * \nu) = \alpha\mu * \nu = \mu * \alpha\nu$ for all $\alpha \in \mathbb{C}$. (Thus, $M$, with multiplication defined by $\mu \cdot \nu = \mu * \nu$, is a commutative algebra over $\mathbb{C}$. Since we already know that $M$ is a Banach space, the inequality in (1) now means that $M$ is in fact a complex Banach algebra.)

(4) Let $\mu, \nu \in M$. Prove that for every Borel set $E \subset \mathbb{R}$,

$$(\mu * \nu)(E) = \int_{\mathbb{R}} \mu\big(\{x - t \colon x \in E\}\big) \, d\nu(t).$$

(5) Say that $\mu \in M$ is *discrete* if there is a countable set $S \subset \mathbb{R}$ such that $\mathbb{R} \setminus S$ is $\mu$-null, and say that $\mu \in M$ is *continuous* if $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}$. Prove that if $\mu, \nu \in M$ are both discrete, then $\mu * \nu$ is discrete. Prove that if $\mu, \nu \in M$ and $\mu$ is continuous, then $\mu * \nu$ is continuous.

(6) As usual, let $m$ be Lebesgue measure on $\mathbb{R}$. (Note that $m \notin M$.) Prove that if $\mu, \nu \in M$ and $\mu \ll m$, then $\mu * \nu \ll m$.

(7) Prove that the discrete measures form a closed subalgebra of $M$ and that the continuous measures form a closed ideal in $M$.

(8) Recall that if $\lambda$ is a (nonnegative) measure on $(X, \mathcal{M})$ and $f\colon X \to \mathbb{C}$ is integrable or nonnegative, then $f \cdot \lambda$ is the (complex or nonnegative) measure on $X$ defined by $(f \cdot \lambda)(E) = \int_E f \, d\lambda$. Prove that $f \mapsto f \cdot m$ defines an isometric linear map which preserves the multiplication given by convolution from $L^1(\mathbb{R})$ to $\big\{\mu \in M \colon \mu \ll m\big\}$. Use this fact to prove that the operation $(f, g) \mapsto f * g$ makes $L^1(\mathbb{R})$ a commutative Banach algebra. Also prove that $\big\{\mu \in M \colon \mu \ll m\big\}$ is a closed ideal in $M$.

*Date*: 21 February 2024.

(9) Prove that the algebra $M$ is unital, but that $L^1(\mathbb{R})$ is not unital.

You may use without proof the obvious analog of Fubini's Theorem for complex measures. It is proved by writing each complex measure as a linear combination of four nonnegative measures, but the product then has 16 terms. Alternatively

*Hint for Part (9).* The easiest way to show that $L^1(\mathbb{R})$ is not unital is to see what would happen to an identity element under the Fourier transform map $f \mapsto \widehat{f}$. Feel free to use that, even if we haven't yet discussed Fourier transforms.

A more useful method, because it generalizes better, to to prove (using the definitions in a previous homework problem) a sufficient special case of the fact that if $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, then $f * g$ is continuous.

Almost all of this works for any locally compact Hausdorff group $G$ in place of $\mathbb{R}$, and with a (left) Haar measure $\mu$ in place of $m$. The algebra is then called $M(G)$, the *measure algebra* of $G$. The algebra $L^1(G)$ is unital if and only if $G$ has the discrete topology; in this case, the Haar measure can be taken to be counting measure, and $M(G) = L^1(G, \mu)$. If $G$ is not commutative, one must be a little more careful with the formulas. The outcome is that commutativity of $G$, of $M(G)$, and of $L^1(G)$, are all equivalent.

*Solution to Part (1).* We need to know that $\mu * \nu$ is defined, that is, that $s^{-1}(E)$ is in the product $\sigma$-algebra of two copies of the Borel subsets of $\mathbb{R}$ when $E$ is Borel. This is immediate from two facts: first, the product $\sigma$-algebra of two copies of the Borel subsets of $\mathbb{R}$ is exactly the Borel subsets of $\mathbb{R}^2$, and if $E$ is Borel then $s^{-1}(E)$ is Borel. The first fact was proved in the proof that Lebesgue measure on $\mathbb{R}^2$ is the product of two copies of Lebesgue measure on $\mathbb{R}$. The second follows from continuity of $s$.

That $\mu * \nu$ is a complex measure is now immediate from the fact that inverse images preserve countable disjoint unions and $\varnothing$.

For the norm estimate, we first claim that $|\mu \times \nu| = |\mu| \times |\nu|$ To prove the claim, write $\mu = h \cdot |\mu|$ and $\mu = k \cdot |\nu$ for Borel functions $h, k \colon \mathbb{R} \to \mathbb{C}$ with $|h(x)| = 1$ and $|k(x)| = 1$ for all $x \in \mathbb{R}$. Then, using Fubini's Theorem at the first and third steps (the measurability criterion is immediate), if $f \colon \mathbb{R}^2 \to \mathbb{C}$ is a bounded Borel function, then

$$\begin{aligned} \int_{\mathbb{R}\times\mathbb{R}} f \, d(\mu \times \nu) &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x, y) \, d\mu(x) \right) \, d\nu(y) \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x, y) h(x) \, d|\mu|(x) \right) k(y) \, d|\nu|(y) = \int_{\mathbb{R}\times\mathbb{R}} fl \, d(|\mu| \times |\nu|). \end{aligned}$$

In particular, if $E \subset \mathbb{R} \times \mathbb{R}$ is Borel, taking $f = \chi_E$ gives

$$(\mu \times \nu)(E) = \int_E l \, d(|\mu| \times |\nu|).$$

Since $E$ is an arbitrary Borel set and $|l(x, y)| = 1$ for all $x, y \in \mathbb{R}$, this implies the claim.

To prove the norm estimate, we need to show that if $(E_n)_{n \in \mathbb{Z}_{>0}}$ is a family of disjoint Borel sets such that $\mathbb{R} = \coprod_{n=1}^{\infty} E_n$, then $\sum_{n=1}^{\infty} |(\mu * \nu)(E_n)| \leq \|\mu\| \|\nu\|$. We have

$$\begin{aligned}\sum_{n=1}^{\infty}|(\mu*\nu)(E_n)| &= \sum_{n=1}^{\infty}|(\mu\times\nu)(s^{-1}(E_n))| \leq \sum_{n=1}^{\infty}|\mu\times\nu|(s^{-1}(E_n)) \\ &= \sum_{n=1}^{\infty}\big(|\mu|\times|\nu|\big)(s^{-1}(E_n)) = \big(|\mu|\times|\nu|\big)(\mathbb{R}^2) \\ &= |\mu|(\mathbb{R})\cdot|\nu|(\mathbb{R}) = \|\mu\|\|\nu\|.\end{aligned}$$

This completes the proof. □

*Alternate proof of the norm estimate in Part (1).* We need to show that if $(E_n)_{n\in\mathbb{Z}_{>0}}$ is a family of disjoint Borel sets such that

$$\mathbb{R} = \coprod_{n=1}^{\infty} E_n, \tag{1}$$

then $\sum_{n=1}^{\infty}|(\mu*\nu)(E_n)| \leq \|\mu\|\|\nu\|$. We have, using the Monotone Convergence Theorem at the sixth step and (1) and countable additivity of $|\mu|$ at the seventh step,

$$\begin{aligned}\sum_{n=1}^{\infty}|(\mu*\nu)(E_n)| &= \sum_{n=1}^{\infty}|(\mu\times\nu)(s^{-1}(E_n))| \\ &= \sum_{n=1}^{\infty}\left|\int_{\mathbb{R}}\nu\big(\{y\in\mathbb{R}\colon (x,y)\in s^{-1}(E_n)\}\big)\,d\mu(x)\right| \\ &= \sum_{n=1}^{\infty}\left|\int_{\mathbb{R}}\nu\big(\{y\in\mathbb{R}\colon x+y\in E_n\}\big)\,d\mu(x)\right| \\ &\leq \sum_{n=1}^{\infty}\int_{\mathbb{R}}\big|\nu\big(\{y\in\mathbb{R}\colon x+y\in E_n\}\big)\big|\,d|\mu|(x) \\ &\leq \sum_{n=1}^{\infty}\int_{\mathbb{R}}|\nu|\big(\{y\in\mathbb{R}\colon x+y\in E_n\}\big)\,d|\mu|(x) \\ &= \int_{\mathbb{R}}\sum_{n=1}^{\infty}|\nu|\big(\{y\in\mathbb{R}\colon x+y\in E_n\}\big)\,d|\mu|(x) \\ &= \int_{\mathbb{R}}|\nu|(\mathbb{R})\,d|\mu|(x) = |\mu|(\mathbb{R})\cdot|\nu|(\mathbb{R}) = \|\mu\|\|\nu\|.\end{aligned}$$

This completes the proof. □

*Solution to Part (2).* The Riesz Representation Theorem for $C_0(\mathbb{R})$ implies the such a measure $\lambda$ is unique.

We must therefore prove that $\lambda$ satisfies the condition. For a Borel set $E\subset\mathbb{R}$, we have $\chi_{s^{-1}(E)}(x,y) = \chi_E(x+y)$. Therefore, using Fubini's Theorem at the last step,

$$\begin{aligned}\int_{\mathbb{R}}\chi_E\,d(\mu*\nu) &= (\mu*\nu)(E) = (\mu\times\nu)(s^{-1}(E)) \\ &= \int_{\mathbb{R}\times\mathbb{R}}\chi_{s^{-1}(E)}\,d(\mu\times\nu) = \int_{\mathbb{R}}\left(\int_{\mathbb{R}}\chi_E(x+y)\,d\mu(x)\right)d\nu(y).\end{aligned} \tag{2}$$

Thus, the required formula holds for characteristic functions of Borel sets. Therefore it holds for Borel simple functions. Since every bounded Borel function is a uniform limit of Borel simple functions and the measures are finite, the formula holds for all bounded Borel functions, and in particular for all functions in $C_0(\mathbb{R})$. $\square$

*Solution to Part (3).* No solution has been written yet, but, using Fubini's Theorem multiple times, everything here is essentially immediate. (Commutativity uses the fact that the group $\mathbb{R}$ is commutative. Associativity uses associativity of addition in $\mathbb{R}$.) $\square$

*Solution to Part (4).* This follows from (2) (as written: for the characteristic function of a Borel set $E$) together with the fact that

$$\int_{\mathbb{R}} \chi_E(x+y)\, d\mu(x) = \mu\big(\{x - t \colon x \in E\}\big)$$

for every Borel set $E$. $\square$

*Solution to Part (5).* I didn't find a definition of $\mu$-null in the edition of Rudin's book I am using. Here, however, are two equivalence characterizations (for any complex measure on $(X, \mathcal{M})$:

(1) $E \in \mathcal{M}$ is $\mu$-null if $|\mu|(E) = 0$.
(2) $E \in \mathcal{M}$ is $\mu$-null if whenever $F \subset E$ is in $\mathcal{M}$, then $\mu(F) = 0$.

That (1) implies (2) follows from the fact that $|\mu|(E)$ is the supremum over all measurable partitions $E = \coprod_{n=1}^{\infty} E_n$ of $\sum_{n=1}^{\infty} |\mu(E_n)|$, by takinf $E_1 = F$. The reverse implication is immediate from the same relation.

For the part about discrete measures, choose subsets $E, F \subset \mathbb{R}$ such that $\mathbb{R} \setminus E$ is $\mu$-null and $\mathbb{R} \setminus F$ is $\nu$-null. Set $S = E + F$, which is also countable. If $Y \subset \mathbb{R} \setminus S$, then $x \in Y$ and $y \in F$ imply $x - y \in \mathbb{R} \setminus E$. Thus, $x \in Y$ implies that $\big\{x - y \colon x \in Y\big\} \subset \mathbb{R} \setminus F$. Therefore, by Part (4),

$$(\mu * \nu)(Y) = \int_{\mathbb{R}} \mu\big(\{x - y \colon x \in Y\}\big)\, d\nu(y) = \int_{\mathbb{R}} 0\, d\nu(y) = 0.$$

For the part about continuous measures, let $z \in \mathbb{R}$. Then, by Part (4),

$$(\mu * \nu)(\{z\}) = \int_{\mathbb{R}} \mu\big(\{x - t \colon x \in \{z\}\}\big)\, d\nu(t) = \int_{\mathbb{R}} \mu(\{z - t\})\, d\nu(t) = \int_{\mathbb{R}} 0\, d\nu(t) = 0.$$

This completes the solution. $\square$

*Solution to Part (6).* Let $E \subset \mathbb{R}$ be a Borel set such that $m(E) = 0$. By translation invariance of $m$, for all $y \in \mathbb{R}$, we also have $m\big(\{x - y \colon x \in E\}\big) = 0$, so that also $\mu\big(\{x - y \colon x \in E\}\big) = 0$. By Part (4),

$$(\mu * \nu)(E) = \int_{\mathbb{R}} \mu\big(\{x - y \colon x \in E\}\big)\, d\nu(y) = \int_{\mathbb{R}} 0\, d\nu(y) = 0.$$

This completes the solution. $\square$

*Solution to Part (7).* It is easy to see that both the sets of discrete and continuous measues are vector subspaces of $M$; proofs are omitted. The algebraic statements involving products in both parts are immediate from Part (5).

We claim that that the set of discrete measures is closed. Let $(\mu_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of discrete measures which converges (in norm) to a measure $\mu$. For $n \in \mathbb{Z}_{>0}$ choose a countable set $S_n \subset \mathbb{R}$ such that $\mathbb{R} \setminus S_n$ is $\mu_n$-null. Set $S = \bigcup_{n=1}^{\infty} S_n$, which is a countable set such that $\mathbb{R} \setminus S$ is $\mu_n$-null for all $n \in \mathbb{Z}_{>0}$. Then for $E \subset \mathbb{R} \setminus S$, we have $\mu(E) = \lim_{n\to\infty} \mu_n(E) = \lim_{n\to\infty} 0 = 0$. Thus $\mathbb{R} \setminus S$ is $\mu$-null, so $\mu$ is discrete. The claim is proved.

We claim that the set of continuous measures is closed. Let $(\mu_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence of continuous measures which converges (in norm) to a measure $\mu$. For every $x \in \mathbb{R}$, we have $\mu(\{x\}) = \lim_{n\to\infty} \mu_n(\{x\}) = \lim_{n\to\infty} 0 = 0$. So $\mu$ is continuous. The claim is proved. □

*Solution to Part (8) (sketch).* It follows from Theorem 6.13 of Rudin's book that $f \mapsto f \cdot m$ is linear and isometric from $L^1(\mathbb{R})$ to $M$. Therefore its range is a closed subspace.

Preservation of convolution is a computation (which requires Fubini's Theorem); not yet written. Since we already showed that $M$ is a commutative Banach algebra, this shows that $L^1(\mathbb{R})$ is isometrically isomorphic to a closed subalgebra of $M$, and is hence a commutative Banach algebra.

The ideal property follows from Part (6). □

It is also easy to prove directly that $\{\mu \in M \colon \mu \ll m\}$ is closed in $M$.

*Solution to Part (9).* For $x \in M$ let $\delta_x$ be the "point mass measure at $x$", that is, for a Borel set $E \subset \mathbb{R}$,

$$\delta_x(E) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$$

(The notation, and its generalizations, is fairly standard.) Then $\delta_x \in M$ for all $x \in \mathbb{R}$.

We claim that $\delta_0$ is an identity for $M$. By commutativity, it is enough to show that $\mu * \delta_0 = \mu$ for every $\mu \in M$. Recall from Part (4) that is $\mu, \nu \in M$, $E \subset \mathbb{R}$ is a Borel set, and for $t \in \mathbb{R}$ we define $F_t \subset \mathbb{R}$ by $F_t = \{x - t \colon x \in E\}$, then

$$(\mu * \nu)(E) = \int_{\mathbb{R}} \mu(F_t)\, d\nu(t).$$

Putting $\nu = \delta_0$ and using $F_0 = E$, we get $(\mu * \delta_0)(E) = \mu(F_0) = \mu(E)$.

We now show that $L^1(\mathbb{R})$ does not have an identity. The easiest way to do this is to consider the map $\varphi \colon L^1(\mathbb{R}) \to C_0(\mathbb{R})$ given by $\varphi(f) = \widehat{f}$ for $f \in L^1(\mathbb{R})$. This map is well defined by Theorem 9.6 of Rudin's book, it is a homomorphism by Theorem 9.2(c) of Rudin's book, and it is injective by Theorem 9.12 of Rudin's book. If $L^1(\mathbb{R})$ had an identity $e$, then $\varphi(e) \in C_0(\mathbb{R})$ would be a nonzero element satisfying $\varphi(e)^2 = \varphi(e)$. Clearly no such nonzero element exists. □

*Alternate solution to Part (9).* We first claim that if $f \in L^1(\mathbb{R})$ and $a, b \in \mathbb{R}$ satisfy $a < b$, then $f * \chi_{[a,b]}$ is continuous. To prove the claim, let $\varepsilon > 0$, and choose $\delta_0 > 0$ so small that whenever $E \subset \mathbb{R}$ is measurable and $m(E) < \delta_0$, then $\int_E |f|\, dm < \varepsilon$. (That this can be done is a standard fact for $L^1(X, \mu)$ for any positive measure $\nu$, and should have been done in Math 616.) Set $\delta = \frac{1}{2}\delta_0$.

For any $t \in \mathbb{R}$, we have

$$(f * \chi_{[a,b]})(t) = \int_{\mathbb{R}} f(s)\chi_{[a,b]}(t-s)\, dm(s) = \int_{t-b}^{t-a} f\, dm.$$

Therefore, if $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < \delta$, the symmestric difference

$$E = [t_1 - b,\, t_1 - a] \triangle [t_2 - b,\, t_2 - a]$$

satisfies $m(E) < 2\delta = \delta_0$, so

$$\left|(f * \chi_{[a,b]})(t_1) - (f * \chi_{[a,b]})(t_2)\right| = \left|\int_E f \, dm\right| \leq \int_E |f| \, dm < \varepsilon.$$

This proves the claim.

If now $f \in L^1(\mathbb{R})$ is an identity for $L^1(\mathbb{R})$, then $f * \chi_{[-1,1]} = \chi_{[-1,1]}$ almost everywhere. The function $f * \chi_{[-1,1]}$ is continuous. But there is no continuous function $g$ such that $g = \chi_{[-1,1]}$, since, if there were, the indefinite integral $F$ of $\chi_{[-1,1]}$ would be differentiable everywhere, which is not the case. $\square$