MATH 618 (SPRING 2010): FINAL EXAM SOLUTIONS

Instructions: All lemmas, claims, examples, counterexamples, etc. require proof, except when explicitly stated otherwise.

Closed book: No notes, books, calculators, cell phones, or other electronic devices.

1. (a) (10 points) State Morera's Theorem.

Solution. Theorem 10.17 of Rudin: Let $\Omega \subset \mathbb{C}$ be open, and let $f\colon \Omega \to \mathbb{C}$ be continuous. Suppose that for every closed triangle in Ω with boundary path γ, one has $\int_\gamma f(\zeta)\, d\zeta = 0$. Then f is holomorphic on Ω. □

The continuity hypothesis is essential.

(b) (10 points) State Cauchy's Formula for a convex set.

Solution. This is 10.15 of Rudin: Let $\Omega \subset \mathbb{C}$ be a convex open set. Let γ be a closed path in Ω, and let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function. Then for every $z \in \Omega \setminus \mathrm{Ran}(\gamma)$, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \mathrm{Ind}_\gamma(z)\cdot f(z).$$

□

(c) (10 points) State the Fourier Inversion Theorem.

Solution. Theorem 9.11 of Rudin: Let $f \in L^1(\mathbb{R})$, and suppose that also $\widehat{f} \in L^1(\mathbb{R})$. For $x \in \mathbb{R}$ set

$$g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{f}(t)e^{itx}\, dt.$$

Then $g = f$ almost everywhere. □

Rudin also includes the statement that $g \in C_0(\mathbb{R})$.

Substantial partial credit will be given for the version for $L^2(\mathbb{R})$, Theorem 9.13(d) of Rudin.

2. (30 points) Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$f(z + 2010) = f(z) \quad \text{and} \quad f(z + i) = f(z)$$

for all $z \in \mathbb{C}$. Prove that f is constant.

Solution. Let

$$R = \{x + iy\colon x \in [0,\, 2010] \text{ and } y \in [0, 1]\} \quad \text{and} \quad M = \sup_{z\in R} |f(z)|.$$

This number is finite because R is compact and f is continuous. We show $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Liouville's Theorem will then imply that f is constant.

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Let $z \in \mathbb{C}$. Choose $m, n \in \mathbb{Z}$ such that

$$\mathrm{Re}(z) - 2010\, m \in [0,\, 2010) \quad \text{and} \quad \mathrm{Im}(z) - n \in [0, 1).$$

Then $z - (2010\, m + in) \in R$, so (using the periodicity hypotheses)

$$|f(z)| = |f(z - (2010\, m + in))| \leq M.$$

This completes the proof. □

3. (25 points) Give an example of a measurable function $f \colon \mathbb{R} \to \mathbb{C}$ such that there is $g \in L^2(\mathbb{R})$ with $\widehat{g} = f$, but such that there is no $g \in L^1(\mathbb{R})$ with $\widehat{g} = f$.

Solution. Set $f = \chi_{[-1,1]}$. Then f is not the Fourier transform of a function in $L^1(\mathbb{R})$, because f is not continuous. However, f is the Fourier transform of a function in $L^2(\mathbb{R})$, because $f \in L^2(\mathbb{R})$.

Of course, there are many other possible choices for f. □

4. (a) (40 points) Evaluate

$$\int_{-\infty}^{\infty} \frac{e^{-(x-i)^2}}{x - i}\, dx - \int_{-\infty}^{\infty} \frac{e^{-(x+i)^2}}{x + i}\, dx.$$

Solution. Set

$$I_1 = \int_{-\infty}^{\infty} \frac{e^{-(x-i)^2}}{x - i}\, dx \quad \text{and} \quad I_2 = \int_{-\infty}^{\infty} \frac{e^{-(x+i)^2}}{x + i}\, dx,$$

so we are to find $I_1 - I_2$.

First, let's check that these integrals actually exist. We have

$$\left| \frac{e^{-(x-i)^2}}{x - i} \right| = \frac{|e^{-x^2 + 2ix + 1}|}{\sqrt{x^2 + 1}} = \frac{|e^{-x^2 + 1}|}{\sqrt{x^2 + 1}} \leq e \cdot e^{-x^2},$$

and e^{-x^2} is integrable on $(-\infty, \infty)$, so the integrand for I_1 is in $L^1(\mathbb{R})$. The same estimate holds for I_2.

Set $f(z) = \frac{1}{z} e^{-z^2}$ for $z \in \mathbb{C} \setminus \{0\}$. For $r > 0$ let $\gamma_{r,1}$ be the straight line path from $-r - i$ to $r - i$, with domain $[0, 2r]$, let $\gamma_{r,3}$ be the straight line path from $r - i$ to $r + i$, with domain $[2r,\, 2r + 2]$, let $\gamma_{r,2}$ be the straight line path from $r + i$ to $-r + i$, with domain $[2r + 2,\, 4r + 2]$, and let $\gamma_{r,4}$ be the straight line path from $-r + i$ to $-r - i$, with domain $[4r + 2,\, 4r + 4]$. (The indexing is out of sequence, to match the names I_1 and I_2 already chosen.) Let γ_r be the concatenation of these paths, which is a piecewise C^1 closed path in $\mathbb{C} \setminus \{0\}$ with domain $[0,\, 4r + 4]$.

We claim that $\mathrm{Ind}_{\gamma_r}(0) = 1$. We use Theorem 10.37 of Rudin. Note that $\mathrm{Ind}_{\gamma_r}(si)$ has the same value for all $s < -1$, by continuity of the index. Since $\{si \colon s \in (-\infty, -1)$ is unbounded, this value must be zero. Set $\rho = \min(r, 1)$. Apply Theorem 10.37 of Rudin, with $a = -i$ and $b = \rho$. We have

$$D_+ = \{z \in B_\rho(-i) \colon \mathrm{Im}(z) > -1\} \quad \text{and} \quad D_- = \{z \in B_\rho(-i) \colon \mathrm{Im}(z) < -1\}.$$

(These sets are both connected because they are convex.) It follows that for all $\varepsilon \in (0, \rho)$, we have $\mathrm{Ind}_{\gamma_r}((-1 + \varepsilon)i) = 1$. The set

$$U = \big\{ z \in \mathbb{C} \colon |\mathrm{Re}(z)| < r \text{ and } |\mathrm{Im}(z)| < 1 \big\}$$

is a convex, hence connected, open set contained in $\mathbb{C} \setminus \mathrm{Ran}(\gamma_r)$. Therefore $\mathrm{Ind}_{\gamma_r}(z)$ has the same value for all $z \in U$. So $\mathrm{Ind}_{\gamma_r}(0) = 1$, proving the claim.

(One can also use a homotopy from γ to a positively oriented circle with center zero.)

Therefore

$$\int_{\gamma_r} f(z)\,dz = 2\pi i \operatorname{Res}(f;0)$$

by the Residue Theorem. Using the series expansion

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-z^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n-1}}{n!},$$

we calculate $\operatorname{Res}(f;0) = 1$.

To simplify the notation, set

$$I_k(r) = \int_{\gamma_{r,k}} f(z)\,dz.$$

Observe that

$$I_1(r) = \int_{-r}^{r}\frac{e^{-(x-i)^2}}{x-i}\,dx \quad \text{and} \quad I_2(r) = -\int_{-r}^{r}\frac{e^{-(x+i)^2}}{x+i}\,dx.$$

(The sign in the second one comes from the negative orientation.) Therefore $\lim_{r\to\infty} I_1(r) = I_1$ and $\lim_{r\to\infty} I_2(r) = -I_2$. Furthermore,

$$|I_3(r)| = \left|\int_{-1}^{1}\frac{e^{-(r+it)^2}}{r+it}\,i\,dt\right| \leq \int_{-1}^{1}\frac{|e^{-r^2-irt+t^2}|}{\sqrt{r^2+t^2}}\,dt \leq \frac{2e^{-r^2+1}}{r}.$$

Therefore $\lim_{r\to\infty} I_3(r) = 0$. The same estimate shows that $\lim_{r\to\infty} I_4(r) = 0$. Now

$$2\pi i = \lim_{r\to\infty}\int_{\gamma_r} f(z)\,dz = \lim_{r\to\infty}[I_1(r) + I_2(r) + I_3(r) + I_4(r)] = I_1 - I_2 + 0 + 0.$$

So $I_1 - I_2 = 2\pi i$. □

(b) (10 points) Use the result of Part (a) to evaluate

$$\int_{-\infty}^{\infty}\frac{e^{-(x-i)^2}}{x-i}\,dx \quad \text{and} \quad \int_{-\infty}^{\infty}\frac{e^{-(x+i)^2}}{x+i}\,dx.$$

Solution. As in the previous solution, call these integrals I_1 and I_2. We calculate the real and imaginary parts of I_1:

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty}\frac{(x+i)e^{-x^2+2ix+1}}{(x+i)(x-i)}\,dx = \int_{-\infty}^{\infty}\frac{(x+i)[\cos(2x)+i\sin(2x)]e^{-x^2+1}}{x^2+1}\,dx \\ &= \int_{-\infty}^{\infty}\frac{[x\cos(2x)-\sin(2x)]e^{-x^2+1}}{x^2+1}\,dx + i\int_{-\infty}^{\infty}\frac{[\cos(2x)+x\sin(2x)]e^{-x^2+1}}{x^2+1}\,dx. \end{aligned}$$

The integrand in the real part is an odd function, so that integral is zero.

Since the integrands are complex conjugates of each other, one gets $I_2 = \overline{I_1}$. Now combining the equations

$$\operatorname{Re}(I_1) = \operatorname{Re}(I_2) = 0, \quad \operatorname{Im}(I_2) = -\operatorname{Im}(I_1), \quad \text{and} \quad I_1 - I_2 = 2\pi i,$$

we get $I_1 = \pi i$ and $I_2 = -\pi i$. □

5. (30 points) Let $D = \{z \in \mathbb{C} \colon |z| < 1\}$. Let $A(D) \subset C(\overline{D})$ be the disk algebra, the closed subspace of $C(\overline{D})$ given by

$$A(D) = \{ f \in C(\overline{D}) \colon f|_D \text{ is holomorphic} \}.$$

(You need not prove that $A(D)$ is a subspace or that it is closed in $C(\overline{D})$.)

Prove that there exists a bounded linear functional $\omega \colon C(\overline{D}) \to \mathbb{C}$ such that $\omega(f) = f'\left(\frac{1}{2}\right)$ for all $f \in A(D)$.

Solution. Define $\omega_0 \colon A(D) \to \mathbb{C}$ by $\omega_0(f) = f'\left(\frac{1}{2}\right)$. We claim that ω_0 is continuous. Suppose $(f_n)_{n \in \mathbb{Z}_{>0}}$ is a sequence in $A(D)$ such that $f_n \to f$ in $A(D)$. Then $f_n|_D \to f|_D$ uniformly, and in particular $f_n|_D \to f|_D$ uniformly on compact sets. Therefore $f_n'|_D \to f'|_D$ uniformly on compact sets. In particular, $\lim_{n\to\infty} f_n'\left(\frac{1}{2}\right) = f'\left(\frac{1}{2}\right)$. This shows that ω_0 is continuous.

The Hahn-Banach Theorem now implies that there is a bounded linear functional $\omega \colon C(\overline{D}) \to \mathbb{C}$ such that $\omega|_{A(D)} = \omega_0$. □

Alternate solution. Let ω_0 be as in the first solution. Instead of proving that ω_0 is continuous, we give an explicit bound on $\|\omega_0\|$. Let $f \in A(D)$. Since f is holomorphic on $B_{1/2}\left(\frac{1}{2}\right)$, Cauchy's Estimates show that

$$\left| f'\left(\tfrac{1}{2}\right) \right| \leq 2 \sup\left(|f(z)| \colon z \in B_{1/2}\left(\tfrac{1}{2}\right) \right\} \leq 2\|f\|.$$

Thus $\|\omega_0\| \leq 2$.

Now apply the Hahn-Banach Theorem as in the first solution. □

Second alternate solution (sketch). We give an explicit formula for ω. Specifically, define $\gamma \colon [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = \frac{1}{2} + \frac{1}{4}e^{it}$. Then define

$$\omega(f) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{\left(\frac{1}{2} - z\right)^2} \, dz$$

for $f \in C(\overline{D})$. Then ω is obviously linear. The computation, valid for $f \in C(\overline{D})$,

$$|\omega(f)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(\gamma(t))| \cdot |\gamma'(t)|}{\left|\frac{1}{2} - \gamma(t)\right|^2} \, dt = \frac{1}{2\pi} \int_0^{2\pi} 4^2 \cdot \tfrac{1}{4} \cdot |f(\gamma(t))| \, dt \leq 4\|f\|$$

implies that $\|\omega\| \leq 4$. That $\omega(f) = f'\left(\frac{1}{2}\right)$ for all $f \in A(D)$ follows from the form of Cauchy's Formula that gives derivatives of f in terms of path integrals, as in one of the homework problems. You would need to prove the appropriate formula, but a fair amount of partial credit will be given even if you don't. □

Remark: The optimal estimate $\|\omega\| \leq 2$ is obtained by the method of the last solution by taking $\gamma(t) = \frac{1}{2} + \frac{1}{2}e^{it}$ or $\gamma(t) = e^{it}$. A bit more work is needed, since these paths do not satisfy $\operatorname{Ran}(\gamma) \subset D$. One can show, however, that they do give $f'\left(\frac{1}{2}\right)$.

6. (35 points) Let $f \colon \mathbb{R} \to \mathbb{R}$ be an integrable function such that $f(x) > 0$ for all $x \in \mathbb{R}$. Prove that for all $t \neq 0$, we have $\operatorname{Re}\left(\widehat{f}(t)\right) < \widehat{f}(0)$.

Solution. We have

$$\widehat{f}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \, dx.$$

In particular, $\widehat{f}(0)$ is real and nonnegative.

Now let $t \in \mathbb{R} \setminus \{0\}$. Then

$$\begin{aligned}\operatorname{Re}\bigl(\widehat{f}(t)\bigr) &= \operatorname{Re}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} f(x)\,dx\right)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(-tx)f(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(tx)f(x)\,dx.\end{aligned}$$

Set $I = \left[\frac{\pi}{3t}, \frac{2\pi}{3t}\right]$. Then $\cos(tx) \le -\frac{1}{2}$ for all $x \in I$.

We claim that there is $\varepsilon > 0$ and a subset $E \subset I$ with Lebesgue measure $m(E) > 0$ such that $f(x) > \varepsilon$ for all $x \in E$. If not, for $n \in \mathbb{Z}_{>0}$ set set $E_n = \bigl\{x \in I \colon f(x) > \frac{1}{n}\bigr\}$. Then $m(E_n) = 0$. Therefore the set

$$\bigl\{x \in I \colon f(x) > 0\bigr\} = \bigcup_{n=1}^{\infty} E_n$$

has measure zero, which is impossible because $m(I) > 0$ and $f(x) > 0$ for all $x \in \mathbb{R}$. This contradiction proves the claim.

Now we have

$$\begin{aligned}\operatorname{Re}\bigl(\widehat{f}(t)\bigr) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\cos(tx)f(x)\,dx \le \frac{1}{2\pi}\int_{\mathbb{R}\setminus E} f(x)\,dx + \frac{1}{2\pi}\int_{E}\bigl(-\tfrac{1}{2}\bigr)\varepsilon\,dx\\ &\le \frac{1}{2\pi}\int_{\mathbb{R}} f(x)\,dx - \frac{\varepsilon m(E)}{4\pi} = \widehat{f}(0) - \frac{\varepsilon m(E)}{4\pi} < \widehat{f}(0).\end{aligned}$$

This completes the proof. □

Remark: In fact, it is true that $\bigl|\widehat{f}(t)\bigr| < \widehat{f}(0)$ for $t \neq 0$, although this takes a bit more work to prove.

Extra Credit. (40 extra credit points) Let $D = \{z \in \mathbb{C} \colon |z| < 1\}$. Prove that the series

$$\sum_{n=1}^{\infty} \frac{z^{2^n+1}}{n^2}$$

converges to a continuous function $f(z)$ on $\overline{D}$ which is holomorphic on D. Further prove (almost all the credit is for this part) that there does not exist any pair (Ω, g) in which Ω is a region with $\Omega \cap \partial D \neq \varnothing$ and g is a holomorphic function on Ω such that $g|_{\Omega\cap D} = f|_{\Omega\cap D}$.

Solution. The series converges uniformly on $\overline{D}$ because $|z^{2^n+1}/n^2| \le \frac{1}{n^2}$ for $z \in \overline{D}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty$. Therefore f is continuous.

If we write $f(z) = \sum_{n=0}^{\infty} c_n z^n$, then $|c_n| \le 1$ for all n. It is immediate that the series has radius of convergence at least 1. (This also follows from the previous paragraph.) Therefore f is holomorphic on D.

The main step in proving the last statement is to show that $\lim_{r\to 1^-}\operatorname{Re}(f'(rz)) = \infty$ for every z of the form $\exp(2\pi i k/2^l)$ with $k \in \mathbb{Z}$ and $l \in \mathbb{Z}_{>0}\cup\{0\}$. By the theorem on term by term differentiation of power series, we have

$$f'(z) = \sum_{n=1}^{\infty}\frac{(2^n+1)z^{2^n}}{n^2}$$

for all $z \in D$. Fix $k \in \mathbb{Z}$ and $l \in \mathbb{Z}_{>0} \cup \{0\}$, and set $z = \exp(2\pi i k / 2^l)$. Let $0 < r < 1$. Then for $n \geq l$,

$$\frac{(2^n + 1)(rz)^{2^n}}{n^2} = \frac{(2^n + 1)r^{2^n}}{n^2}.$$

Set

$$M_0 = \sum_{n=1}^{l} \frac{(2^n + 1)}{n^2}.$$

For any $M \in \mathbb{R}$ there is $r_0 < 1$ and $n \geq l$ such that

$$\frac{(2^n + 1)r_0^{2^n}}{n^2} > M + M_0,$$

and for $r_0 < r < 1$ we have

$$\begin{aligned} \operatorname{Re}(f'(rz)) &\geq \frac{(2^n + 1)r_0^{2^n}}{n^2} - \operatorname{Re}\left(\sum_{n=1}^{l} \frac{(2^n + 1)(rz)^{2^n}}{n^2} \right) \\ &\geq \frac{(2^n + 1)r_0^{2^n}}{n^2} - \sum_{n=1}^{l} \frac{(2^n + 1)|rz|^{2^n}}{n^2} > (M + M_0) - M_0 = M. \end{aligned}$$

This completes the proof that $\lim_{r \to 1^-} \operatorname{Re}(f'(rz)) = \infty$.

Now suppose Ω is a region with $\Omega \cap \partial D \neq \varnothing$, and suppose g is a holomorphic function on Ω such that $g|_{\Omega \cap D} = f|_{\Omega \cap D}$. Then we can choose $k \in \mathbb{Z}$ and $l \in \mathbb{Z}_{>0} \cup \{0\}$ such that $z = \exp(2\pi i k / 2^l) \in \Omega$. It follows from the previous paragraph that g' is not bounded on any neighborhood of z. This contradicts continuity of g' at z. □

Remark: For the last part, it is *not* enough to show that the radius of convergence of the power series is at most 1. Indeed, the series $\sum_{n=1}^{\infty} z^n$ has radius of convergence 1, but one can take $\Omega = \mathbb{C} \setminus \{1\}$ and $g(z) = (1 - z)^{-1}$.