MATH 617 FINAL EXAM (WINTER 2007)

1. (10 points) State the Radon-Nikodym Theorem and the Lebesgue Decomposition Theorem.

(These are usually considered two separate theorems. They were combined in the book, and you can give the combined statement if you like.)

2. (a) (10 points) State Fubini's Theorem.

(b) (35 points) Define $f\colon (0,1)\times(0,1)\to\mathbb{R}$ by

$$f(x,y) = \begin{cases} 0 & 0 < x \leq y < 1 \\ x^{-3/2}\sin(1/(xy)) & 0 < y < x < 1. \end{cases}$$

Prove that

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy = \int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx.$$

3. (a) (5 points) State the definition of a Lebesgue point.

(b) (35 points) Let E be the subset of $\mathbb{R}^2$ given by

$$E = \left\{ (x,y) \in \mathbb{R}^2 \colon \, -1 \leq x \leq 1, \, -1 \leq y \leq \sqrt{|x|} \right\}.$$

Consider the points in $\mathbb{R}^2$:

$$(-7,\,-17), \quad (0,\,-1), \quad \left(0,\,-\tfrac{1}{2}\right), \quad (0,0), \quad \left(\tfrac{1}{2},\sqrt{\tfrac{1}{2}}\right).$$

For each of the five points c listed above, determine, with proof, whether there exists a number $\lambda_c \in \mathbb{C}$ such that c is a Lebesgue point of the function

$$f(p) = \begin{cases} \lambda_c & p = c \\ \chi_E(p) & p \neq c. \end{cases}$$

4. (15 points) Let $(X,\mathcal{M},\mu)$ be a measure space. Let $M(X)$ be the Banach space of all complex measures defined on the σ-algebra $\mathcal{M}$. Let $E \subset M(X)$ be the set of all measures in $M(X)$ which are absolutely continuous with respect to μ. Prove that E is a closed subspace of $M(X)$.

5. (20 points) Let E and F be Banach spaces, and let $a\colon E \to F$ be an injective bounded linear map whose range $a(E) \subset F$ is closed. Prove that there is $\delta > 0$ such that $\|a\xi\| \geq \delta\|\xi\|$ for all $\xi \in E$.

6. (40 points) Let E be the set of bounded complex sequences $\xi = (\xi(n))_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\xi(n)$ exists. For $\xi \in E$ define $\|\xi\| = \sup_{n\in\mathbb{N}}|\xi(n)|$.

Prove carefully that E is a vector space, that $\|\cdot\|$ is a norm on E, and that E is a Banach space. (A large part of the credit is for the last part.)

7. (30 points) Let E be a Banach space. Prove or disprove: If $\omega\colon E\to\mathbb{C}$ is a linear functional such that $|\omega(\xi)| < 1$ for all $\xi \in E$ with $\|\xi\| = 1$, then $\|\omega\| < 1$.