MATH 618 (SPRING 2010): FINAL EXAM

Instructions: All lemmas, claims, examples, counterexamples, etc. require proof, except when explicitly stated otherwise.

Closed book: No notes, books, calculators, cell phones, or other electronic devices.

1. (a) (10 points) State Morera's Theorem.
(b) (10 points) State Cauchy's Formula for a convex set.
(c) (10 points) State the Fourier Inversion Theorem.

2. (30 points) Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$f(z + 2010) = f(z) \quad \text{and} \quad f(z + i) = f(z)$$

for all $z \in \mathbb{C}$. Prove that f is constant.

3. (25 points) Give an example of a measurable function $f\colon \mathbb{R} \to \mathbb{C}$ such that there is $g \in L^2(\mathbb{R})$ with $\widehat{g} = f$, but such that there is no $g \in L^1(\mathbb{R})$ with $\widehat{g} = f$.

4. (a) (40 points) Evaluate

$$\int_{-\infty}^{\infty} \frac{e^{-(x-i)^2}}{x-i}\, dx - \int_{-\infty}^{\infty} \frac{e^{-(x+i)^2}}{x+i}\, dx.$$

(b) (10 points) Use the result of Part (a) to evaluate

$$\int_{-\infty}^{\infty} \frac{e^{-(x-i)^2}}{x-i}\, dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{e^{-(x+i)^2}}{x+i}\, dx.$$

5. (30 points) Let $D = \{z \in \mathbb{C}\colon |z| < 1\}$. Let $A(D) \subset C(\overline{D})$ be the disk algebra, the closed subspace of $C(\overline{D})$ given by

$$A(D) = \{f \in C(\overline{D})\colon f|_D \text{ is holomorphic}\}.$$

(You need not prove that $A(D)$ is a subspace or that it is closed in $C(\overline{D})$.)

Prove that there exists a bounded linear functional $\omega\colon C(\overline{D}) \to \mathbb{C}$ such that $\omega(f) = f'\left(\frac{1}{2}\right)$ for all $f \in A(D)$.

6. (35 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ be an integrable function such that $f(x) > 0$ for all $x \in \mathbb{R}$. Prove that for all $t \neq 0$, we have $\mathrm{Re}\big(\widehat{f}(t)\big) < \widehat{f}(0)$.

Extra Credit. (40 extra credit points) Let $D = \{z \in \mathbb{C}\colon |z| < 1\}$. Prove that the series

$$\sum_{n=1}^{\infty} \frac{z^{2^n+1}}{n^2}$$

converges to a continuous function $f(z)$ on $\overline{D}$ which is holomorphic on D. Further prove (almost all the credit is for this part) that there does not exist any pair (Ω, g) in which Ω is a region with $\Omega \cap \partial D \neq \varnothing$ and g is a holomorphic function on Ω such that $g|_{\Omega\cap D} = f|_{\Omega \cap D}$.

Date: 7 June 2010.