THE MODERN THEORY OF CUNTZ SEMIGROUPS OF C*-ALGEBRAS

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Abstract. We give a detailed introduction to the theory of Cuntz semigroups for C*-algebras. Beginning with the most basic definitions and technical lemmas, we present several results of historical importance, such as Cuntz's theorem on the existence of quasitraces, Rørdam's proof that $\mathcal{Z}$-stability implies strict comparison, and Toms' example of a non $\mathcal{Z}$-stable simple, nuclear C*-algebra. We also give the reader an extensive overview of the state of the art and the modern approach to the theory, including the recent results for C*-algebras of stable rank one (for example, the Blackadar-Handelman conjecture and the realization of ranks), as well as the abstract study of the Cuntz category **Cu**.

Dedicated to the memory of Eberhard Kirchberg.

Contents

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1. Introduction

The Cuntz semigroup is an invariant for C*-algebras whose origin can be traced back to the seminal work of Joachim Cuntz [38] on the existence of quasitraces on simple, stably finite C*-algebras. The Cuntz semigroup $\mathrm{Cu}(A)$ of a C*-algebra A resembles the Murray-von Neumann semigroup $\mathrm{V}(A)$, but is constructed using positive elements instead of projections (and a suitable equivalence relation). The comparison between the Cuntz semigroup and the K-groups (particularly K_0) puts in perspective the advantages and disadvantages of these invariants. Arguably one of the biggest advantages of K-theory is its computability, as there are for instance several 6-term exact sequences that are very useful in a number of situations. On the other hand, many C*-algebras do not contain any nontrivial projections, and thus $\mathrm{V}(A)$ and $\mathrm{K}_0(A)$ may in general contain very little information about A outside the class of real rank zero C*-algebras, which is a class where projections abound. For example, the complex numbers $\mathbb{C}$, continuous functions on the Hilbert cube $[0,1]^{\mathbb{N}}$, the Jiang-Su algebra $\mathcal{Z}$, and the suspension of the Calkin algebra $S\mathcal{Q}$ all have the same K-theory. Moreover, the K-groups of a C*-algebra do not contain any information about its ideal structure, which helps explain why virtually all classification results that only use the K-theory of the algebra must assume simplicity. The Cuntz semigroup, on the other hand, always contains plenty of information about the C*-algebra, essentially because every C*-algebra contains a great deal of positive elements. (Exactly what kind of information about A is encoded in $\mathrm{Cu}(A)$ is not completely clear, but we give many concrete instances in the theorems below.) The Cuntz semigroup is unfortunately rather difficult to compute, which makes its rich information sometimes difficult to access. This is perhaps not entirely surprising, given how intrincate its structure is. Interestingly, in certain instances enough information suffices to distinguish algebras without the need of a full computation. For example, the Cuntz semigroup distinguishes the algebras $\mathbb{C}$, $C([0,1]^{\mathbb{N}})$, $\mathcal{Z}$ and $S\mathcal{Q}$ mentioned above. (We do not have an explicit computation of either $\mathrm{Cu}(C([0,1]^{\mathbb{N}}))$ or $\mathrm{Cu}(S\mathcal{Q})$, but enough about them is known to claim that they are different.)

The Cuntz semigroup is intimately related with classification, particularly with the classification program of simple, nuclear C*-algebras initiated by George Elliott. The original conjecture aimed at classifying all simple, separable, unital, nuclear C*-algebras using K-theoretical data, conveniently encoded in the Elliott invariant Ell that, loosely speaking, consists of the K_0-group, the topological K_1-group, the trace simplex, and the pairing between projections and traces. More precisely then, it was asked whether an isomorphism of invariants $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ could be lifted to an isomorphism of the algebras $A \cong B$. The relevance of the Cuntz semigroup in the modern theory of C*-algebras was made evident in the celebrated work of Toms [91], where he constructed two simple, separable, nuclear, unital C*-algebras A and B which satisfy $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ and $A \ncong B$. These algebras constitute a counterexample to Elliott's conjecture, and were distinguished using the Cuntz semigroup; see Section 9 for an exposition of Toms' examples and more details of the classification program.[1] The work of Toms motivated the systematic study of the Cuntz semigroup, which was initiated by Coward, Elliott and Ivanescu in [37]. Since

[1]Toms' counterexample to the Elliott conjecture was not the first one. Indeed, Rørdam had earlier constructed [80] a simple, nuclear C*-algebra A containing a finite and an infinite projection. It follows that A and $A \otimes \mathcal{Z}$ have the same Elliott invariant, but are not isomorphic. Prior to Rørdam's construction, Villadsen [96] had constructed examples of simple, separable, nuclear C^*-algebras which agreed on the Elliott invariant but were not isomorphic. The relevance of Toms' example in the context of these notes stems from the fact that he used the Cuntz semigroup to distinguish C*-algebras with isomorphic Elliott invariants.

then, numerous papers have been written about the Cuntz semigroup developing a categorical framework for their study, thus unveiling fascinating features of this invariant.

The Cuntz semigroup has been successfully used to classify certain classes of C*-algebras, as well as maps between them. Ciuperca and Elliott's classification [34] of AI-algebras was ultimately greatly generalized by Robert [73], who classified certain direct limits of one-dimensional NCCW-complexes. These classification results are obtained as consequences (via an intertwining argument) of general theorems classifying homomorphisms from said algebras into arbitrary C*-algebras of stable rank one. A basic form of these classification results for maps is the work [76] of Robert and Santiago, where they show that two homomorphisms from $C_0((0,1])$ into a C*-algebra of stable rank one are approximately unitarily equivalent if and only if they are equal at the level of the Cuntz semigroup. It should be noted that none of these results requires either the domain or codomain algebra to be simple; this is perhaps not so surprising considering the fact that the Cuntz semigroup encodes the ideal lattice of the algebra (see below) as well as other structural aspects (see, for example, [66] and also [77]).

The goal of this survey is to introduce the reader to this rich theory, beginning with a detailed exposition of the basics, and proving some of the most celebrated results. We will also give an overview of the modern theory of Cuntz semigroups, particularly the spectacular recent developments for C*-algebras of stable rank one.

In the rest of this introduction, we give a summary of the main results discussed in this work. As it turns out, the Cuntz semigroup of a C*-algebra is quite a special kind of ordered semigroup: for example, suprema of increasing sequences always exist, and addition is compatible with suprema and with the so-called compact containment relation $\ll$; see Section 4. Thus, the Cuntz semigroup $\mathrm{Cu}(A)$ naturally belongs to a subcategory of positively ordered monoids:

Theorem A. (Coward-Elliott-Ivanescu [37]). There is a category **Cu** of positively ordered monoids to which $\mathrm{Cu}(A)$ belongs for every C*-algebra A. Moreover, the Cuntz semigroup determines a functor $\mathrm{Cu}\colon \mathbf{C}^* \to \mathbf{Cu}$ which respects (countable) direct limits.

In fact, considering the Cuntz semigroup as an ordered set, it becomes an ω-*domain*, that is, a sequentially complete partially ordered set which is also ω-*continuous*; see [60], and also [49, 84]. While the work [37] only considered *countable* direct limits, it was later shown in [8] that this assumption is not necessary: **Cu** posseses arbitrary direct limits, and Cu preserves them. More properties of the category **Cu** and the functor Cu will be addressed in Theorem H.

The objects in the category **Cu** are partially ordered semigroups satisfying certain axioms (see Definition 4.5) which are enjoyed by $\mathrm{Cu}(A)$ for all C*-algebras A. The goal of describing precisely which partially ordered semigroups arise from C*-algebras has led to the discovery of five additional axioms, but even these do not entirely describe the range of the invariant. Obtaining a complete exlicit description is an extremely complicated task, and we are very far from achieving it. This should be compared to the situation with K-theory, where it is not so hard to show that every pair of abelian groups arises as the K-groups of a C*-algebra.

Theorem A is important since it grants us access to categorical methods in the study of (abstract) Cuntz semigroups. This perspective has been extremely fruitful, and some of the most recent applications are discussed in Section 10; see also Theorem H below.

As mentioned before, the ideal structure of A can be read off of $\mathrm{Cu}(A)$. (In this work, by an *ideal* in a C*-algebra we will always mean a closed, two-sided ideal.)

Theorem B. (Ciuperca-Robert-Santiago [36]). The Cuntz semigroup of a C*-algebra encodes its ideal-lattice structure, as well as the Cuntz semigroups of every ideal and every quotient. More explicitly, the assignment $I \mapsto \mathrm{Cu}(I)$ defines a lattice isomorphism between the ideals of A and the ideals of $\mathrm{Cu}(A)$, and there is a canonical isomorphism $\mathrm{Cu}(A)/\mathrm{Cu}(I) \cong \mathrm{Cu}(A/I)$.

The above theorem should also be compared to analogous statements in K-theory: in general, the K-groups of A do not contain any information about the ideal structure of A.

Another major part of the structure of a C*-algebra which is encoded in its Cuntz semigroup is its (quasi)tracial state space.

Theorem C. (Blackadar-Handelman [17]). Let A be a unital C^*-algebra. Then there is a natural affine bijection between the set of all quasitracial states on A and the set of all normalized functionals on $\mathrm{Cu}(A)$. Given a quasitrace $\tau \in \mathrm{QT}(A)$, the corresponding functional is

$$d_\tau([a]) = \lim_{n\to\infty} \tau\big(a^{\frac{1}{n}}\big)$$

for all $a \in A_+$ (and extended naturally to positive elements in $A \otimes \mathcal{K}$).

By the work of Elliott, Robert and Santiago, the natural bijection described in the theorem above extends to a bijection between the set of all lower-semicontinuous quasitraces on A and the set of all functionals on $\mathrm{Cu}(A)$; see [40].

Recall (see also Definition 2.9) that a unital C*-algebra A is said to be *stably finite* if for all $n \in \mathbb{N}$ and all $s \in M_n(A)$ with $s^*s = 1$, we have $ss^* = 1$. In other words, all isometries in matrix algebras over A are automatically unitaries. Very loosely speaking, one may think of stably finite, simple, unital C*-algebras as the C*-analogues of the finite von Neumann factors.

The fact that a finite von Neumann factor admits a faithful trace is a fundamental result in their study. In the C*-algebra setting, Cuntz used a precursor of Theorem C to show the following version of that result:

Theorem D. (Cuntz [38]). Let A be a simple, unital C*-algebra. Then A is stably finite if and only if it admits a faithful quasitrace.

For the most part of the last two decades, classification has revolved around the Jiang-Su algebra $\mathcal{Z}$ and C*-algebras that absorb it tensorially; such algebras are called *$\mathcal{Z}$-stable*. The main reason for this is the fact (see Remark 9.2) that the Elliott invariant cannot distinguish between A and $A \otimes \mathcal{Z}$. Thus, only $\mathcal{Z}$-stable C*-algebras can be expected to be classified using Ell. The computation of the Cuntz semigroup of $\mathcal{Z}$ was used by Rørdam to show that Cuntz semigroups of $\mathcal{Z}$-stable C*-algebras are well-behaved:

Theorem E. (Rørdam [81]). Let A be a separable, unital $\mathcal{Z}$-stable C*-algebra. Then $\mathrm{Cu}(A)$ is almost unperforated; equivalently, A has strict comparison.

Perhaps surprisingly, it is conjectured that the converse of Theorem E is true in the simple, nuclear setting: this is the only implication that remains open in the Toms-Winter conjecture. The conditions of $\mathcal{Z}$-stability and strict comparison should be regarded as C*-algebraic counterparts of McDuffness and the fact that in a II_1-factor, the order on projections is determined by the unique trace. While those conditions are always satisfied for hyperfinite factors, $\mathcal{Z}$-stability and strict comparison are not automatic for simple, nuclear C*-algebras, as Toms showed:

Theorem F. (Toms [91]). There exists a simple, separable, nuclear, unital C*-algebra A which satisfies $\mathrm{Ell}(A) \cong \mathrm{Ell}(A \otimes \mathcal{Z})$ and $\mathrm{Cu}(A) \not\cong \mathrm{Cu}(A \otimes \mathcal{Z})$, so in particular $A \not\cong A \otimes \mathcal{Z}$.

The way that Toms distinguished $\mathrm{Cu}(A)$ from $\mathrm{Cu}(A\otimes\mathcal{Z})$ was using the property of almost unperforation via Theorem E. The explicit computation of $\mathrm{Cu}(A)$, for the C*-algebra A constructed by Toms, is still unknown; see Problem 16.1. On the other hand, the difficult task of computing Cuntz semigroups becomes much simpler for $\mathcal{Z}$-stable C*-algebras:

Theorem G. (Brown-Toms [27], Brown-Perera-Toms [26]). Let A be a simple, separable, unital, stably finite $\mathcal{Z}$-stable C*-algebra. Then

$$\mathrm{Cu}(A) \cong \mathrm{V}(A) \sqcup \mathrm{LAff}(\mathrm{QT}(A))_{++}.$$

In particular, the pair $(\mathrm{Cu}(A), \mathrm{K}_1(A))$ is equivalent to $\mathrm{Ell}(A)$.

In the theorem above, $\mathrm{LAff}(\mathrm{QT}(A))_{++}$ denotes the set of all lower semicontinuous affine functions $\mathrm{QT}(A) \to (0,\infty]$. For the class of algebras in Theorem G, the group $\mathrm{K}_1(A)$ can also be recovered from Cuntz semigroup data, namely from $\mathrm{Cu}(A\otimes C(\mathbb{T}))$; see Theorem 9.10.

A recurrent theme in these notes is the fact that C*-algebras of stable rank one have particularly well behaved Cuntz semigroups. This is made already apparent in Subsection 2.2, where we show that Cuntz comparison in the stable rank one case takes a form which is very close to Murray-von Neumann subequivalence for projections; see Proposition 2.16. We also report on some of the most recent results on Cuntz semigroups of C*-algebras of stable rank one, including the following:

Theorem H. (Antoine-Perera-Robert-Thiel [5]). Let A be a separable, unital C*-algebra of stable rank one without finite-dimensional quotients. Then $\mathrm{Cu}(A)$ has the Riesz interpolation property and it is an inf-semillatice ordered semigroup. Moreover:

(i) (Realization of ranks). For every $f \in \mathrm{LAff}(\mathrm{QT}(A))_{++}$, there exists $a \in (A\otimes\mathcal{K})_+$ such that $d_\tau([a]) = f(\tau)$ for all $\tau \in \mathrm{QT}(A)$.
(ii) (Blackadar-Handelman conjecture). The space $\mathrm{DF}(A)$ of dimension functions on A is a Choquet simplex.

The developments that led to Theorem H ran parallel to, and also benefited from, the advances made in the systematic study of the category **Cu**. Indeed, Theorem A opened the doors for an abstract study of the category **Cu** and of the functor Cu. In this setting, it is particularly important to establish the existence of some standard categorical constructions in **Cu** such as direct limits, tensor products or products. This naturally leads one to consider two larger auxiliary categories **W** and **Q**, which contain **Cu** as a full subcategory. These larger categories have the advantage that the constructions that we are interested in exist in them (for example, it is not so difficult to prove that **W** has arbitrary direct limits and tensor products). One can show (see Theorem 12.9 and Theorem 14.5) that there exist functors $\gamma\colon \mathbf{W}\to\mathbf{Cu}$ and $\tau\colon \mathbf{Q}\to\mathbf{Cu}$ which are, respectively, a reflector and coreflector to the canonical inclusions. Using these, it is possible to transfer constructions from **W** and **Q** back to **Cu**. This is an essential ingredient in the following:

Theorem I. (Antoine-Perera-Thiel [8]). The category **Cu** is closed, symmetric, monoidal, and bicomplete. The Cuntz semigroup functor preserves directed limits and coproducts, and, suitably interpreted, also products and ultraproducts.

We almost exclusively work with the picture of the Cuntz semigroup of a C*-algebra A which uses positive elements in $A\otimes\mathcal{K}$. We should, however, point out that there are at least two alternative presentations of $\mathrm{Cu}(A)$ which may be more convenient in some contexts: the one using Hilbert modules, which was first considered in [37] and is only very briefly presented here in the comments before Proposition 11.12, and the one using open projections, which was given in [67], and is not covered here. We also mention that most of the results presented hold both for separable and nonseparable C*-algebras, but some of them do require separability, an assumption that will be made as needed.

We have made these notes as self-contained as possible, assuming only a very elementary background in C*-algebras and functional calculus. As such, we hope that this survey will be useful to young mathematicians who wish to learn the theory of Cuntz semigroups, as well as to those researchers who want to see a streamlined presentation of the latest results in the structure theory of Cuntz semigroups. Even though we made an effort to cover a large variety of topics on Cuntz semigroups, space constraints and the magnitude of the existent literature make it impossible to be exhaustive, and it was inevitable to make some omissions. Among others, we do not go into details to describe the many statement in the literature which, despite not referring to the Cuntz semigroup, depend on it for its proof. One example is Theorem D, but there are many others including the proof in [33] that $\mathcal{Z}$-stability implies finiteness of the nuclear dimension. On the other hand, there are two other introductions to the subject [12, 84], with somewhat different approaches. Indeed, [12], which was written over a decade ago, focuses primarily on the classical (or uncompleted) Cuntz semigroup W, and makes extensive use of the Hilbert module picture. On the other hand, [84], which is more recent, regards Cu-semigroups as domains with additional structure, and uses methods from lattice and category theory. Both references are good complements to this survey.

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2. Comparison of positive elements

The following is the definition on which the theory of Cuntz semigroups is based.

Definition 2.1. Let A be a C*-algebra and let $a, b \in A_+$. We say that a is *Cuntz subequivalent* to b (in A), denoted $a \precsim b$ (or $a \precsim_A b$ if there is a need to specify the ambient C*-algebra), if there exists a sequence $(r_n)_{n\in\mathbb{N}}$ in A such that $\lim\limits_{n\to\infty} \|r_n b r_n^* - a\| = 0$. Equivalently, for every $\varepsilon > 0$ there exists $r \in A$ such that $\|rbr^* - a\| < \varepsilon$.

We say that a and b are *Cuntz equivalent*, written $a \sim b$, if $a \precsim b$ and $b \precsim a$.

It is easy to see that if $a \precsim b$ and $b \precsim c$, then $a \precsim c$.

In order to give a feeling for the relation $\precsim$, we first look at the case of commutative C*-algebras. For $f \in C(X)$, we denote its *open support* by

$$\mathrm{supp_o}(f) = \{x \in X : f(x) \neq 0\}.$$

Proposition 2.2. Let X be a locally compact, Hausdorff space, and let $f, g \in C_0(X)$ be positive functions. Then $f \precsim g$ if and only if

$$\mathrm{supp_o}(f) \subseteq \mathrm{supp_o}(g).$$

Equivalently, $g(x) = 0$ implies $f(x) = 0$, for all $x \in X$.

Proof. Assuming that $f \precsim g$, choose a sequence $(r_n)_{n\in\mathbb{N}}$ in $C_0(X)$ such that $\lim\limits_{n\to\infty} r_n g r_n^* = f$ uniformly on X. For $x \in X$, it is then easy to see that $g(x) = 0$ implies $f(x) = 0$.

Conversely, suppose that $\mathrm{supp}_{\mathrm{o}}(f) \subseteq \mathrm{supp}_{\mathrm{o}}(g)$ and let $\varepsilon > 0$. Set $K = \{x \in X \colon f(x) \geq \varepsilon\}$, which is a compact set on which g is strictly positive. Find $\delta > 0$ such that $g(x) \geq \delta$ for all $x \in K$, and set

$$U = \left\{x \in X \colon g(x) > \tfrac{\delta}{2}\right\}.$$

Then U is open in X and $K \subseteq U$. By Urysohn's lemma, there is a positive function $h \in C_0(X)$ which is identically equal to 1 on K and vanishes precisely outside of U. Define $s \in C_0(X)$ by

$$s(x) = \begin{cases} \frac{f(x)}{g(x)} h(x), & \text{if } x \in U; \\ 0, & \text{else.} \end{cases}$$

Setting $r = s^{\frac{1}{2}}$, we get $r = r^*$ and $\|f - rgr\| < \varepsilon$, as desired. □

For an element a in a C*-algebra A, we denote by $\mathrm{sp}(a)$ its spectrum. We now extract some very useful consequences of Proposition 2.2:

Corollary 2.3. Let A be a C*-algebra, let $a \in A_+$ and let $f\colon [0,\infty) \to [0,\infty)$ be continuous and satisfy $f(0) = 0$. Then:

(i) We have $f(a) \precsim a$.
(ii) If $f(t) > 0$ for $t > 0$, then $f(a) \sim a$.
(iii) We have $a \sim a^\lambda$ and $a \sim \lambda a$ for $\lambda \in (0,\infty)$.
(iv) For $x \in A$, we have $x^*x \sim xx^*$.

Proof. (i) Note that a and $f(a)$ belong to $C^*(a) \cong C_0(\mathrm{sp}(a))$, and under this identification these elements correspond to $\mathrm{id}_{\mathrm{sp}(a)}$ and $f|_{\mathrm{sp}(a)}$, respectively. The conclusion then follows from Proposition 2.2 since $x = 0$ implies $f(x) = 0$.

(ii) Follows from (i) by taking f^{-1}.

(iii) Follows from (ii) by taking either $f(t) = t^\lambda$ or $f(t) = \lambda t$, respectively.

(iv) We have

$$xx^* \overset{\text{(iii)}}{\sim} (xx^*)^2 = x(x^*x)x^* \precsim x^*x.$$

Analogously, we get $x^*x \precsim xx^*$, as desired. □

Recall that a subalgebra B of a C*-algebra A is said to be *hereditary* if whenever $b \in B_+$ and $a \in A_+$ satisfy $a \leq b$, then $a \in B$. Given $b \in A_+$, the smallest hereditary subalgebra of A containing b is $A_b := \overline{bAb}$. We will use, without proof, the fact that $(b^{\frac{1}{n}})_{n\in\mathbb{N}}$ is an approximate identity for A_b. For arbitrary elements a, b in a C*-algebra A, we shall as customary write $a \approx_\varepsilon b$ to mean that $\|a - b\| < \varepsilon$.

Proposition 2.4. Let A be a C*-algebra and let $a, b \in A_+$. If $a \in A_b$, then $a \precsim b$. In particular, if $a \leq b$ then $a \precsim b$.

Proof. Suppose that $a \in A_b$, and let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $a \approx_{\frac{\varepsilon}{2}} b^{\frac{1}{n}} a b^{\frac{1}{n}}$. Using part (iv) of Corollary 2.3 at the second step, and part (iii) of at the fourth step, we get

$$b^{\frac{1}{n}} a b^{\frac{1}{n}} = (b^{\frac{1}{n}} a^{\frac{1}{2}})(a^{\frac{1}{2}} b^{\frac{1}{n}}) \sim a^{\frac{1}{2}} b^{\frac{2}{n}} a^{\frac{1}{2}} \precsim b^{\frac{2}{n}} \sim b.$$

Choose $r \in A$ with $rbr^* \approx_{\frac{\varepsilon}{2}} b^{\frac{1}{n}} a b^{\frac{1}{n}}$. Then

$$a \approx_{\frac{\varepsilon}{2}} b^{\frac{1}{n}} a b^{\frac{1}{n}} \approx_{\frac{\varepsilon}{2}} rbr^*,$$

and hence $a \precsim b$, as desired. □

Given $\varepsilon > 0$, let $f_\varepsilon \colon [0,\infty) \to [0,\infty)$ be given by $f_\varepsilon(t) = \max\{t-\varepsilon, 0\}$. Given $a \in A_+$, we write $(a-\varepsilon)_+$ for $f_\varepsilon(a)$. The element $(a-\varepsilon)_+$ is usually referred to as the ε *cut-down* of a. Since $(a-\varepsilon)_+ \leq a$ for any $\varepsilon > 0$, we get from Proposition 2.4 that $(a-\varepsilon)_+ \precsim a$. The following lemma is a generalization of this fact.

Lemma 2.5. Let A be a C*-algebra, let $\varepsilon > 0$, and let $a, b \in A_+$ with $\|a-b\| < \varepsilon$. Then there exists a contraction $r \in A$ with $rbr^* = (a-\varepsilon)_+$. In particular, $(a-\varepsilon)_+ \precsim b$.

Proof. We only prove the last assertion; the first one is rather involved, and a proof can be found in [62, Lemma 2.2].

Note that $\|a-b\| < \varepsilon$ implies that $a - \varepsilon \leq b$. Multiplying on both sides by $(a-\varepsilon)_+$, we get

$$(a-\varepsilon)_+(a-\varepsilon)(a-\varepsilon)_+ \leq (a-\varepsilon)_+ b (a-\varepsilon)_+. \tag{2.1}$$

Using part (iii) of Corollary 2.3 at the first step, we get

$$(a-\varepsilon)_+ \sim (a-\varepsilon)_+^3 = (a-\varepsilon)_+(a-\varepsilon)(a-\varepsilon)_+ \overset{(2.1)}{\leq} (a-\varepsilon)_+ b (a-\varepsilon)_+ \precsim b,$$

as desired. □

The reader is encouraged to prove the first assertion in Lemma 2.5 in the case where a and b commute.

We will sometimes need the following strengthening of part (iv) of Corollary 2.3. We omit its proof, which can be found in [84, Corollary 2.53].

Lemma 2.6. Let A be a C*-algebra, let $x \in A$ and let $\varepsilon > 0$. Then we have

$$(x^*x - \varepsilon)_+ \sim (xx^* - \varepsilon)_+$$

inside of $A \otimes \mathcal{K}$.

The following is one of the most used technical results about Cuntz comparison; see [79, Proposition 2.4].

Theorem 2.7. (Rørdam's lemma.) Let A be a C*-algebra and let $a, b \in A_+$. Then the following are equivalent:

(i) $a \precsim b$;
(ii) For every $\varepsilon > 0$ we have $(a-\varepsilon)_+ \precsim b$;
(iii) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $(a-\varepsilon)_+ \precsim (b-\delta)_+$;
(iv) For every $\varepsilon > 0$ there exist $\delta > 0$ and $x \in A$ such that $(a-\varepsilon)_+ = x^*x$ and $xx^* \in A_{(b-\delta)_+}$;
(v) For every $\varepsilon > 0$ there exist $\delta > 0$ and $r \in A$ such that $(a-\varepsilon)_+ = r(b-\delta)_+ r^*$.

Proof. (i) implies (ii) since $(a-\varepsilon)_+ \precsim a$, as observed above. Conversely, let $\varepsilon > 0$. Since $(a-\frac{\varepsilon}{2})_+ \precsim b$, there is $r \in A$ with

$$rbr^* \approx_{\frac{\varepsilon}{2}} \left(a - \tfrac{\varepsilon}{2}\right)_+ \approx_{\frac{\varepsilon}{2}} a,$$

so (ii) implies (i). That (iii) implies (ii) follows from $(b-\delta)_+ \leq b$.

We now show that (iv) implies (iii). Let $\varepsilon > 0$ and find $\delta > 0$ and $x \in A$ with $(a-\varepsilon)_+ = x^*x$ and $xx^* \in A_{(b-\delta)_+}$. Using part (iv) of Corollary 2.3 at the second step and using Proposition 2.4 at the third step, we get

$$(a-\varepsilon)_+ = x^*x \sim xx^* \precsim (b-\delta)_+,$$

as desired. To show that (v) implies (iv), let $\varepsilon > 0$ and find $\delta > 0$ and $r \in A$ with $(a-\varepsilon)_+ = r(b-\delta)_+ r^*$. Set $x = (b-\delta)_+^{\frac{1}{2}} r^*$. Then

$$(a-\varepsilon)_+ = x^*x \quad \text{and} \quad xx^* = (b-\delta)^{\frac{1}{2}} r^* r (b-\delta)^{\frac{1}{2}} \in A_{(b-\delta)_+},$$

as desired. Finally, we show that (i) implies (v). Let $\varepsilon > 0$ and choose $c \in A$ with $cbc^* \approx_{\frac{\varepsilon}{2}} a$. By continuity of functional calculus, there is $\delta > 0$ such that $cbc^* \approx_{\frac{\varepsilon}{2}} c(b-\delta)_+c^*$. By Lemma 2.5, there is $d \in A$ such that $(a-\varepsilon)_+ = dc(b-\delta)_+c^*d^*$. Then $r = dc$ satisfies $(a-\varepsilon)_+ = r(b-\delta)_+r^*$. □

We will later want to compute the Cuntz semigroup of $\mathbb{C}$, for which we will need the following useful lemma. Recall that projections p, q in a C*-algebra A are said to be *Murray-von Neumann equivalent*, in symbols $p \sim_{\mathrm{MvN}} q$, provided there is a partial isometry $v \in A$ such that $p = v^*v$ and $q = vv^*$. We also say that p is *Murray-von Neumann subequivalent* to q, written $p \precsim_{\mathrm{MvN}} q$, if there is a projection $p' \leq q$ such that $p \sim_{\mathrm{MvN}} p'$.

Lemma 2.8. Let A be a C*-algebra and let $p, q \in A$ be projections. Then $p \precsim q$ if and only if $p \precsim_{\mathrm{MvN}} q$.

Proof. It is clear that $p \precsim_{\mathrm{MvN}} q$ implies $p \precsim q$. Conversely, assume that $p \precsim q$. For $\varepsilon = \frac{1}{2}$, use part (iv) of Rørdam's lemma (Theorem 2.7) to find $\delta > 0$ and $x \in A$ such that $x^*x = (p - \frac{1}{2})_+$ and $xx^* \in A_{(q-\delta)_+}$. Since p and q are projections, we have $(p-\frac{1}{2})_+ = \frac{1}{2}p$ and similarly $(q-\delta)_+ = (1-\delta)q \sim q$. In particular, $A_{(q-\delta)_+} = A_q = qAq$. It follows that $v = \sqrt{2}x$ is a partial isometry satisfying $v^*v = p$ and $vv^* \in qAq$, so $vv^* \leq q$ as desired. □

Note, however, that the previous lemma does not imply that $p \sim q$ if and only if $p \sim_{\mathrm{MvN}} q$, since in general $p \sim_{\mathrm{MvN}} q$ is not equivalent to $p \precsim_{\mathrm{MvN}} q \precsim_{\mathrm{MvN}} p$. This *is* the case if A is *stably finite*, as we explain in the following subsection; see Proposition 2.10.

2.1. **Stably finite C*-algebras.** Stable finiteness is central to the study of the Cuntz semigroup, and we therefore isolate it here together with some consequences for Cuntz comparison.

For a C*-algebra A, its *minimal unitization* $\widetilde{A}$ is defined to be A when A is unital. For nonunital A, we have $\widetilde{A} = A \oplus \mathbb{C}$ as a vector space, whilst the product is given by $(a,\lambda)(b,\mu) = (ab + \mu a + \lambda b, \lambda\mu)$, so that $(0,1)$ becomes the unit and A sits inside $\widetilde{A}$ as a closed, two-sided ideal.

Definition 2.9. A unital C*-algebra A is said to be *finite* if for all $s \in A$ with $s^*s = 1$ we automatically have $ss^* = 1$. In other words, every isometry in A is a unitary. We say that A is *stably finite* if $M_n(A)$ is finite for all $n \in \mathbb{N}$.

A general (possibly nonunital) C*-algebra A is *stably finite* if $\widetilde{A}$ is stably finite; see [13, V.2.2.1].

In the unital case, one can show that finiteness of a C*-algebra is equivalent to the fact that no non-zero projection has a proper subprojection equivalent to it. Whilst for a nonunital algebra A, this is still a consequence of $\widetilde{A}$ being finite, it is no longer equivalent, as testified by taking $A = C^*(S - I)$, where S is the unilateral shift; see [13, V.2.2.14].

We warn the reader the stable finiteness for nonunital C*-algebras is less well-behaved than for unital ones, and that even for unital C*-algebras, stable finiteness is often too weak of a property outside of the simple case, since it does not imply stable finiteness of all of its quotients. (For example, the unitization of the suspension of $\mathcal{O}_2$ is stably finite according to the definition above.) We have chosen to avoid these complications in this survey, and will consider almost exclusively unital C*-algebras throughout, additionally assuming simplicity when necessary.

Examples of stably finite C*-algebras are easy to come by. For example, all commutative C*-algebras are stably finite. Since direct limits of stably finite algebras are easily seen to be stably finite, and subalgebras of stably finite algebras are as well stably finite, we deduce that all ASH-algebras are stably finite. On the other hand, purely infinite C*-algebras, for example the Cuntz algebra $\mathcal{O}_2$, are not finite, so in particular not stably finite. There exist (simple, unital, nuclear) C*-algebras which are finite but not stably finite, as was shown by Rørdam in [80].

Stable fininiteness is the key assumption in Cuntz's celebrated theorem on the existence of (quasi)traces (see Theorem 6.11), whose proof used a precursor of the Cuntz semigroup. There are other reasons why stable finiteness is fundamental in the study of the Cuntz semigroup, and one piece of evidence is given by the following result, which should be compared to Lemma 2.8.

Proposition 2.10. Let A be a unital, finite C*-algebra. For projections $p, q \in A$, we have $p \sim q$ if and only if $p \sim_{\mathrm{MvN}} q$.

Proof. Since the "if" implication is true in full generality, we only prove the "only if" direction. Assume that $p \sim q$, that is, $p \precsim q \precsim p$. By Lemma 2.8, we have $p \precsim_{\mathrm{MvN}} q \precsim_{\mathrm{MvN}} p$. Choose partial isometries $v, w \in A$ with

$$v^*v = p, \quad vv^* \leq q, \quad \text{and} \quad w^*w = q, \quad ww^* \leq p.$$

Then it follows that $w^*v^*vw = p$ and $vww^*v^* \leq p$. Note that the corner pAp is finite as well, and that vw is an isometry in pAp. By finiteness we must have $vww^*v^* = p$, which implies that $vv^* = q$ and $ww^* = q$, so $p \sim_{\mathrm{MvN}} q$. □

Recall that the *Murray-von Neumann semigroup* $\mathrm{V}(A)$ of a unital C*-algebra A is defined as the set of Murray-von Neumann equivalence classes of projections in $\bigcup_{n=1}^{\infty} M_n(A)$. (The Grothendieck enveloping group of $\mathrm{V}(A)$, for unital A, is $\mathrm{K}_0(A)$.) The following remark will be needed later.

Remark 2.11. Murray-von Neumann subequivalence does not in general induce an order on the semigroup $\mathrm{V}(A)$, since $p \precsim_{\mathrm{MvN}} q \precsim_{\mathrm{MvN}} p$ does not imply $p \precsim_{\mathrm{MvN}} q$. On the other hand, the proof of Proposition 2.10 shows that this is indeed the case if A is stably finite. In other words, $\mathrm{V}(A)$ is an ordered semigroup whenever A is unital and stably finite.

We close this section by showing that Proposition 2.10 fails in general for C*-algebras which are not finite.

Example 2.12. For example, consider the Cuntz algebra $\mathcal{O}_n$, for $n > 2$, with canonical generating isometries $s_1, \ldots, s_n$. It is known that $\mathrm{V}(\mathcal{O}_n) \cong \mathbb{Z}_{n-1}$ with the unit representing the canonical generator $1 \in \mathbb{Z}_{n-1}$. Set $p = s_1s_1^* + s_2s_2^*$, which is a projection in $\mathcal{O}_n$. Since $s_js_j^*$ is clearly Murray-von Neumann equivalent to 1 for all $j = 1, \ldots, n$, it follows that p represents $2 \in \mathbb{Z}_{n-1}$, and is therefore not Murray-von Neumann equivalent to 1 (since $n > 2$): $p \not\sim_{\mathrm{MvN}} 1$.

On the other hand, since $\mathcal{O}_n$ is purely infinite and simple and $p \neq 0$, there exists $x \in \mathcal{O}_n$ such that $xpx^* = 1$. This implies that $1 \precsim p$. Since $p \precsim 1$ is immediate, we conclude that $p \sim 1$.

2.2. **Cuntz comparison and stable rank one.** C*-algebras of stable rank one play an important role in these notes, since their Cuntz semigroups are particularly well-behaved. In this subsection, we show that Cuntz comparison in C*-algebras of stable rank one takes a form which is very similar to Murray-von Neumann subequivalence for projections; see Proposition 2.16.

We begin with a general discussion for the reader to develop some intuition around the notion of stable rank one.

Definition 2.13. A unital C*-algebra A is said to have *stable rank one* if the set $\mathrm{GL}(A)$ of invertible elements in A is dense in A. Moreover, a nonunital C*-algebra is said to have stable rank one if its minimal unitization does.

The class of C*-algebras with stable rank one is pleasantly large. It was shown by Rørdam in [81] that a unital, simple, stably finite and $\mathcal{Z}$-stable C*-algebra has stable rank one, hence this applies to all classifiable stably finite C*-algebras. Stable rank one also holds outside of the classifiable class, for example for crossed products of the form $C(X) \rtimes \mathbb{Z}^n$ for a free, minimal action of $\mathbb{Z}^n$ on a compact Hausdorff space X, regardless of whether $C(X) \rtimes \mathbb{Z}^n$ is $\mathcal{Z}$-stable or not; see [54]. (See also Section 7 below for more information on the C*-algebra $\mathcal{Z}$.)

As the name indicates, the stable rank is an integer valued number that is associated to any C*-algebra and can be with advantage thought of as a noncommutative dimension theory. In fact, one has that $\mathrm{sr}(C(X)) = \left[\frac{\dim(X)}{2}\right] + 1$; see [72]. The notion of stable rank in [72] agrees with that of Bass, as shown in [56].

We record here, without proof, some general facts about C*-algebras of stable rank one that will be needed throughout.

Proposition 2.14. Let A be a C*-algebra of stable rank one.

(i) A is automatically stably finite.
(ii) If $B \subseteq A$ is a hereditary subalgebra, then B has stable rank one.
(iii) $A \otimes \mathcal{K}$ has stable rank one.
(iv) If $p, q \in A$ satisfy $p \sim_{\mathrm{MvN}} q$, then there is a unitary $u \in \widetilde{A}$ satisfying $p = uqu^*$.

We need a technical lemma; see [35, Lemma 2.4] for the proof.

Lemma 2.15. Let A be a C*-algebra and let $B \subseteq A$ be a hereditary subalgebra of stable rank one. Let $\delta > 0$ and let $x, y \in A$ satisfy

$$xx^*, yy^* \in B, \quad x^*x \in A_{y^*y} \quad \text{and} \quad \|x^*x - y^*y\| < \delta.$$

Then there is a unitary $u \in \widetilde{B}$ such that $\|x - uy\| < \sqrt{\delta}$.

The following is one of the basic technical results that makes the study of Cuntz comparison in C*-algebras of stable rank one particularly accessible. (Another one will be given in Theorem 3.7.)

Proposition 2.16. ([35, Proposition 2.5]). Let A be a C*-algebra of stable rank one, and let $a, b \in A_+$. Then $a \precsim b$ if and only if there exists $x \in A$ satisfying $x^*x = a$ and $xx^* \in A_b$.

Proof. If there exists $x \in A$ as in the statement, then part (iv) of Corollary 2.3 gives

$$a = x^*x \sim xx^* \in A_b,$$

and thus $a \precsim b$ by Proposition 2.4.

Conversely, assume that $a \precsim b$. By Rørdam's lemma (Theorem 2.7), for every $n \in \mathbb{N}$ there exists $y_n \in A$ such that

$$\left(a - \tfrac{1}{2^{2n}}\right)_+ = y_n^* y_n \quad \text{and} \quad y_n y_n^* \in A_b.$$

Given $n \in \mathbb{N}$, apply Lemma 2.15 to y_n and y_{n+1} to obtain a unitary $u_n \in \widetilde{A}$ satisfying $\|y_n - u_n y_{n+1}\| < \frac{1}{2^n}$. Set $x_n = u_1 \cdots u_{n-1} y_n \in A$. Then $\|x_n - x_{n+1}\| < \frac{1}{2^n}$, and thus the sequence $(x_n)_{n\in\mathbb{N}}$ has a limit $x \in A$. Moreover,

$$x^*x = \lim_{n\to\infty} x_n^* x_n = \lim_{n\to\infty} \left(a - \tfrac{1}{2^{2n}}\right)_+ = a,$$

and similarly $xx^* = \lim\limits_{n\to\infty} x_n x_n^* \in A_b$, as desired. $\square$

3. The Cuntz semigroup

The following is the object we will study in these notes.

Definition 3.1. Let A be a C*-algebra. The *Cuntz semigroup* of A is defined as

$$\mathrm{Cu}(A) = (A \otimes \mathcal{K})_+/\sim,$$

where $\sim$ stands for the Cuntz equivalence relation. For a positive element $a \in (A \otimes \mathcal{K})_+$, we denote by $[a]$ its Cuntz equivalence class, hence $\mathrm{Cu}(A) = \big\{[a] \colon a \in (A \otimes \mathcal{K})_+\big\}$. There is a natural partial order defined on $\mathrm{Cu}(A)$, namely $[a] \leq [b]$ if $a \precsim b$. (The element $0 := [0]$ is the minimal element in $\mathrm{Cu}(A)$.) We define an addition on $\mathrm{Cu}(A)$ by setting[2]

$$[a] + [b] = \big[\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\big].$$

We also denote $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ by $a \oplus b$.

Lemma 3.2. Let A be a C*-algebra, and let $a, b \in A_+$. Then $a + b \precsim a \oplus b$. If $ab = 0$, then $a + b \sim a \oplus b$.

Proof. Note that

$$\begin{pmatrix} (aa^*)^{\frac{1}{n}} & (bb^*)^{\frac{1}{n}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} (a^*a)^{\frac{1}{n}} & 0 \\ (b^*b)^{\frac{1}{n}} & 0 \end{pmatrix} \to \begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix},$$

hence $a + b \sim \left(\begin{smallmatrix} a+b & 0 \\ 0 & 0 \end{smallmatrix}\right) \precsim a \oplus b$. On the other hand, if $ab = 0$ and we let $x = \left(\begin{smallmatrix} a^{1/2} & b^{1/2} \end{smallmatrix}\right)$, then $xx^* = a + b$ and $x^*x = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$. ☐

It is not difficult to check that $[a] + [b] = [b] + [a]$ and that $[a] + 0 = [a]$ in $\mathrm{Cu}(A)$ for all $[a], [b] \in \mathrm{Cu}(A)$. More interestingly, addition and order are compatible in $\mathrm{Cu}(A)$, in the sense that $[a_1] \leq [b_1]$ and $[a_2] \leq [b_2]$ imply $[a_1] + [a_2] \leq [b_1] + [b_2]$. This gives $\mathrm{Cu}(A)$ the structure of a positively ordered monoid[3].

We turn to the first computation of a Cuntz semigroup.

Example 3.3. Let us compute $\mathrm{Cu}(\mathbb{C}) \cong \mathrm{Cu}(M_n) \cong \mathrm{Cu}(\mathcal{K})$. We will show that the rank map $\mathrm{rk} \colon \mathcal{K}_+ \to \{0, 1, \ldots, \infty\} =: \overline{\mathbb{N}}$ induces an ordered semigroup isomorphism

$$\mathrm{rk} \colon \mathrm{Cu}(\mathcal{K}) \to \overline{\mathbb{N}}.$$

To show this, given $a, b \in \mathcal{K}_+$ we will prove that $\mathrm{rk}(a) \leq \mathrm{rk}(b)$ if and only if $a \precsim b$.

Let $a \in \mathcal{K}_+$. By the Spectral Theorem, there are scalars $\lambda_n \geq 0$ and finite rank projections $p_n \in \mathcal{K}$ for $n \in \mathbb{N}$, such that

$$a = \sum_{n=0}^{\infty} \lambda_n p_n.$$

Moreover, $\mathrm{rk}(a) < \infty$ if and only if there is m such that $\lambda_n = 0$ for all $n \geq m$. In this case, part (iii) of Corollary 2.3 implies that $a \sim \sum_{n=1}^{m} p_n =: p_a$. Note that $\mathrm{rk}(p_a) = \mathrm{rk}(a)$, and recall that Murray-von Neumann subequivalence for projections in $\mathcal{K}$ is determined by the rank. Using Lemma 2.8 at the second step, for finite rank elements $a, b \in \mathcal{K}_+$ we have

$$a \precsim b \quad \Leftrightarrow \quad p_a \precsim p_b \quad \Leftrightarrow \quad p_a \precsim_{\mathrm{MvN}} p_b \quad \Leftrightarrow \quad \mathrm{rk}(p_a) \precsim \mathrm{rk}(p_b) \quad \Leftrightarrow \quad \mathrm{rk}(a) \precsim \mathrm{rk}(b),$$

as desired. Now, if $\mathrm{rk}(b) = \infty$ and $a \in \mathcal{K}_+$ is arbitrary, we will show that $a \precsim b$. By part (ii) of Rørdam's lemma (Theorem 2.7), it suffices to show that $(a - \varepsilon)_+ \precsim b$ for

[2]Here, just like it is done in K-theory, we are implicitly fixing an isomorphism $M_2 \otimes \mathcal{K} \cong \mathcal{K}$ and using it to identify $\mathrm{Cu}(A)$ with $\mathrm{Cu}(M_2(A))$; one can check that this identification does not depend on the isomorphism we fixed.

[3]A *monoid* is a semigroup with a neutral element, and it is said to be *positively ordered* if every element dominates the neutral element. The reasons for having termed $\mathrm{Cu}(A)$ a semigroup are only historical.

all $\varepsilon > 0$. Note that $(a-\varepsilon)_+$ has finite rank and is therefore Cuntz equivalent to a finite-rank projection $p \in \mathcal{K}$, by the previous part. Since $\mathcal{K}_b$ contains projections of arbitrary large rank, there is $q \in \mathcal{K}_b$ with $p \sim_{\mathrm{MvN}} q$. Then

$$(a-\varepsilon)_+ \sim p \precsim_{\mathrm{MvN}} q \precsim b,$$

which implies the result by Lemma 2.8.

We isolate the following convenient corollary. Recall that an *order-embedding*[4] $\phi\colon S \to T$ between ordered sets is a map satisfying $\phi(s) \leq \phi(s')$ if and only if $s \leq s'$.

Corollary 3.4. Let A be a unital C*-algebra. Then there is a natural semigroup map $\iota\colon \mathrm{V}(A) \to \mathrm{Cu}(A)$. If A is stably finite, then ι is an order embedding.

Proof. The fact that a semigroup map $\iota\colon \mathrm{V}(A) \to \mathrm{Cu}(A)$ exists is obvious since every projection is a positive element. Moreover, if A is stably finite, then $\mathrm{V}(A)$ is an ordered semigroup by Remark 2.11, and the last claim follows immediately from Lemma 2.8 and Proposition 2.10. □

It is also possible to describe which positive elements are Cuntz-equivalent to projections in a stably finite, unital C*-algebra:

Remark 3.5. If A is unital and stably finite, then the image of ι in $\mathrm{Cu}(A)$ is precisely the set of the so-called *compact elements* of $\mathrm{Cu}(A)$; see Definition 4.1 and Remark 4.13.

Notation 3.6. Given $a, b \in A_+$, we introduce the following notation:

- $a \sim_u b$ if there is a unitary $u \in \widetilde{A}$ with $uau^* = b$;
- $a \subseteq b$ if $a \in A_b$;
- $a \subseteq_u b$ if there is a unitary $u \in \widetilde{A}$ such that $uau^* \in A_b$.

The following theorem shows that Cuntz comparison in stable C*-algebras is *unitarily implemented*. The result holds more generally for C*-algebras of *weak stable rank one*: by definition, a C*-algebra A has weak stable rank one if $A \subseteq \overline{\mathrm{GL}(\widetilde{A})}$. By [18, Lemma 4.3.2] every stable C*-algebra has weak stable rank one.

Theorem 3.7. Let A be a C*-algebra with weak stable rank one and let $a, b \in A_+$. Then $a \precsim b$ if and only if for every $\varepsilon > 0$ we have $(a-\varepsilon)_+ \subseteq_u b$.

Proof. Note that the "only if" implication is true in full generality: indeed, given $\varepsilon > 0$, find $u \in \mathcal{U}(\widetilde{A})$ as in the statement. With $x = u(a-\varepsilon)_+^{\frac{1}{2}} \in A$, we use part (iv) of Corollary 2.3 at the second step, and Proposition 2.4 at the last one to get

$$(a-\varepsilon)_+ = x^*x \sim xx^* = u(a-\varepsilon)_+u^* \precsim b.$$

Then $a \precsim b$ by Rørdam's lemma (Theorem 2.7).

We only sketch the proof of the converse, so assume that A has weak stable rank one. For $c, d \in A_+$, we write $c \sim_u d$ if there is a unitary $u \in \mathcal{U}(\widetilde{A})$ such that $c = udu^*$. From this, one shows that for every $x \in A$ and every $\varepsilon > 0$ we have

$$(x^*x-\varepsilon)_+ \sim_u (xx^*-\varepsilon)_+. \tag{2.2}$$

(Note that this is a strengthening of Lemma 2.6.) Assume now that $a \precsim b$ and let $\varepsilon > 0$. By Rørdam's lemma (Theorem 2.7), there exists $x \in A$ such that

$$(a-\tfrac{\varepsilon}{2})_+ = x^*x \quad \text{and} \quad xx^* \in A_b.$$

Then

$$(a-\varepsilon)_+ = (x^*x-\tfrac{\varepsilon}{2})_+ \overset{(2.2)}{\sim_u} (xx^*-\tfrac{\varepsilon}{2})_+ \in A_b.$$

Therefore there is $u \in \mathcal{U}(\widetilde{A})$ with $u(a-\varepsilon)_+u^* \in A_b$, as desired. □

[4] This is stronger than being order-preserving and injective.

The following theorem, originally obtained by Coward, Elliott and Ivanescu in [37] using Hilbert C*-modules, was the first result about the internal structure of Cuntz semigroups, and shows that they are rather special ordered semigroups.

Theorem 3.8. Let A be a C*-algebra. Then every increasing sequence in $\mathrm{Cu}(A)$ has a supremum.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $(A\otimes\mathcal{K})_+$ satisfying $a_n \precsim a_{n+1}$ for all $n\in\mathbb{N}$.

Case 1: *$(a_n)_{n\in\mathbb{N}}$ is increasing in $A\otimes\mathcal{K}$ and has a limit $a = \lim\limits_{n\to\infty} a_n$. Then $[a]$ is the supremum of $([a_n])_{n\in\mathbb{N}}$.* In this case $[a]\in\mathrm{Cu}(A)$ is an upper bound for $([a_n])_{n\in\mathbb{N}}$. Let $[b]\in\mathrm{Cu}(A)$ is another upper bound; we want to show that $[a]\leq[b]$. By Rørdam's lemma, it suffices to show that for every $\varepsilon>0$ we have $(a-\varepsilon)_+\precsim b$. For the $\varepsilon>0$ given, find $n\in\mathbb{N}$ such that $\|a-a_n\|<\varepsilon$, so that we have $(a-\varepsilon)_+\precsim a_n$ by Lemma 2.5. Then $(a-\varepsilon)_+\precsim a_n\precsim b$, as desired.

Case 2: *for every $n\in\mathbb{N}$ we have $a_n\subseteq a_{n+1}$ (which means $a_n\in A_{a_{n+1}}$).* In this case, for $n\in\mathbb{N}$ we set

$$b_n=\sum_{k=1}^{n}\frac{a_k}{\|a_k\|2^k} \quad \text{and} \quad b=\sum_{k=1}^{\infty}\frac{a_k}{\|a_k\|2^k}.$$

It is clear that $b_n\leq b_{n+1}$ and that $b=\lim\limits_{n\to\infty} b_n$. Note that

$$a_n\sim\frac{a_n}{\|a_n\|2^n}\leq b_n,$$

and hence $a_n\precsim b_n$ by Proposition 2.4. On the other hand, since $a_1,\ldots,a_n$ belong to the hereditary subalgebra generated by a_n, the same is true for b_n. Thus $b_n\precsim a_n$ by Proposition 2.4 and therefore $a_n\sim b_n$. It follows from Case 1 that $[b]$ is the supremum of $([a_n])_{n\in\mathbb{N}}$.

Case 3: *for every $n\in\mathbb{N}$ we have $a_n\subseteq_u a_{n+1}$.* This case is easy to reduce to the previous one. For each $n\in\mathbb{N}$, choose $u_n\in\mathcal{U}(\widetilde{A\otimes\mathcal{K}})$ such that $u_na_nu_n^*\in A_{a_{n+1}}$. Set $b_1=a_1$ and

$$b_n=u_1^*u_2^*\cdots u_{n-1}^*a_nu_{n-1}\cdots u_2u_1.$$

Then $b_n\sim_u a_n$, so $[b_n]=[a_n]$. One readily checks that $b_n\in A_{b_{n+1}}$, and hence Case 2 implies the result in this case.

Case 4: *the sequence $(a_n)_{n\in\mathbb{N}}$ is arbitrary.* Using Rørdam's lemma repeatedly together with Theorem 3.7, for every $n\in\mathbb{N}$ we can find a sequence $(\varepsilon_k^{(n)})_{k\in\mathbb{N}}$ which decreases to zero and such that

$$(a_n-\varepsilon_k^{(n)})_+\subseteq_u(a_{n+1}-\varepsilon_k^{(n+1)})_+$$

for all $k, n \in \mathbb{N}$. We represent this graphically as follows:

$$
\begin{array}{ccccccc}
a_1 & \precsim & a_2 & \precsim & a_3 & \precsim & \cdots \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
\vdots & & \vdots & & \vdots & & \cdots \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
(a_1 - \varepsilon_3^{(1)})_+ & \subseteq_u & (a_2 - \varepsilon_3^{(2)})_+ & \subseteq_u & (a_3 - \varepsilon_3^{(3)})_+ & \subseteq_u & \cdots \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
(a_1 - \varepsilon_2^{(1)})_+ & \subseteq_u & (a_2 - \varepsilon_2^{(2)})_+ & \subseteq_u & (a_3 - \varepsilon_2^{(3)})_+ & \subseteq_u & \cdots \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
(a_1 - \varepsilon_1^{(1)})_+ & \subseteq_u & (a_2 - \varepsilon_1^{(2)})_+ & \subseteq_u & (a_3 - \varepsilon_1^{(3)})_+ & \subseteq_u & \cdots
\end{array}
$$

Set $b_n = (a_n - \varepsilon_n^{(n)})_+$. Then $b_n \subseteq_u b_{n+1}$ for all $n \in \mathbb{N}$, and by Case 3 the supremum of $([b_n])_{n\in\mathbb{N}}$ exists in $\mathrm{Cu}(A)$, say $[b]$. Then $[a_n] \leq [b]$, since

$$[a_n] \overset{\textbf{Case 1}}{=} \sup_{k\in\mathbb{N}} [(a_n - \varepsilon_k^{(n)})_+] \ \leq \ \sup_{k\in\mathbb{N}} [(a_k - \varepsilon_k^{(k)})_+] \ = \ [b].$$

(To justify the second step: one can without loss of generality assume that $k > n$. Then $(a_n - \varepsilon_k^{(n)})_+ \precsim (a_k - \varepsilon_k^{(n)})_+$ since $a_n \precsim a_k$, and $(a_k - \varepsilon_k^{(n)})_+ \precsim (a_k - \varepsilon_k^{(k)})_+$ because $\varepsilon_k^{(n)} \geq \varepsilon_k^{(k)}$.) Thus $[b]$ is an upper bound of the sequence. To see that it is the smallest, let $[c]$ be another upper bound. Then

$$(a_n - \varepsilon)_+ \precsim a_n \precsim c$$

for any $\varepsilon > 0$, and thus $b \precsim c$. We conclude that $[b]$ is the supremum of $([b_n])_{n\in\mathbb{N}}$. □

Remark 3.9. It follows from Case 1 above that for all $a \in (A \otimes \mathcal{K})_+$ we have

$$[a] = \sup_{\varepsilon>0} [(a - \varepsilon)_+].$$

Remark 3.10. The proof of Theorem 3.8 shows that if $(a_n)_{n\in\mathbb{N}}$ is any sequence in $(A \otimes \mathcal{K})_+$ which is increasing in $\mathrm{Cu}(A)$, then there exists an increasing sequence $(b_n)_{n\in\mathbb{N}}$ in $(A \otimes \mathcal{K})_+$ which converges to an element representing $\sup_{n\in\mathbb{N}} [a_n]$, and satisfies $b_n \precsim a_n$ for all $n \in \mathbb{N}$. (However, one cannot in general arrange that $a_n \sim b_n$.)

Proposition 3.11. Let A be a C*-algebra, let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence in $(A \otimes \mathcal{K})_+$, let $a \in (A \otimes \mathcal{K})_+$ and let $\varepsilon > 0$. If

$$[a] \leq \sup_{n\in\mathbb{N}} [a_n],$$

then there exists $n \in \mathbb{N}$ with $[(a - \varepsilon)_+] \leq [a_n]$.

Proof. Use Remark 3.10 to find an increasing sequence $(b_n)_{n\in\mathbb{N}}$ in $(A \otimes \mathcal{K})_+$ with limit b satisfying $[b] = \sup_{n\in\mathbb{N}} [a_n]$ and $b_n \precsim a_n$ for all $n \in \mathbb{N}$. Since $a \precsim b$, by Rørdam's lemma (Theorem 2.7) there exists $\delta > 0$ such that $(a - \varepsilon)_+ \precsim (b - \delta)_+$. Find $n \in \mathbb{N}$ such that $\|b - b_n\| < \delta$, so that $(b - \delta)_+ \precsim b_n$ by Lemma 2.5. Then

$$(a - \varepsilon)_+ \precsim (b - \delta)_+ \precsim b_n \precsim a_n,$$

as desired. □

The cut-down is necessary in Proposition 3.11: for example, let $a_n \in C([0,1])$ be a positive function supported on $\left[\frac{1}{n}, 1-\frac{1}{n}\right]$, such that $(a_n)_{n\in\mathbb{N}}$ converges to a function a vanishing only on 0 and 1. Then $[a] = \sup\limits_{n\in\mathbb{N}}[a_n]$, but there is no n such that $[a] \leq [a_n]$ by Proposition 2.2.

4. Compact containment and the category **Cu**

In this section, we will explore the order-theoretic aspects of $\mathrm{Cu}(A)$ in more detail. For this, some abstraction will be necessary, and we will often work with (partially) ordered semigroups $(S, \leq)$, or even just ordered sets; the example we will always have in mind is $(\mathrm{Cu}(A), \leq)$. A more general version of the following definition appears in [49, Definition I-1.1].

Definition 4.1. Let $(S, \leq)$ be an ordered set. We define an additional relation $\ll$ on S, called *(sequential) compact containment*, as follows: for $s, t \in S$, we set $s \ll t$ if whenever $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence in S with supremum x satisfying $t \leq x$, then there exists $n \in \mathbb{N}$ such that $s \leq x_n$.

We say that $s \in S$ is *compact* if $s \ll s$.

Remark 4.2. Proposition 3.11 shows precisely that $[(a-\varepsilon)_+] \ll [a]$ for all $\varepsilon > 0$ and all $a \in (A \otimes \mathcal{K})_+$.

It is easy to see that $s \ll t$ implies $s \leq t$, but the converse is not true, even in Cuntz semigroups of (commutative) C*-algebras; see Proposition 4.4. In fact, the relation $\ll$ is an example of what is called an *auxiliary relation*; see Definition 12.6. We now show that compact containment in C*-algebras can be characterized using cut-downs:

Proposition 4.3. Let A be a C*-algebra and let $a, b \in (A \otimes \mathcal{K})_+$. Then $[a] \ll [b]$ if and only if there exists $\varepsilon > 0$ such that $[a] \leq [(b-\varepsilon)_+]$. In particular, $[a]$ is compact if and only if there exists $\varepsilon > 0$ with $a \sim (a-\varepsilon)_+$.

Proof. Suppose that $[a] \ll [b]$. Since $\sup\limits_{n\in\mathbb{N}}[(b-\frac{1}{n})_+] = [b]$ by Remark 3.9, it follows that there exists $n \in \mathbb{N}$ with $[a] \leq [(b-\frac{1}{n})_+]$. The converse follows from Remark 4.2. $\square$

Using the above proposition together with Proposition 2.2, compact containment in commutative C*-algebras can be easily characterized in terms of open supports. We leave the proof as an exercise:

Proposition 4.4. Let X be a compact Hausdorff space and let $a, b \in C(X)_+$. Then $[a] \ll [b]$ if and only if

$$\overline{\mathrm{supp}_{\mathrm{o}}(a)} \subseteq \mathrm{supp}_{\mathrm{o}}(b).$$

In particular, for a as above, $[a]$ is compact if and only if $\mathrm{supp}_{\mathrm{o}}(a)$ is compact in the lattice of open sets of X.

Definition 4.5. Let $(S, \leq)$ be a positively ordered monoid. We say that S is an *(abstract) Cuntz semigroup*, or just a Cu-*semigroup*, if it satisfies the following so-called axioms:

(O1) Every increasing sequence has a supremum.
(O2) For every $s \in S$, there is a sequence $(s_n)_{n\in\mathbb{N}}$ in S with $s_n \ll s_{n+1}$ and $s = \sup\limits_{n\in\mathbb{N}} s_n$.
(O3) If $s \ll t$ and $s' \ll t'$, then $s + s' \ll t + t'$.

(O4) If $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are increasing sequences, then

$$\sup_{n\in\mathbb{N}}(s_n+t_n)=\sup_{n\in\mathbb{N}}s_n+\sup_{n\in\mathbb{N}}t_n.$$

Given Cu-semigroups S and T, a Cu-*morphism* between them is a map $f\colon S\to T$ preserving addition, neutral element, order $\leq$, suprema of increasing sequences, and also the compact containment relation $\ll$. Maps that preserve all the structure except possibly for $\ll$ are called *generalized* Cu-*morphisms.* They are also relevant, as we will see in Section 14.

We denote by **Cu** the category whose objects are Cu-semigroups and whose morphisms are Cu-morphisms. The set of Cu-morphisms between two semigroups S and T will be denoted by $\mathbf{Cu}(S,T)$, and the set of generalized Cu-morphisms will be denoted by $\mathbf{Cu}[S,T]$.

The following result, due to Coward, Elliott and Ivanescu [37], was arguably the beginning of the systematic study of Cuntz semigroups.

Theorem 4.6. Let A be a C*-algebra. Then Cu(A) is a Cu-semigroup. Moreover, if $\varphi\colon A\to B$ is a $*$-homomorphism between C*-algebras, then φ naturally induces a Cu-morphism $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A)\to\mathrm{Cu}(B)$. In other words, Cu is a functor from the category $\mathbf{C}^*$ of C*-algebras to **Cu**.

Proof. Most of the work has already been done. (O1) is Theorem 3.8, while (O2) follows from Remark 3.9 and Remark 4.2. To verify (O3), let $a,a',b,b'\in(A\otimes\mathcal{K})_+$ satisfy $[a]\ll[b]$ and $[a']\ll[b']$. Upon identifying $[a]$ with the class of $\left(\begin{smallmatrix} a & 0\\ 0 & 0\end{smallmatrix}\right)\in A\otimes\mathcal{K}\otimes M_2\cong A\otimes\mathcal{K}$, and similarly for a',b,b', we may assume that $a\perp a'$ and $b\perp b'$. Use Proposition 4.3 to find $\varepsilon>0$ such that $[a]\leq[(b-\varepsilon)_+]$ and $[a']\leq[(b'-\varepsilon)_+]$. Using that $b\perp b'$ at the second step, we get

$$a+a'\precsim(b-\varepsilon)_++(b'-\varepsilon)_+=(b+b'-\varepsilon)_+.$$

Note that $[(b+b'-\varepsilon)_+]\ll[b+b']$ by Proposition 3.11. Using this at the second step, that $a\perp a'$ at the first step, and that $b\perp b'$ at the last step (both in combination with Lemma 3.2, we get

$$[a]+[a']=[a+a']\ll[b+b']=[b]+[b'],$$

as desired.

Finally, to verify (O4), let $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be increasing sequences in Cu(A). Since $s_n+t_n\leq\sup_{n\in\mathbb{N}}s_n+\sup_{n\in\mathbb{N}}t_n$, we get

$$\sup_{n\in\mathbb{N}}(s_n+t_n)\leq\sup_{n\in\mathbb{N}}s_n+\sup_{n\in\mathbb{N}}t_n.$$

To show the converse inequality, use Remark 3.10 to find increasing, norm-convergent sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in $(A\otimes\mathcal{K})_+$, with limits a and b, satisfying $[a_n]\leq s_n$ and $[b_n]\leq t_n$ for all $n\in\mathbb{N}$, and such that $[a]=\sup_{n\in\mathbb{N}}s_n$ and $[b]=\sup_{n\in\mathbb{N}}t_n$. Then

$$\sup_{n\in\mathbb{N}}s_n+\sup_{n\in\mathbb{N}}t_n=[a\oplus b]=\sup_{n\in\mathbb{N}}[a_n\oplus b_n]=\sup_{n\in\mathbb{N}}\big([a_n]+[b_n]\big)\leq\sup_{n\in\mathbb{N}}\big(s_n+t_n\big),$$

as desired. This finishes the proof. ☐

Remark 4.7. One can define a natural order-topology on Cu-semigroups, called the *Scott topology.* A base for this topology is given by those upward-hereditary sets U (that is, $s\in U$ and $s\leq t$ imply $t\in U$) such that for every $s\in U$ there exists $s'\in U$ with $s'\ll s$. With respect to this topology, an increasing sequence in Cu(A) actually *converges* to its supremum. Regarding increasing sequences as the order-theoretic analogues of Cauchy sequences, (O1) states that Cu-semigroups are complete. We will not need or use this topology in this work.

Definition 4.8 ([102, Definition 1.3])**.** We say that a linear map $\varphi\colon A \to B$ between C*-algebras A and B is a *completely positive contractive map of order-zero*, in short a cpc$_\perp$ map, in case the natural extensions of φ to matrices are all positive and contractive, and φ preserves orthogonality, in the sense that $\varphi(a)\varphi(b) = 0$ whenever $a, b \in A_+$ satisfy $ab = 0$.

Just as homomorphisms are the natural models for Cu-morphisms, cpc$_\perp$ maps are natural models for generalized Cu-morphisms, as the result below shows (we omit its proof):

Proposition 4.9 (see [102, Corollary 3.5] and [8, Proposition 2.2.7])**.** Let $\varphi\colon A \to B$ be a cpc$_\perp$ map. Then φ induces a generalized Cu-morphism $\mathrm{Cu}(\varphi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$, given by $\mathrm{Cu}(\varphi)([a]) = [\varphi(a)]$ for all $a \in (A \otimes \mathcal{K})_+$.

Perhaps a natural question at this point is whether *every* Cu-semigroup is the Cuntz semigroup of a C*-algebra. The answer is unfortunately negative. The following, due to Bosa and Petzka, is the smallest example:

Example 4.10. ([24, Example 5.3]). Set $S = \{0, 1, \infty\}$ with the usual order and addition ($1+1 = \infty$). Then there is no C*-algebra A with $\mathrm{Cu}(A) = \{0, 1, \infty\}$. This is however surprisingly difficult to prove.

In fact there are more axioms that $\mathrm{Cu}(A)$ always satisfies, and some of these will be discussed in Section 10. Describing precisely which Cu-semigroups arise as $\mathrm{Cu}(A)$ is extremely difficult and currently considered to be out of reach, although this is possible (and usually tedious) for some specific classes of C*-algebras such as AF-algebras [8], AI-algebras [95], certain commutative C*-algebras with 2-dimensional spectrum [75], and simple $\mathcal{Z}$-stable C*-algebras (see Theorem 9.7).

The functor $\mathrm{Cu}\colon \mathbf{C}^* \to \mathbf{Cu}$ has many nice properties and preserves a number of constructions, including direct limits, short exact sequences, direct sums, direct products, and ultraproducts. These claims have to be suitably interpreted: there are categorical versions of the above notions, which can be shown to always exist in $\mathbf{Cu}$ (their existence in $\mathbf{C}^*$ is known), and the functor Cu preserves these. The study of the category $\mathbf{Cu}$ in itself (or some subcategory of it) is crucial in this setting, and allows one to better understand the functor Cu; this is explored in detail in Section 14. Thus, and even if one is only interested in $\mathrm{Cu}(A)$, studying abstract Cu-semigroups is often necessary.

Knowing that the functor Cu preserves a number of constructions is unfortunately not very useful without understanding how to actually construct these objects in $\mathbf{Cu}$. We will only focus on direct limits in this section:

Theorem 4.11. The category $\mathbf{Cu}$ has direct limits, and the functor Cu satisfies

$$\mathrm{Cu}(\varinjlim A_n) \cong \varinjlim \mathrm{Cu}(A_n).$$

Proof. We only briefly describe how to show that $\mathbf{Cu}$ has direct limits; a more conceptual approach is given in Theorem 12.9 and the comments after it. Let

$$(\phi_n\colon S_n \to S_{n+1})_{n\in\mathbb{N}}$$

be an inductive sequence in $\mathbf{Cu}$. Denote by W the direct limit of this sequence in the category of positively ordered monoids. (For example, take the direct limit as semigroups, and define an order by declaring that two sequences compare if they eventually compare.) This will in general not be a Cu-semigroup, since increasing sequences may not necessarily have suprema. The correct object to consider is a certain “completion” of W; see Remark 4.7 for a more formal interpretation of what completion means, and also Theorem 12.9 and the comments after it. Intuitively speaking, we want to add the suprema of all increasing sequences, similarly to how one adds the limits of all Cauchy sequences when completing a metric space. Since we also want (O2) to be satisfied, we will take sequences in W which are $\ll$-increasing.

We define an order on the space of $\ll$-increasing sequences in W by setting $(x_n)_{n\in\mathbb{N}} \precsim (y_n)_{n\in\mathbb{N}}$ if for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ with $x_n \leq y_m$. Write $\sim$ for the symmetrization of this order: $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$ if $(x_n)_{n\in\mathbb{N}} \precsim (y_n)_{n\in\mathbb{N}} \precsim (x_n)_{n\in\mathbb{N}}$. The space S of equivalence classes of increasing sequences in W is a Cu-semigroup, and one can show that $S = \varinjlim(S_n, \phi_n)$. □

In retrospect, the construction of direct limits in **Cu** is very similar to the construction of direct limits in $\mathbf{C}^*$. Indeed, one first considers the algebraic direct limit, and then suitably completes the resulting direct limit to get the desired object. The completion procedure described in the proof of Theorem 4.11 is actually a functor from a suitable category **W** to **Cu**, and this will be explored in detail in Section 12; see Theorem 12.8 and Theorem 12.9.

We will use the above to compute the Cuntz semigroup of the CAR-algebra:

Example 4.12. We denote by M_{2^∞} the UHF-algebra of type 2^∞, which is the direct limit of M_{2^n} with connecting maps of the form $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$. By Example 3.3, we have $\mathrm{Cu}(M_{2^n}) \cong \overline{\mathbb{N}}$, and via the isomorphism given in that example (the rank), the connecting maps are easily seen to be multiplication by 2. The algebraic direct limit is $S = \mathbb{N}\big[\frac{1}{2}\big] \cup \{\infty\}$, with the order inherited from $[0,\infty]$. Note that $w \ll w$ for every $w \in S \setminus \{\infty\}$. It is easy to see that S is not a Cu-semigroup, since not every increasing sequence has a supremum.[5] (Note, however, that the supremum exists in $[0,\infty]$.)

Let $(a_n)_{n\in\mathbb{N}}$ be an $\ll$-increasing sequence in S. Regarding it as an increasing sequence in $[0,\infty]$, it has a limit $x \in [0,1]$. We claim that the $\sim$-class of $(x_n)_{n\in\mathbb{N}}$ only depends on $x \in [0,\infty]$ and whether $(x_n)_{n\in\mathbb{N}}$ is constant or strictly increasing (both interpreted as eventual behaviors). It is not difficult to see that any two strictly increasing sequences in S whose limits in $[0,\infty]$ agree are automatically equivalent, and similarly for constant sequences. Finally, one also checks that a constant sequence cannot be equivalent to a strictly increasing sequence. Since equivalent sequences must have the same limit in $[0,\infty]$, the claim follows.

Given a $\ll$-increasing sequence $(a_n)_{n\in\mathbb{N}}$ with limit $x \in [0,\infty]$, we abbreviate its $\sim$-equivalence class as

$$[(a_n)_{n\in\mathbb{N}}] = \begin{cases} c_x, & \text{if } (a_n)_{n\in\mathbb{N}} \text{ is constant (in which case } x \in S \setminus \{\infty\}), \\ s_x, & \text{if } (a_n)_{n\in\mathbb{N}} \text{ is strictly increasing (in which case } x \in (0,\infty]). \end{cases}$$

It follows that

$$\mathrm{Cu}(M_{2^\infty}) = \{c_x \colon x \in W \setminus \{\infty\}\} \sqcup \{s_x \colon x \in (0,\infty]\} \cong \mathbb{N}\big[\tfrac{1}{2}\big] \sqcup (0,\infty].$$

Addition and order work as one would expect on each component. For mixed terms, we have:

- $s_x \leq c_y$ if and only if $x \leq y$;
- $c_x \leq s_y$ if and only if $x < y$;
- $c_x + s_y = s_{x+y}$.

In particular, $c_x \leq s_x$ but $s_x \not\leq c_x$ (otherwise they would be equal).

Remark 4.13. The example above shows some phenomena that can be seen in the Cuntz semigroups of more general C*-algebras:

[5]Take, for example, any increasing sequence of dyadic numbers converging to a non-dyadic number.

(i) The compact elements of $\mathrm{Cu}(M_{2^\infty})$ are $\mathbb{N}\left[\frac{1}{2}\right] \cong \mathrm{V}(M_{2^\infty})$; see Corollary 3.4 and the comments after it, and see also Proposition 4.14 below. The fact that $\mathrm{V}(A)$ can be identified with the compact elements in $\mathrm{Cu}(A)$ is a general fact about stably finite C*-algebras, due to Brown-Ciuperca; see [25].

(ii) The component $(0, \infty]$ of $\mathrm{Cu}(M_{2^\infty})$ corresponds to the values of positive elements, not equivalent to projections, on the unique trace.

More generally, the Cuntz semigroup of a simple, unital, stably finite, $\mathcal{Z}$-stable C*-algebra A can be computed in terms of $\mathrm{V}(A)$ and $\mathrm{T}(A)$ (or rather, $\mathrm{QT}(A)$) in a similar fashion; see Theorem 9.7.

The first part of Remark 4.13 admits a somewhat more direct proof for C*-algebras of stable rank one, which is the case that we will need in these notes. Recall from [19] that an element z in a C*-algebra A is called a *scaling element* provided $zz^* \neq z^*z$ and $(z^*z)(zz^*) = zz^*$. It was shown in [19, Theorem 4.1] and the arguments after it that if a C*-algebra A contains a scaling element, then $M_n(A)$ contains an infinite projection for some $n \geq 1$.

Proposition 4.14. Let A be a C*-algebra of stable rank one, and let $x \in \mathrm{Cu}(A)$ be a compact element. Then there exists a projection $p \in A \otimes \mathcal{K}$ such that $[p] = x$.

In particular, the subsemigroup of $\mathrm{Cu}(A)$ consisting of compact elements is order-isomorphic to $\mathrm{V}(A)$.

Proof. By part (iii) of Proposition 2.14, we may assume that A is stable. Write $x = [a]$ for some $a \in A_+$. For each $\varepsilon > 0$, define

$$g_\varepsilon(t) = \begin{cases} 0, & \text{if } t \leq \frac{\varepsilon}{2} \\ \text{linear}, & \text{if } t \in [\frac{\varepsilon}{2}, \varepsilon] \\ 1, & \text{if } t \geq \varepsilon. \end{cases}$$

Note that $g_{\frac{\varepsilon}{2}} g_\varepsilon = g_\varepsilon$ and that $g_\varepsilon(a) \sim (a - \frac{\varepsilon}{2})_+ \leq a$ by Proposition 2.2. Since $x = \sup\limits_{\varepsilon > 0}[g_\varepsilon(a)]$ and x is compact, there is $\varepsilon > 0$ such that

$$a \sim g_\varepsilon(a) \sim g_{\varepsilon'}(a)$$

for all $\varepsilon' < \varepsilon$. Since A has stable rank one, we can use Proposition 2.16 to find $z \in A$ such that $g_{\frac{\varepsilon}{2}}(a) = z^*z$ and $zz^* \in A_{g_\varepsilon(a)}$. Thus $(z^*z)(zz^*) = g_{\frac{\varepsilon}{2}}(a)zz^* = zz^*$.

Assume that $zz^* = z^*z$. Then $g_{\frac{\varepsilon}{2}}(a) \in A_{g_\varepsilon(a)}$. Let $\delta > 0$ be such that $\frac{\varepsilon}{4}(1+\delta) < \frac{\varepsilon}{2}$. Then there is $w \in A$ such that $\|g_{\frac{\varepsilon}{2}}(a) - g_\varepsilon(a)wg_\varepsilon(a)\| < \frac{\delta}{4}$. It follows that $\|g_{\frac{\varepsilon}{2}}(a)^2 - g_{\frac{\varepsilon}{2}}(a)\| < \frac{\delta}{2}$. A standard application of functional calculus allows us to find a projection $p \in A_{g_\varepsilon(a)}$ with $\|p - g_{\frac{\varepsilon}{2}}(a)\| < \delta$; see, for example [99, Lemma 5.1.6].

By Lemma 2.5, we have $(g_{\frac{\varepsilon}{2}}(a) - \delta)_+ \precsim p$. By our choice of δ, and since $(g_{\frac{\varepsilon}{2}}(a) - \delta)_+ \sim g_{\frac{\varepsilon}{2(1+\delta)}}(a)$, we obtain that

$$g_{\frac{\varepsilon}{2(1+\delta)}}(a) \precsim p \precsim g_\varepsilon(a) \precsim g_{\frac{\varepsilon}{2(1+\delta)}}(a),$$

and thus $a \sim p$.

Assume now that $zz^* \neq z^*z$, so that z is a scaling element. As mentioned before this proposition, this implies that $M_n(A)$ contains an infinite projection, in contradiction with the fact that A has stable rank one.

The last statement follows from Lemma 2.8. $\square$

5. Ideals and quotients

The goal of this and the following sections is to show how some very important information about A is completely encoded in $\mathrm{Cu}(A)$. In this section, we will show how to recover the ideal lattice of A from its Cuntz semigroup; as a byproduct, we will see that the Cuntz semigroup of every ideal and every quotient of A can be read off of $\mathrm{Cu}(A)$. The upcoming section deals with (quasi)traces on A.

Definition 5.1. Let S be a Cu-semigroup. An *ideal* in S is a submonoid $I \subseteq S$ which is closed under suprema of increasing sequences and is hereditary, in the sense that $a \leq b$ and $b \in I$ imply $a \in I$. We denote by $\mathrm{Lat}(S)$ the lattice of ideals of S, which is ordered by inclusion. We say that S is *simple* if $\mathrm{Lat}(S)$ consists only of $\{0\}$ and S.

Next, we will show that ideals in A naturally induce ideals in $\mathrm{Cu}(A)$. For Cu-morphisms which are order-embeddings, the ordered structure of the domain and the induced structure in the codomain agree.

Lemma 5.2. Let A be a C*-algebra and let J be an ideal in A. Denote by $\iota\colon J \to A$ the canonical inclusion. Then

$$\mathrm{Cu}(\iota)\colon \mathrm{Cu}(J) \to \mathrm{Cu}(A)$$

is an order-embedding, and its image $\{[x] \in \mathrm{Cu}(A)\colon x \in (J \otimes \mathcal{K})_+\}$ is an ideal in $\mathrm{Cu}(A)$.

Proof. Without loss of generality, assume that A and J are stable. Let $x, y \in J$ satisfy $x \precsim y$ in A. Given $\varepsilon > 0$ choose $r \in A$ such that $ryr^* \approx_{\frac{\varepsilon}{2}} x$. Find $e \in J_+$ such that $eye \approx_{\frac{\varepsilon}{2\|r\|^2}} y$. Then $re \in J$ and

$$(re)y(re)^* \approx_{\frac{\varepsilon}{2}} ryr^* \approx_{\frac{\varepsilon}{2}} x,$$

showing that $x \precsim y$ in J. It remains to show that the image of $\mathrm{Cu}(\iota)$ is an ideal, for which it suffices to show that it is hereditary. Let $a \in A_+$ and $b \in J_+$ satisfy $a \precsim b$ in A, and choose a sequence $(r_n)_{n\in\mathbb{N}}$ in A satisfying $a = \lim\limits_{n\to\infty} r_n b r_n^*$. Since J is an ideal, it follows that $r_n b r_n^* \in J$ and thus $a \in J$. ☐

In view of the lemma above, whenever J is an ideal in A, we will identify $\mathrm{Cu}(J)$ with an ideal in $\mathrm{Cu}(A)$. We isolate the following observations for future use:

Remark 5.3. Let A be a C*-algebra and let $x, y \in A$. Using that $(x-y)^*(x-y) \geq 0$, one gets

$$(x+y)^*(x+y) \leq 2x^*x + 2y^*y.$$

Thus $x^*y + y^*x \leq x^*x + y^*y$ and also $[(x+y)^*(x+y)] \leq [x^*x] + [y^*y]$ in $\mathrm{Cu}(A)$.

Remark 5.4. Let A be a C*-algebra and let J be an ideal in A. For $a \in A$, one can use the polar decomposition to show that $a \in J$ if and only if $a^*a \in J$.

The following is the main result of this section; see [36].

Theorem 5.5. Let A be a C*-algebra. Let $\Phi\colon \mathrm{Lat}(A) \to \mathrm{Lat}(\mathrm{Cu}(A))$ be given by $\Phi(J) = \mathrm{Cu}(J)$. Then Φ is an order-isomorphism. Its inverse $\Psi\colon \mathrm{Lat}(\mathrm{Cu}(A)) \to \mathrm{Lat}(A)$ is given by

$$\Psi(I) = \{x \in A\colon [x^*x] \in I\}.$$

Proof. We begin by showing that Ψ is well-defined, namely that $\Psi(I)$ is an ideal in A. Given $x, y \in \Psi(I)$, by Remark 5.3 we have

$$[(x+y)^*(x+y)] \leq [x^*x] + [y^*y] \in I.$$

Since I is hereditary, it follows that $[(x+y)^*(x+y)] \in I$ and thus $x + y \in \Psi(I)$. That $\Psi(I)$ is closed under scalar multiplication is clear. To show that it is a left and right ideal, let $x \in \Psi(I)$ and let $a \in A$. Since

$$(ax)^*(ax) = x^*a^*ax \leq \|a\|^2 x^*x,$$

it follows from Proposition 2.4 that $(ax)^*(ax) \precsim x^*x$. Also, $(xa)^*(xa) = a^*x^*xa \precsim x^*x$. Since I is hereditary, it follows as before that $ax, xa \in \Psi(I)$.

It remains to show that $\Psi(I)$ is closed. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\Psi(I)$ converging to $a \in A$. Then $(x_n^*x_n)_{n\in\mathbb{N}}$ converges to a^*a. Given $\varepsilon > 0$, find $n \in \mathbb{N}$ such that $\|x_n^*x_n - a^*a\| < \varepsilon$. By Proposition 3.11, we have $[(a^*a-\varepsilon)_+] \leq [x_n^*x_n] \in I$. Since I is hereditary, we deduce that $[(a^*a-\varepsilon)_+] \in I$. Since $[a^*a] = \sup\limits_{m>0}[(a^*a-\frac{1}{m})_+]$ and I is closed under suprema, we conclude that $[a^*a] \in I$ and hence $a \in \Psi(I)$.

It is also clear that both Φ and Ψ are order-preserving. To show that they are mutual inverses, let $I \in \mathrm{Lat}(\mathrm{Cu}(A))$. Given $x \in A_+$, we have $[x] \in \Phi(\Psi(I))$ if and only if $x \in \Psi(I)$, if and only if $[x^*x] = [x] \in I$. Thus $I = \Phi(\Psi(I))$. For the converse, given $J \in \mathrm{Lat}(A)$ we want to show that $J = \Psi(\Phi(J))$. For $a \in A$, we have $a \in \Psi(\Phi(J))$ if and only if $[a^*a] \in \Phi(J)$, if and only if $[a^*a] \in \mathrm{Cu}(J)$. Using the definition of Cuntz equivalence, it is clear that the above is equivalent to $a^*a \in J$, which is equivalent to $a \in J$ by Remark 5.4. Thus $J = \Psi(\Phi(J))$ and the proof is complete. □

Notation 5.6. Let S be a Cu-semigroup. Given $a \in S$, write ∞_a for the supremum of $(na)_{n\in\mathbb{N}}$.

It is easy to see that the ideal generated by a is precisely $\{x \in S : x \leq \infty_a\}$.

Corollary 5.7. Let S be a Cu-semigroup. Then S is simple if and only if $\infty_a = \infty_b$ for all $a, b \in S \setminus \{0\}$. Equivalently, for all nonzero $a, b \in S$, we have $a \leq \infty_b$.

In view of the above corollary, there is a unique infinity in every simple Cu-semigroup, which we will denote simply by ∞.

In the remainder of this section, we explain how to detect the Cuntz semigroups of all quotients of a given C*-algebra in its own Cuntz semigroup.

Definition 5.8. Let S be a Cu-semigroup and let I be an ideal. Given $a, b \in S$, we set $a \leq_I b$ if there exists $c \in I$ such that $a \leq b + c$. We set $a \sim_I b$ if $a \leq_I b$ and also $b \leq_I a$.

It is easy to see that $\sim_I$ is an equivalence relation; we write S/I for the associated quotient. For $a \in S$, we write a_I for its equivalence class. We define an addition on S/I by setting $a_I + b_I = (a+b)_I$, and an order by declaring $a_I \leq b_I$ if $a \leq_I b$. It can be shown that this gives S/I the structure of a Cu-semigroup. The following is proved in [36].

Theorem 5.9. Let A be a C*-algebra, let J be an ideal, and let $\pi\colon A \to A/J$ denote the quotient map. Given $a, b \in \mathrm{Cu}(A)$, we have

$$\mathrm{Cu}(\pi)(a) \leq \mathrm{Cu}(\pi)(b) \text{ in } \mathrm{Cu}(A/J) \quad \text{if and only if} \quad a \leq_{\mathrm{Cu}(J)} b \text{ in } \mathrm{Cu}(A).$$

In particular, $\mathrm{Cu}(\pi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(A/J)$ induces an order-isomorphism

$$\mathrm{Cu}(A)/\mathrm{Cu}(J) \cong \mathrm{Cu}(A/J).$$

Proof. Let $a, b \in \mathrm{Cu}(A)$ satisfy $a \leq_{\mathrm{Cu}(J)} b$, and choose $c \in \mathrm{Cu}(J)$ with $a \leq b + c$. Since $\mathrm{Cu}(\pi)(c) = 0$, we get

$$\mathrm{Cu}(\pi)(a) \leq \mathrm{Cu}(\pi)(b+c) = \mathrm{Cu}(\pi)(b),$$

as desired. The converse is somewhat more involved, and we omit it. □

The results of this section should be compared to similar statements in K-theory: while $\mathrm{Cu}(A)$ encodes the lattice of ideals of A, as well as the Cuntz semigroups of all ideals, in general K-theory does not contain any of this information. (There is one notable exception: the ordered K_0-group of an AF-algebra encodes the ideal structure and the K_0-groups of all ideals and quotients. This helps explain why the classification of AF-algebras works also in the non-simple case, without a larger invariant.)

6. Functionals, quasitraces, and Cuntz's theorem

In this section, we will show that there is a natural bijective correspondence between (quasi)traces on A and functionals on $\mathrm{Cu}(A)$; see Theorem 6.9. We will then present Cuntz's theorem, stating that a simple, unital, stably finite C*-algebra always admits a quasitrace; see Theorem 6.11.

The motivating observation for this section comes from the Riesz theorem:

Theorem 6.1. (Riesz). Let X be a compact Hausdorff space. Then there is a natural bijection between the set of all Radon Borel probability measures on X and all tracial states on $C(X)$. Given a Radon probability measure μ on X, the corresponding tracial state $\tau_\mu \colon C(X) \to \mathbb{C}$ is given by

$$\tau_\mu(f) = \int_X f(x)\ d\mu(x) = \int_0^\infty \mu(\{x \in X \colon f(x) > t\})\ dt$$

for all non-negative functions $f \in C(X)_+$, and extended linearly.

We want to express the trace τ_μ in a different manner. Note that

$$\{x \in X \colon f(x) > t\} = \mathrm{supp_o}((f-t)_+),$$

which only depends on the Cuntz class of $(f-t)_+$. By writing τ_μ as

$$\tau_\mu(f) = \int_0^\infty \mu\big(\mathrm{supp_o}((f-t)_+)\big)\ dt,$$

we have expressed $\tau_\mu(f)$ entirely in terms of Cuntz classes in $\mathrm{Cu}(C(X))$.

Hence, if we are given a "nice" map $\lambda \colon \mathrm{Cu}(A) \to [0,\infty]$, we may attempt to define a trace on A via

$$\tau_\lambda(a) = \int_0^\infty \lambda\big(\mathrm{supp_o}((a-t)_+)\big)\ dt$$

for all $a \in A$. This is the essential idea behind the correspondence between traces on A and functionals on $\mathrm{Cu}(A)$. There is, however, one problem: even for nice maps λ, it is not at all clear whether the map τ_λ defined above is additive. We are therefore led to consider non-linear maps, which brings us to the definition of a *quasitrace*.

Definition 6.2. Let A be a unital C^*-algebra. A *1-quasitracial state* (or just *1-quasitrace*) is a function $\tau \colon A \to \mathbb{C}$ such that

(i) $\tau(a+ib) = \tau(a) + i\tau(b)$ for all $a, b \in A_{\mathrm{sa}}$;
(ii) τ is linear on *commutative subalgebras* of A;
(iii) $\tau(a^*a) = \tau(aa^*) \geq 0$ for all A;
(iv) $\tau(1) = 1$.

We say that τ is a *quasitracial state* (or just a *quasitrace*) if it extends to a 1-quasitrace on $M_n(A)$ for all $n \in \mathbb{N}$.[6] We denote by $\mathrm{QT}(A)$ the space of all quasitraces on A.

There exist 1-quasitraces that are not quasitraces; such examples were first constructed by Kirchberg.

Remark 6.3. Note that a quasitrace is a trace if and only if it is additive on all positive elements of A, not just on commuting ones.

[6] It was shown by Blackadar and Handelman that, in order for a 1-quasitrace to be a quasitrace, it is enough for it to extend to $M_2(A)$; see [17, Proposition II.4.1].

It is a major open problem in C*-algebra theory whether every quasitrace is automatically a trace. By a very deep result of Haagerup [55], this is the case whenever the C*-algebra in question is *exact*. In particular, this is always the case for nuclear C*-algebras.

We now define the *dimension function* (or functional) associated to a quasitrace.

Definition 6.4. Let A be a unital C*-algebra. Given $\tau \in \mathrm{QT}(A)$, we define its associated *dimension function* $d_\tau \colon \mathrm{Cu}(A) \to [0,\infty]$ by

$$d_\tau([a]) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}})$$

for all $a \in M_\infty(A)_+$, and extended to $(A\otimes\mathcal{K})_+$ by taking suprema.[7]

Remark 6.5. Given a positive element $a \in A$, the sequence $(a^{\frac{1}{n}})_{n\in\mathbb{N}}$ converges in A^{**} in the weak-$*$ topology to the *support projection* p_a of a; this is the smallest projection which acts as a unit on a. Since quasitraces are weak-$*$ continuous on A^{**}, we have

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}}) = \tau(p_a).$$

Implicit in Definition 6.4 is the fact that $d_\tau([a])$ only depends on $[a]$. This is indeed the case. In fact, more is true:

Proposition 6.6. *Let A be a unital C^*-algebra. Given $\tau \in \mathrm{QT}(A)$, the map d_τ is well-defined on $\mathrm{Cu}(A)$. Moreover, we have:*

(i) $d_\tau(0) = 0$;
(ii) $d_\tau([1]) = 1$;
(iii) $d_\tau(s+t) = d_\tau(s) + d_\tau(t)$ for all $s,t \in \mathrm{Cu}(A)$;
(iv) $d_\tau(s) \le d_\tau(t)$ whenever $s \le t$ in $\mathrm{Cu}(A)$;
(v) d_τ preserves suprema of increasing sequences.

The above proposition is not hard to prove (maybe with the exception of (iv) and (v)), and we will omit the argument. Instead, we will show a particular case, which connects back to the motivation we gave after Theorem 6.1.

Proposition 6.7. Let X be a compact Hausdorff space, let μ be a Radon Borel probability measure, and let $\tau_\mu \colon C(X) \to \mathbb{C}$ be the trace it induces. Given $a \in C(X)_+$, we have

$$d_{\tau_\mu}([a]) = \mu(\mathrm{supp}_{\mathrm{o}}(a)).$$

Proof. Without loss of generality, we may assume that $\|a\| \le 1$. Note that $(a^{\frac{1}{n}})_{n\in\mathbb{N}}$ is an increasing sequence which converges pointwise to the indicator function of $\mathrm{supp}_{\mathrm{o}}(a)$. Applying the dominated convergence theorem at the third step, we get

$$d_\tau([a]) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}}) = \lim_{n\to\infty} \int_X a^{\frac{1}{n}}(x)\; d\mu(x) = \int_X \lim_{n\to\infty} a^{\frac{1}{n}}(x)\; d\mu(x) = \mu(\mathrm{supp}_{\mathrm{o}}(a)),$$

as desired. $\square$

Proposition 6.6 shows that the map d_τ is a normalized functional on $\mathrm{Cu}(A)$ in the sense of the following definition.

Definition 6.8. Let S be a Cu-semigroup. A *functional* on S is a map $f \colon S \to [0,\infty]$ which preserves the zero element, addition, order, and suprema of increasing sequences. We denote by $F(S)$ the set of all functionals on S.

If $e \in S$ is a distinguished compact element in S, a functional $\lambda \in F(S)$ is said to be *normalized (at e)* if $\lambda(e) = 1$. We write $F_e(S)$ for the set of all normalized functionals on S.

[7]Explicitly, for a general element $a \in (A\otimes\mathcal{K})_+$, one defines $d_\tau([a])$ to be the supremum in $[0,\infty]$ over $\varepsilon > 0$ of $d_\tau([(a-\varepsilon)_+])$, which is well defined since $(a-\varepsilon)_+$ belongs to $M_\infty(A)_+$.

In this section, we will mostly focus on normalized functionals, but we will come back to non-normalized ones in Section 13.

We write $d\colon \mathrm{QT}(A) \to F_{[1]}(\mathrm{Cu}(A))$ for the map given by $d(\tau) = d_\tau$ for $\tau \in \mathrm{QT}(A)$; see Proposition 6.6. Both $\mathrm{QT}(A)$ and $F_{[1]}(\mathrm{Cu}(A))$ are convex sets, and it is easy to see that d is an affine map.[8]

Theorem 6.9. Let A be a unital C^*-algebra. Then $d\colon \mathrm{QT}(A) \to F_{[1]}(\mathrm{Cu}(A))$ is an affine bijection.

Proof. Given $\lambda \in F_{[1]}(\mathrm{Cu}(A))$, define $\tau_\lambda \in \mathrm{QT}(A)$ by

$$\tau_\lambda(a) = \int_0^\infty \lambda\big([(a-t)_+]\big)dt$$

for all $a \in A_+$, and extended $\mathbb{T}$-linearly to A using real and imaginary parts, and positive and negative parts. We claim that τ_λ is a quasitracial state on A. Condition (i) in Definition 6.2 is satisfied by construction, while (iv) is clear. For (ii), let B be a commutative unital C*-subalgebra, and let X denote the maximal ideal space of B. Note that the restriction of τ_λ to B induces an assignment $\mu_0\colon \mathcal{O}(X) \to [0,1]$, defined on the open subsets $\mathcal{O}(X)$ of X, which is σ-additive on disjoint sets.[9] By Caratheodory's Theorem, μ_0 extends to a Radon Borel probability measure μ on X. It follows that $(\tau_\lambda)|_B = \tau_\mu$, and thus τ_λ is additive on B.

For (iii), let $a \in A$. Using Lemma 2.6 at the second step, we have

$$\tau_\lambda(a^*a) = \int_0^\infty \lambda([a^*a-t]_+)dt = \int_0^\infty \lambda([aa^*-t]_+)dt = \tau_\lambda(aa^*) \geq 0,$$

as desired.

Finally, in order to show that the two assignments $\tau \mapsto d_\tau$ and $\lambda \mapsto \tau_\lambda$ are mutual inverses, it suffices to check this on commutative subalgebras, which follows from Proposition 6.7. ☐

Remark 6.10. Note that the fact that τ_λ is additive on commutative subalgebras is ultimately a consequence of Caratheodory's Theorem. Thus, the question of whether all quasitraces are traces can be reinterpreted as asking whether there is a noncommutative version of Caratheodory's Theorem for C*-algebras.

We have stated and proved Theorem 6.9 only for quasitracial *states* on A and *normalized* functionals on $\mathrm{Cu}(A)$, but the bijective correspondence also extends to lower-semicontinuous $[0,\infty]$-valued quasitraces on A and functionals on $\mathrm{Cu}(A)$, with the same formulas; see [40, Theorem 4.4].

Having established the correspondence between quasitraces on A and functionals on $\mathrm{Cu}(A)$, we are ready to prove Cuntz's theorem [38]: a simple, unital, stably finite C*-algebra always admits a quasitrace. This was historically the first use of the Cuntz semigroup, and the reason why it receives its name. (It should be mentioned that Cuntz used a slightly different semigroup, which is discussed at length in Section 12, using positive elements taken from $M_\infty(A)$ instead of $A \otimes \mathcal{K}$, and considering its Grothendieck enveloping group.)

We will use, without proof, that if $\tau \in \mathrm{QT}(A)$, then

$$\ker(\tau) := \{x \in A\colon \tau(x^*x) = 0\}$$

[8] The sets $\mathrm{QT}(A)$ and $F_{[1]}(\mathrm{Cu}(A))$ also admit natural topologies with respect to which they are compact and Hausdorff: for $\mathrm{QT}(A)$ the topology is induced by pointwise convergence, while the topology on $F_{[1]}(\mathrm{Cu}(A))$ is described at the beginning of Section 13. One can check that d is continuous with respect to these topologies, and hence a homeomorphism. This is worked out in detail in [40].

[9] In order to define μ_0, given an open set U find a continuous function $f \in C(X)$ whose open support is exactly U. Then set $\mu_0(U) = \lambda([f])$.

is an ideal in A. In particular, if A is simple and $a \in A_+$ satisfies $\tau(a) = 0$, then $a = 0$. This follows, for example, from part (3) of Lemma 3.5 in [55].

Theorem 6.11. (Cuntz). Let A be a simple, unital C*-algebra. Then A is stably finite if and only if it admits a quasitracial state.

Proof. Suppose that A admits a quasitracial state τ. Without loss of generality, it suffices to show that A is finite (otherwise consider the quasitracial state $\tau \otimes \mathrm{tr}_n$ on $M_n(A)$ instead of τ). Let $v \in A$ be an isometry. Then $\tau(vv^*) = \tau(v^*v) = \tau(1_A) = 1_A$, and hence $\tau(1_A - vv^*) = 0$. Thus $1_A - vv^*$ is a projection in the kernel of τ, which is an ideal in A. By simplicity, we must have $vv^* = 1_A$, so that v is a unitary.

We turn to the converse, so assume that A is stably finite.

Claim: *for $n, m \in \mathbb{N}$, if $n[1_A] \leq m[1_A]$ then $n \leq m$.* Suppose that $n[1_A] = m[1_A]$ for some $n < m \in \mathbb{N}$. Then $1_{M_m(A)} \precsim 1_{M_n(A)}$. By Lemma 2.8, we deduce that

$$1_{M_m(A)} \precsim_{\mathrm{MvN}} 1_{M_n(A)},$$

so there is an isometry $v \in M_m(A)$ such that $vv^* = \begin{pmatrix} 1_{M_n(A)} & 0 \\ 0 & 0_{m-n} \end{pmatrix}$. This contradicts stable finiteness of A and proves the claim.

Set $S_0 = \{n[1_A] \colon n \in \mathbb{N}\} \subseteq \mathrm{Cu}(A)$. Define an additive map $\lambda_0 \colon S_0 \to [0, \infty]$ by $\lambda_0(n[1_A]) = n$ for all $n \in \mathbb{N}$. Note that λ_0 is order-preserving by the previous claim. Since $[0, \infty]$ is an injective object in the category of positively ordered monoids, we can extend it to a map $\widetilde{\lambda} \colon \mathrm{Cu}(A) \to [0, \infty]$ which preserves the zero, addition and order, but not necessarily suprema of increasing sequences. We fix this by taking its "regularization" $\lambda \colon \mathrm{Cu}(A) \to [0, \infty]$ given by

$$\lambda(a) = \sup\{\widetilde{\lambda}(a') \colon a' \ll a\}$$

for all $a \in \mathrm{Cu}(A)$. One can check that λ preserves suprema of increasing sequences, so it is a functional. Since $[1_A] \ll [1_A]$, we have $\lambda([1_A]) = \widetilde{\lambda}([1_A]) = 1$, so λ is normalized. By Theorem 6.9, λ induces a quasitracial state on A, as desired. ☐

7. A construction of the Jiang-Su algebra $\mathcal{Z}$ using the Cuntz semigroup

In this section, we use the Cuntz semigroup to define the Jiang-Su algebra $\mathcal{Z}$ via generalized dimension drop algebras, and we compute its Cuntz semigroup.

Recall that a function $f \colon X \to S$ from a topological space X into an Cu-semigroup S is said to be *lower semicontinuous* if for every $s \in S$, the set

$$f^{-1}(\{t \in S \colon s \ll t\})$$

is open in X. We write $\mathrm{Lsc}(X, S)$ for the set of all lower-semicontinuous functions $X \to S$. In some relevant cases, $\mathrm{Lsc}(X, S)$ is a Cu-semigroup; see [7].

Definition 7.1. We define the *generalized dimension drop algebra* (of type 2,3) by

$$\mathcal{Z}_{2^\infty,3^\infty} = \left\{ a \in C\big([0,1], M_{2^\infty} \otimes M_{3^\infty}\big) \colon \begin{array}{l} a(0) \in M_{2^\infty} \otimes 1_{M_{3^\infty}}, \\ a(1) \in 1_{M_{2^\infty}} \otimes M_{3^\infty} \end{array} \right\}$$

The Cuntz semigroup of $\mathcal{Z}_{2^\infty,3^\infty}$ can be computed using [7, Corollary 3.5]:

$$\mathrm{Cu}(\mathcal{Z}_{2^\infty,3^\infty}) \cong \left\{ f \in \mathrm{Lsc}\big([0,1], \mathbb{N}[\tfrac{1}{6}] \cup (0,\infty]\big) \colon \begin{array}{l} f(0) \in \mathbb{N}[\frac{1}{2}] \cup (0,\infty] \\ f(1) \in \mathbb{N}[\frac{1}{3}] \cup (0,\infty] \end{array} \right\},$$

with pointwise order and addition (see, essentially, [7, Example 4.3]). Using these identifications, we define a Cu-morphism

$$\phi \colon \mathrm{Cu}(\mathcal{Z}_{2^\infty,3^\infty}) \to \mathrm{Cu}(\mathcal{Z}_{2^\infty,3^\infty}),$$

by setting

$$\phi(f) = \int_{[0,1]} f(t)\ dt$$

for $f \in \mathrm{Cu}(\mathcal{Z}_{2^\infty,3^\infty})$. This integral has to be interpreted appropriately: if f is compact, then $\phi(f)$ is the constant function with the corresponding compact element as its value, and otherwise it is the constant function with the corresponding non-compact element with that value. The main feature of this map is that it is *trace-collapsing*: this means that if $\lambda, \lambda' \colon \mathrm{Cu}(\mathcal{Z}_{2^\infty,3^\infty}) \to [0,\infty]$ are normalized functionals, then $\lambda \circ \phi = \lambda' \circ \phi$.

By a deep result of Robert [73, Theorem 1.0.1], there exists a unital homomorphism

$$\Phi \colon \mathcal{Z}_{2^\infty,3^\infty} \to \mathcal{Z}_{2^\infty,3^\infty} \quad \text{with} \quad \mathrm{Cu}(\Phi) = \phi.$$

Definition 7.2. The *Jiang-Su algebra* $\mathcal{Z}$ is defined to be the stationary direct limit

$$\mathcal{Z} = \varinjlim(\mathcal{Z}_{2^\infty,3^\infty}, \Phi).$$

This definition of $\mathcal{Z}$ is not the original one by Jiang and Su [59], and this presentation of $\mathcal{Z}$ was essentially obtained by Rørdam and Winter [82].

The type of the generalized dimension drop algebra is actually irrelevant, and the only crucial ingredient is that 2 and 3 are coprime. Indeed, coprimeness implies that $\mathbb{N}[\frac{1}{2}] \cap \mathbb{N}[\frac{1}{3}] = \mathbb{N}$ and guarantees that the direct limit does not have any projections; see the proof of Theorem 7.3.

The tools developed in the previous sections allow us to compute the Cuntz semigroup of $\mathcal{Z}$. The computation of this semigroup was first carried out in [69, Theorem 3.1].

Theorem 7.3. The Cuntz semigroup of $\mathcal{Z}$ can be described as follows. As sets, we have

$$\mathrm{Cu}(\mathcal{Z}) = \mathbb{N} \sqcup (0,\infty].$$

Addition and order are the expected ones on each component. For $n \in \mathbb{N}$, let $c_n \in \mathbb{N}$ be the corresponding compact element, and for $x \in (0,\infty]$ let s_x be the corresponding non-compact element in $(0,\infty]$. For $n \in \mathbb{N}$ and $x \in (0,\infty]$, we have:

- $s_x \leq c_n$ if and only if $x \leq n$;
- $c_n \leq s_x$ if and only if $n < x$;
- $c_n + s_x = s_{n+x}$.

Proof. This is similar to Example 4.12. We first want to understand the algebraic direct limit of the Cuntz semigroups. One can check that the compact elements in $\mathrm{Cu}(\mathcal{Z}_{2^\infty,3^\infty})$ are the constant functions

$$c \colon [0,1] \to \mathbb{N}[\tfrac{1}{6}] \cup (0,\infty]$$

whose constant value is compact (and thus in $\mathbb{N}[\frac{1}{6}]$). The endpoint conditions on c give $c(0) \in \mathbb{N}[\frac{1}{2}]$ and $c(1) \in \mathbb{N}[\frac{1}{3}]$. Since $c(0) = c(1)$ and $\mathbb{N}[\frac{1}{2}] \cap \mathbb{N}[\frac{1}{3}] = \mathbb{N}$, we deduce that the function c must have constant value in $\mathbb{N}$.

It follows that the direct limit of $(\mathrm{Cu}(\mathcal{Z}_{2^\infty,3^\infty}), \phi)$ in the category of partially ordered monoids is $\mathbb{N} \sqcup (0,\infty]$, with order and addition matching the ones in the statement. In principle one would need to complete, as was done in Example 4.12, but one can check that this is already a Cu-semigroup. $\square$

Remark 7.4. From the description above, it follows that for every $n \in \mathbb{N}$ there exists $z \in (\mathcal{Z} \otimes \mathcal{K})_+$ with

$$n[z] \leq [1_{\mathcal{Z}}] \leq (n+1)[z].$$

One can take z to be, for example, any representative of $s_{\frac{1}{n}}$.

Corollary 7.5. The Jiang-Su algebra $\mathcal{Z}$ is simple, nuclear, has a unique trace, and has no projections other than 0 and 1.

Proof. It is easy to see that $\mathrm{Cu}(\mathcal{Z})$ is a simple Cu-semigroup (for example, since $\infty_s = \infty_t$ for all nonzero $s, t \in \mathrm{Cu}(\mathcal{Z})$). Hence $\mathcal{Z}$ is simple by Corollary 5.7. Moreover, it has a unique normalized functional (given by $\lambda(c_n) = n$ for $n \in \mathbb{N}$ and $\lambda(s_x) = x$ for all $x \in (0, \infty]$). Thus $\mathcal{Z}$ has a unique quasitracial state by Theorem 6.9, and since $\mathcal{Z}$ is exact (being nuclear), it follows that $\mathrm{QT}(\mathcal{Z}) = \mathrm{T}(\mathcal{Z})$, as desired. Finally, a projection in $\mathcal{Z}$ other than 0 and 1 would yield a compact element in $\mathrm{Cu}(\mathcal{Z})$ strictly between 0 and 1, which does not exist. □

It was an old question of Kaplansky whether all simple C*-algebras must contain nontrivial projections. After Blackadar constructed the first examples of projectionless simple C*-algebras [14, 15], the question then became whether *nuclear*, simple C*-algebras must contain nontrivial projections, and Blackadar showed that the answer to this question is also negative. The algebra $\mathcal{Z}$ constructed by Jiang and Su in [59] is a further example of a simple, nuclear C*-algebra without projections, and it attracted a great deal of attention due to its prominent role in the classification programme. For example, since (the current version of) the Elliott invariant cannot distinguish between A and $A \otimes \mathcal{Z}$ (see Remark 9.2, and note that this is only the case because we are not including the order on K_0 as part of the Elliott invariant), only $\mathcal{Z}$-stable C*-algebras can be expected to be classified using exclusively the invariant Ell.

8. $\mathcal{Z}$-stability and strict comparison; the Toms-Winter conjecture

The goal of this section is to prove a theorem of Rørdam [81], relating two notions that do not in principle seem to be related: $\mathcal{Z}$-stability on the one hand, and almost unperforation of the Cuntz semigroup on the other hand. We will also show that almost unperforation in the Cuntz semigroup is equivalent to strict comparison, and make connections to the Toms-Winter conjecture.

Definition 8.1. Let A be a C*-algebra. We say that A is *$\mathcal{Z}$-stable* if $A \otimes \mathcal{Z} \cong A$.

For most practical purposes, just knowing that an isomorphism exists will not be of much help. As it turns out, $\mathcal{Z}$ has some remarkable properties that allow one to show that, whenever an isomorphism $A \otimes \mathcal{Z} \cong A$ exists, then a *nice* isomorphism exists, in the sense of the following result.

Theorem 8.2. (Jiang-Su [59]). Let A be a separable C*-algebra. If $A \otimes \mathcal{Z} \cong A$, then there exists an isomorphism $\varphi \colon A \to A \otimes \mathcal{Z}$ which is approximately unitarily equivalent to the map $a \mapsto a \otimes 1_{\mathcal{Z}}$. In particular, φ satisfies $[\varphi(a)] = [a \otimes 1_{\mathcal{Z}}]$ for all $a \in (A \otimes \mathcal{K})_+$.

We will begin by relating $\mathcal{Z}$-stability to the following notion:

Definition 8.3. Let S be a Cu-semigroup. We say that S is *almost unperforated* if whenever $s, t \in S$ and $n \in \mathbb{N}$ satisfy $(n+1)s \leq nt$, then $s \leq t$.

The following is the main result of this section. For $a \in A_+$ and $n \in \mathbb{N}$, recall that $n[a] = [a \otimes 1_n]$ in $\mathrm{Cu}(A)$. The following is

Theorem 8.4. (Rørdam [81]). Let A be a simple, separable, $\mathcal{Z}$-stable unital C*-algebra. Then $\mathrm{Cu}(A)$ is almost unperforated.

Proof. Let $n \in \mathbb{N}$ and $a, b \in (A \otimes \mathcal{K})_+$ satisfy

$$(n+1)[a] \leq n[b] \quad \text{in } \mathrm{Cu}(A). \tag{7.1}$$

Use Remark 7.4 to find $z \in (\mathcal{Z} \otimes \mathcal{K})_+$ such that

$$n[z] \leq [1_{\mathcal{Z}}] \leq (n+1)[z] \ \text{ in } \mathrm{Cu}(\mathcal{Z}) \tag{7.2}$$

Working in (the stabilization of) $A \otimes \mathcal{Z}$, we get

$$a \otimes 1_{\mathcal{Z}} \overset{(7.2)}{\precsim} a \otimes (z \otimes 1_{n+1}) \sim (a \otimes 1_{n+1}) \otimes z \overset{(7.1)}{\precsim} (b \otimes 1_n) \otimes z \overset{(7.2)}{\precsim} b \otimes 1_{\mathcal{Z}}.$$

Use Theorem 8.2 to choose an isomorphism $\varphi\colon A \to A \otimes \mathcal{Z}$ satisfying $[\varphi(a)] = [a \otimes 1_{\mathcal{Z}}]$ for all $a \in (A \otimes \mathcal{K})_+$. Applying φ to the subequivalence above, we deduce that $[\varphi(a)] \leq [\varphi(b)]$. Since φ is an isomorphism, this finishes the proof. □

We put Theorem 8.4 into context, by relating it to the notion of *strict comparison.*

Definition 8.5. Let A be a simple unital C*-algebra. We say that A has *strict comparison (of positive elements by quasitraces)* if whenever $a, b \in (A \otimes \mathcal{K})_+$ are nonzero and satisfy $d_\tau(a) < d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$, then $a \precsim b$.

Strict comparison is essentially a property of $\mathrm{Cu}(A)$. Using Theorem 6.9, one can show that A has strict comparison if and only if whenever $s, t \in \mathrm{Cu}(A)$ are nonzero and satisfy $\lambda(s) < \lambda(t)$ for all $\lambda \in F_{[1]}(\mathrm{Cu}(A))$, then $s \leq t$. Less obvious is the fact that strict comparison is equivalent to almost unperforation; see [81, Proposition 3.2]. (One direction is easy, namely almost unperforation implies strict comparison, while the converse requires an order-semigroup theoretic version of the Hahn-Banach theorem, similar to what was used in the proof of Theorem 6.11.)

In particular, Theorem 8.4 asserts that $\mathcal{Z}$-stable C*-algebras have strict comparison. Perhaps surprisingly, it is conjectured that the converse is true in the simple, nuclear setting:

Conjecture 8.6. (Toms-Winter regularity conjecture, see [94], and also [101, Conjecture 5.2]). Let A be a simple, separable, unital, nuclear C*-algebra. The following are equivalent:

(i) $\dim_{\mathrm{nuc}}(A) < \infty$;
(ii) A is $\mathcal{Z}$-stable;
(iii) A has strict comparison.

Nuclear dimension for C*-algebras is a noncommutative version of the covering dimension for topological spaces which was introduced by Winter and Zacharias [103], and we will not define this notion here.

We state the conjecture in this form for historical reasons, but the fact that (i) and (ii) are equivalent is by now a theorem: that (i) implies (ii) is an impressive result of Winter [100], while the implication from (ii) to (i) has recently been obtained in an equally outstanding work by Castillejos, Evington, Tikuisis, White and Winter [33]. The fact that (ii) implies (iii) is precisely Theorem 8.4. The converse implication remains open, although it is known in some cases, such as whenever $\partial_e T(A)$ is compact and finite-dimensional thanks to the independent works of Kirchberg-Rørdam [63], Toms-White-Winter [92], and Sato [83]; or whenever A has stable rank one and locally finite nuclear dimension thanks to the work of Thiel [85]; or whenever A has uniform property Γ, by the work of Castillejos, Evington, Tikuisis and White [32].

9. Toms' example and the relation with the Elliott invariant

In this section, we present an example, due to Andrew Toms [91], of two C*-algebras that agree on the Elliott invariant (and more), yet they are distinguished by their Cuntz semigroup. This example shows the importance of the Cuntz semigroup outside the class of $\mathcal{Z}$-stable C*-algebras, as a key ingredient for classification. We also relate the Cuntz semigroup with the Elliott invariant.

Definition 9.1. Let A be a unital C*-algebra. The *Elliott invariant* of A, denoted by $\mathrm{Ell}(A)$, consists of the 4-tuple

$$\mathrm{Ell}(A) = \big((\mathrm{K}_0(A), [1_A]), \mathrm{K}_1(A), \mathrm{T}(A), r_A\big),$$

where $r_A\colon \mathrm{K}_0(A) \times \mathrm{T}(A) \to \mathbb{R}$ is the pairing between K-theory and traces, defined as $r_A([p], \tau) = \tau(p)$ for all projections $[p] \in \mathrm{K}_0(A)$ and all $\tau \in \mathrm{T}(A)$.

It should be pointed out that Elliott's original formulation also included the positive cone $\mathrm{K}_0(A)^+$ of $\mathrm{K}_0(A)$ as part of the invariant. The modification we make here is inspired by the most recent approach to classification [31], and has the following convenient consequence:

Remark 9.2. Let A be a unital C*-algebra. Then $\mathrm{Ell}(A) \cong \mathrm{Ell}(A \otimes \mathcal{Z})$.

For $\mathcal{Z}$-stable C*-algebras, $\mathrm{K}_0(A)^+$ can be recovered from the remaining parts of $\mathrm{Ell}(A)$ as follows:

$$\mathrm{K}_0(A)^+ = \{x \in \mathrm{K}_0(A)\colon r_A(x, \tau) > 0 \text{ for all } \tau \in \mathrm{T}(A)\} \cup \{0\}.$$

Thus, for $\mathcal{Z}$-stable C*-algebras, there is no loss of information when dropping $\mathrm{K}_0(A)^+$ from the invariant. For the sake of this discussion, denote by $\widetilde{\mathrm{Ell}}(A)$ the invariant obtained from $\mathrm{Ell}(A)$ by adding the positive cone $\mathrm{K}_0(A)^+$ of $\mathrm{K}_0(A)$ as part of the invariant. While $\mathrm{Ell}(A) \cong \mathrm{Ell}(A \otimes \mathcal{Z})$ for *every* unital C*-algebra A, this is no longer true if one considers $\widetilde{\mathrm{Ell}}$ instead, as $\mathcal{Z}$-stable C*-algebras have weakly unperforated K_0-groups; see [50].

The Elliott conjecture originally predicted that $\widetilde{\mathrm{Ell}}$ would be a complete invariant for the class of simple, separable, unital, nuclear C*-algebras. The counterexamples to this conjecture were obtained by Rørdam first, and later by Toms, which is the one we present below. These examples showed that $\mathcal{Z}$-stability is not automatic for simple, separable, nuclear C*-algebras, and this strongly suggested the need to add this condition as an assumption in the conjecture. After many years of work by many researchers, which culminated in [61, 70, 51, 52, 41, 90], the following impressive classification theorem was obtained; see also [101, Theorem D]. Note that, since $\mathcal{Z}$-stability is assumed, $\widetilde{\mathrm{Ell}}$ can be replaced by Ell in the following statement:

Theorem 9.3. Let A and B be simple, separable, unital, nuclear, $\mathcal{Z}$-stable C*-algebras satisfying the UCT. Then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.

Here, the UCT stands for the so-called Universal Coefficient Theorem and, in the theorem above, its assumption is potentially vacuous. This is one of the most important open problems in the area.

Let us turn the attention now to the construction offered by Toms. This was in turn based on previous work of Villadsen; see [97, 98]. Let $(k_i)_{i\in\mathbb{N}}$ and $(n_i)_{i\in\mathbb{N}}$ be sequences of natural numbers, to be specified later. For each $i \in \mathbb{N}$, set $N_i = \prod_{j\leq i} n_j$ and

$$A_i = M_{k_i} \otimes C\left([0,1]^{6N_i}\right).$$

Identify $[0,1]^{6N_i}$ with $\left([0,1]^{6N_{i-1}}\right)^{n_i}$ and for each i and l such that $1 \leq l \leq n_i$, let

$$\pi_l^{(i)}\colon [0,1]^{6N_{i+1}} \to [0,1]^{6N_i}$$

be the coordinate projection, given by $\pi_l^{(i)}(x_1, \ldots, x_{n_i}) = x_l$ for all $(x_1, \ldots, x_{n_i}) \in [0,1]^{6N_i}$. For ease of notation, write $X_i = [0,1]^{6N_i}$. For each $i \in \mathbb{N}$, choose a dense sequence $(z_l^{(i)})$ in X_i, and choose points $x_1^{(i)}, \ldots, x_i^{(i)} \in X_i$ by setting $x_i^{(i)} = z_i^{(i)}$ and, if $1 \leq j \leq i-1$, choose $x_j^{(i)}$ such that $\pi_1^{(j)}\pi_1^{(j+1)} \ldots \pi_1^{i-2}\pi_1^{i-1}(x_j^{(i)}) = z_{i+1-j}^{(i)}$.

Let us define $\phi_{i-1}\colon A_{i-1} \to A_i$ as follows. Given $f \in A_{i-1}$ and $x \in [0,1]^{6N_i}$, set

$$\phi_{i-1}(f)(x) = \mathrm{diag}\Big(f\big(\pi_1^{(i)}(x)\big), \ldots, f\big(\pi_{n_i}^{(i)}(x)\big), f\big(x_1^{(i-1)}\big), \ldots, f\big(x_{i-1}^{(i-1)}\big)\Big).$$

We now choose n_i in such a way that n_i is much larger than i as $i \to \infty$ and such that for each $r \in \mathbb{N}$, there is i_0 with $r|(n_{i_0} + i_0)$. Set $n_1 = 1$, $k_1 = 4$ and $k_{i+1} = k_i(i + 6N_i)$, and let $A = \varinjlim(A_i, \phi_i)$.

Proposition 9.4. The C*-algebra A just constructed is simple, separable, unital, nuclear, satisfies the UCT, has real rank one and stable rank one, and $A \otimes \mathcal{Z}$ is isomorphic to an AI algebra.

Proof. The choice of the points $x_j^{(i)}$, for $1 \leq j \leq i$ and for each i, ensure that A is simple, as the arguments in [97] show. That A is separable, unital, nuclear, and that it satisfies the UCT, are clear by construction.

Since X_i is contractible for all $i \in \mathbb{N}$, we have

$$(\mathrm{K}_0(A_i), [1_{A_i}], \mathrm{K}_1(A_i)) \cong (\mathbb{Z}, k_i, \{0\}).$$

This, coupled with the fact that K-theory is continuous and the choice of $(n_i)_{i\in\mathbb{N}}$, ensures that $(\mathrm{K}_0(A), [1_A], \mathrm{K}_1(A)) \cong (\mathbb{Q}, 1, \{0\})$. It follows that there is a simple AI-algebra B whose Elliott invariant is that of A. Moreover, both $A \otimes \mathcal{Z}$ and B satisfy the assumptions of Theorem 9.3, so they are isomorphic.

By the results in [97], the real rank and the stable rank of A are both equal to one, and thus the same is true for B. □

Proposition 9.5. The Cuntz semigroup of the C*-algebra A in Proposition 9.4 is not almost unperforated.

Proof. It is enough to find positive elements $x, y \in A_1$ such that for all $i \in \mathbb{N}$ and some $\delta > 0$, we have $11[\phi_{1,i}(x)] \leq 10[\phi_{1,i}(y)]$ in $\mathrm{Cu}(A_i)$, but $\|r\phi_{1,i}(y)r^* - \phi_{1,i}(x)\| > \delta$ for all $r \in A_i$.

To give the idea, we do this for $i = 1$, that is, we show that $\mathrm{Cu}(A_1)$ is not almost unperforated, and partly reproduce the argument in [91, Proof of Theorem 1.1]. Notice that $A_1 = C([0,1]^3 \times [0,1]^3, M_4)$. Set

$$S = \left\{x \in [0,1]^3 \colon \tfrac{1}{8} < \mathrm{dist}\big(x, \big(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\big)\big) < \tfrac{3}{8}\right\},$$

so that $M_4(C_0(S \times S))$ is a hereditary subalgebra of A_1.

Let ξ be a line bundle over S^2 with nonzero Euler class, and use θ_1 to denote the trivial line bundle. Since $\xi \times \xi$ does not have zero Euler class, θ_1 is not a sub-bundle of $\xi \times \xi$ over $S^2 \times S^2$ (see [97, Lemma 1]). Considering $\xi \times \xi$ and θ_1 as projections in $M_4(C_0(S^2 \times S^2))$, we have $\|x(\xi \times \xi)x^* - \theta_1\| \geq \frac{1}{2}$ for all $x \in M_4(C_0(S^2 \times S^2))$. Stability properties of vector bundles yield, on the other hand, that $11[\theta_1] \leq 10[\xi \times \xi]$. Set

$$S' = \left\{x \in S \colon \mathrm{dist}\big(x, \big(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\big)\big) < \tfrac{1}{4}\right\} \subseteq S$$

and let $f \in A_1$ be a positive scalar function supported on $S \times S$ which equals one on $S' \times S'$. Let ρ denote the projection of $\overline{S}$ onto S'. One has, by restricting from $S \times S$ to $S' \times S'$, that

$$\|xf(\rho^*(\xi) \times \rho^*(\xi))x^* - f\theta_1\| \geq \tfrac{1}{2},$$

for any $x \in A_1$, where ρ^* is the pullback of ξ via ρ, and also

$$11[f\theta_1] \leq 10[f(\rho^*(\xi) \times \rho^*(\xi))]$$

in $\mathrm{Cu}(A_1)$. This shows that $\mathrm{Cu}(A_1)$ is not almost unperforated. □

Theorem 9.6. The C*-algebras A and $B = A \otimes \mathcal{Z}$ from Proposition 9.4 are not isomorphic, yet they have the same stable and real rank, and satisfy $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.[10]

[10] In fact, they even satisfy $\widetilde{\mathrm{Ell}}(A) \cong \widetilde{\mathrm{Ell}}(B)$, which is a stronger statement since A is not $\mathcal{Z}$-stable.

Proof. We know from Proposition 9.5 that $\mathrm{Cu}(A)$ is not almost unperforated. On the other hand, $\mathrm{Cu}(B)$ is almost unperforated as B is $\mathcal{Z}$-stable, by Theorem 8.4. Hence, A and B cannot be isomorphic.

The argument in Proposition 9.4 shows that A and B have stable rank and real rank one, and they have the same Elliott invariant by Remark 9.2. □

We now proceed to discuss the connection of the Cuntz semigroup with the Elliott invariant for classifiable C*-algebras. The first connection is given by the following computation of the Cuntz semigroup of a $\mathcal{Z}$-stable C*-algebra, obtained in [26, 27]. For each element $x = [p] \in \mathrm{V}(A)$, we denote by $\widehat{x}\colon \mathrm{QT}(A) \to \mathbb{R}_{++}$ the continuous function defined by $\widehat{x}(\tau) = \tau(p)$ for $\tau \in \mathrm{QT}(A)$. We also denote by $\mathrm{LAff}(\mathrm{QT}(A))_{++}$ the semigroup of lower semicontinuous, affine functions defined on $\mathrm{QT}(A)$ with values on $(0, \infty]$.

Theorem 9.7. Let A be a simple, separable, unital, stably finite $\mathcal{Z}$-stable C*-algebra. Then

$$\mathrm{Cu}(A) \cong \underbrace{\mathrm{V}(A)}_{\text{compacts}} \sqcup \mathrm{LAff}(\mathrm{QT}(A))_{++},$$

with addition and order defined as follows:

(i) The addition in $\mathrm{V}(A)$ is the usual addition and in $\mathrm{LAff}(\mathrm{QT}(A))_{++}$ is given by pointwise addition of functions. If $x \in \mathrm{V}(A)$ and $f \in \mathrm{LAff}(\mathrm{QT}(A))_{++}$, then $x + f = \widehat{x} + f$.

For $x \in \mathrm{V}(A)$ and $f \in \mathrm{LAff}(\mathrm{QT}(A))_{++}$, we have

(ii) $x \leq f$ if $\widehat{x}(\tau) < f(\tau)$ for every $\tau \in \mathrm{QT}(A)$.

(iii) $f \leq x$ if $f(\tau) \leq \widehat{x}(\tau)$ for every $\tau \in \mathrm{QT}(A)$.

Remark 9.8. For stably finite, nuclear $\mathcal{Z}$-stable C*-algebras, the above allows one to recover $\mathrm{K}_0(A)$ as the Grothendieck group of $\mathrm{V}(A) = \mathrm{Cu}(A)_c$, as well as $\mathrm{QT}(A) = F(\mathrm{Cu}(A))$. The pairing r_A can be recovered then by evaluation of a functional on a compact element. In particular, the pair $(\mathrm{Cu}(A), \mathrm{K}_1(A))$ is equivalent to $\mathrm{Ell}(A)$.

In contrast to Theorem 9.7, the Cuntz semigroup records essentially *no* information in the purely infinite case. Indeed, if A is simple and purely inifnite, then $\mathrm{Cu}(A) \cong \{0, \infty\}$ regardless of the K_0-group of A. This is easy to see: given $a, b \in A_+$ nonzero, find $r \in A$ such that $rbr^* = a$, which implies $a \precsim b$. It follows that all nonzero positive elements in (matrices over) A are Cuntz-equivalent.

As we see from Theorem 9.7, the Cuntz semigroup of A does not in general contain any information about its K_1-group. We remark that, in the non-unital case, it is not even obvious that K_0 is encoded. One may have, for example, a simple stably projectionless C*-algebra A, hence $\mathrm{V}(A) = 0$, yet $\mathrm{K}_0(A) \neq 0$. In this case we would consider $\mathrm{Cu}(\tilde{A})$ and recover $\mathrm{K}_0(A)$ from $\mathrm{V}(\tilde{A})$.

We may also recover $\mathrm{K}_1(A)$, but using the Cuntz semigroup of a different algebra, namely $A \otimes C(\mathbb{T})$.[11] We will need the following construction.

Let S be a Cu-semigroup. Assume that the subset S_{nc} of non-compact elements is an absorbing subsemigroup, in the sense that $S_{nc} + S \subseteq S_{\mathrm{nc}}$. (This is always the case if $S = \mathrm{Cu}(A)$ for a simple, separable, unital, stably finite $\mathcal{Z}$-stable C*-algebra by Theorem 9.7.) Denote by S_{c} the subsemigroup of compact elements and $S_{\mathrm{c}}^* = S_{\mathrm{c}} \setminus \{0\}$. Let G be an abelian group and consider the semigroup

$$S_G = (\{0\} \sqcup (G \times S_{\mathrm{c}}^*)) \sqcup S_{\mathrm{nc}}\,,$$

with natural operations in both components, and $(g, x) + y = x + y$ whenever $x \in S_{\mathrm{c}}^*$, $y \in S_{\mathrm{nc}}$, and $g \in G$. This semigroup can be ordered as follows:

[11] Thinking of the Cuntz semigroup as a refined version of K_0, the idea behind this is that $\mathrm{K}_0(A \otimes C(\mathbb{T})) \cong \mathrm{K}_0(A) \oplus \mathrm{K}_1(A)$.

(i) For $x, y \in S_{\mathrm{c}}^*$, and $g, h \in G$, we have $(g, x) \leq (h, y)$ if and only if $x = y$ and $g = h$, or else $x < y$.
(ii) For $x \in S_{\mathrm{c}}^*$, $y \in S_{\mathrm{nc}}$, $g \in G$, (g, x) is comparable with y if x is comparable with y.

The proof of the following is left as an exercise.

Lemma 9.9. Let S be an object of **Cu** such that S_{nc} is an absorbing subsemigroup. If G is an abelian group, then S_G is also an object of **Cu**.

For a C*-algebra A, let us denote $\mathrm{Cu}_{\mathbb{T}}(A) = \mathrm{Cu}(A \otimes C(\mathbb{T}))$.

Theorem 9.10. ([3, Theorem 3.8]). Let A be a separable, unital, stably finite, $\mathcal{Z}$-stable C*-algebra. Then, there is an order-isomorphism

$$\mathrm{Cu}_{\mathbb{T}}(A) \cong (\{0\} \sqcup (\mathrm{K}_1(A)) \times \mathrm{V}(A)^*) \sqcup \mathrm{Lsc}_{\mathrm{nc}}(\mathbb{T}, \mathrm{Cu}(A))\,,$$

where $\mathrm{V}(A)^* = \mathrm{V}(A) \setminus \{0\}$, and where the right hand side is ordered as above.

Let us interpret this in terms of the theory of classification. To this end, let us write **Ell** to denote the category whose objects are 4-tuples of the form

$$((G_0, u), G_1, X, r)\,,$$

where G_0 is a (countable) abelian group with distinguished element u, G_1 is a (countable) abelian group, X is a (metrizable) Choquet simplex, $r\colon X \to \mathrm{St}(G_0, u)$ is an affine map, where $\mathrm{St}(G_0, u)$ denotes the state space of (G_0, u), such that, if we set

$$G_0^+ = \{g \in G_0 \colon r(x)(g) > 0 \text{ for all } x \in X\} \cup \{0\},$$

then (G_0, G_0^+, u) is a simple, partially ordered group with order unit u.

The morphisms in **Ell** between $((G_0, u), G_1, X, r)$ and $((H_0, v), H_1, Y, s)$ are given by triples $(\theta_0, \theta_1, \gamma)$, where $\theta_0\colon G_0 \to H_0$ is a morphism of groups with $\theta_0(u) = v$, the map $\theta_1\colon G_1 \to H_1$ is a morphism of groups, and $\gamma\colon Y \to X$ is an affine and continuous map such that $r \circ \gamma = \theta_0^* \circ s$, where $\theta_0^*\colon \mathrm{St}(H_0, v) \to \mathrm{St}(G_0, u)$ is the naturally induced map at the level of states.

Let $\mathbf{C}^*_{\mathcal{Z}}$ denote the class of simple, separable, unital, nuclear, finite $\mathcal{Z}$-stable C*-algebras. The Elliott invariant then naturally defines a functor

$$\mathrm{Ell}\colon \mathbf{C}^*_{\mathcal{Z}} \to \mathbf{Ell}.$$

We now define

$$\mathrm{F}\colon \mathbf{Ell} \to \mathbf{Cu}$$

as follows. If $\mathcal{E} = ((G_0, u), G_1, X, r)$ is an object of **Ell** and G_0^+ is defined as above, then notice first that $G_0^+ \sqcup \mathrm{LAff}(X)_{++}$ is an object of **Cu** (see, for example, [1, Lemma 6.3]). Set $G_0^{++} = G_0^+ \setminus \{0\}$ and

$$\mathrm{F}(\mathcal{E}) = (\{0\} \sqcup (G_1 \times G_0^{++})) \sqcup \mathrm{Lsc}_{\mathrm{nc}}(\mathbb{T}, G_0^+ \sqcup \mathrm{LAff}(X)_{++})\,,$$

with addition given by $(g + f)(x) = r(x)(g) + f(x)$, for all $g \in G$, $f \in \mathrm{LAff}(X)$ and $x \in X$. It follows from Lemma 9.9 that $\mathrm{F}(\mathcal{E})$ is also an object of **Cu**.

Lemma 9.11. F is a functor, and its corestriction $\mathrm{F}\colon \mathbf{Ell} \to \mathrm{F}(\mathbf{Ell})$ is full, faithful and dense.

Proof. We only check that, if

$$(\theta_0, \theta_1, \gamma)\colon ((G_0, u), G_1, X, r) \to ((H_0, v), H_1, Y, s)$$

is a morphism in **Ell** and $f\colon \mathbb{T} \to G_0^+ \sqcup \mathrm{LAff}(X)_{++}$ is non-compact, then $(\theta_0 \sqcup \gamma^*) \circ f\colon \mathbb{T} \to H_0^+ \sqcup \mathrm{LAff}(Y)_{++}$ is also non-compact. Here

$$\theta_0 \sqcup \gamma^*\colon G_0^+ \sqcup \mathrm{LAff}(X)_{++} \to H_0^+ \sqcup \mathrm{LAff}(Y)_{++}$$

is defined as θ_0 on G_0^+ and γ^* on $\mathrm{LAff}(X)_{++}$.

By way of contradiction, if $(\theta_0 \sqcup \gamma^*) \circ f$ is compact, then there is $h \in H_0^+$ such that $\theta_0(f(\mathbb{T})) = \{h\}$ and $f(\mathbb{T}) \subseteq G_0^+$. As f is non-compact and lower semicontinuous, there are $s, t \in \mathbb{T}$ with $f(t) < f(s)$, whence $f(s) - f(t) \in G_0^{++}$ is an order-unit. Thus, there exists $n \in \mathbb{N}$ with $f(s) \leq n(f(s) - f(t))$. After applying θ_0, we obtain that $h \leq 0$, hence $h = 0$. But this is not possible since, as f is not constant, it takes some non-zero value a, which will be an order-unit with $\theta_0(a) = 0$, contradicting that $\theta_0(u) = v$. □

Remarks 9.12. In Lemma 9.11, we have considered $\mathrm{F}(\mathbf{Ell})$. We note that the image of a functor G is a category when G is faithful, in the sense that $\mathrm{G}(f) = \mathrm{G}(g)$ implies $f = g$. The reason for this is that if $\mathrm{G}(f)\colon \mathrm{G}(X) \to \mathrm{G}(Y)$ and $\mathrm{G}(g)\colon \mathrm{G}(Z) \to \mathrm{G}(V)$, with $\mathrm{G}(Y) = \mathrm{G}(Z)$, then one has to verify that $\mathrm{G}(g)\mathrm{G}(f)\colon \mathrm{G}(X) \to \mathrm{G}(V)$ has the form $\mathrm{G}(h)$ for some h. Assuming G faithful, from $\mathrm{G}(Y) = \mathrm{G}(Z)$ one has $\mathrm{G}(\mathrm{id}_Y) = \mathrm{id}_{\mathrm{G}(Y)} = \mathrm{id}_{\mathrm{G}(Z)} = \mathrm{G}(\mathrm{id}_Z)$, and thus $\mathrm{id}_Y = \mathrm{id}_Z$. Therefore $Y = Z$, and in this way we may choose $h = gf$.

That F in Lemma 9.11 is faithful was established in [3].

As a consequence, we get the following.

Theorem 9.13 ([89])**.** Upon restriction to the class of unital, simple, separable and stably finite $\mathcal{Z}$-stable algebras, Ell is a classifying functor if, and only if, so is $\mathrm{Cu}_{\mathbb{T}}$.

Proof. Since $\mathrm{F}\colon \mathbf{Ell} \to \mathrm{F}(\mathbf{Ell})$ is a full, faithful and dense functor by Lemma 9.11, it is a general fact in category theory that it yields an equivalence of categories, so that there exists another functor $\mathrm{G}\colon \mathrm{F}(\mathbf{Ell}) \to \mathbf{Ell}$ such that $\mathrm{F} \circ \mathrm{G}$ and $\mathrm{G} \circ \mathrm{F}$ are naturally equivalent to the respective identities. This, together with Theorem 9.10 and [26, Corollary 5.7] implies that there are natural equivalences of functors

$$\mathrm{F} \circ \mathrm{Ell} \simeq \mathrm{Cu}_{\mathbb{T}} \text{ and } \mathrm{Ell} \simeq \mathrm{G} \circ \mathrm{Cu}_{\mathbb{T}},$$

which implies the result. □

We have chosen the algebra $C(\mathbb{T})$ to be tensored with our target algebra A above since the compact part of $A \otimes C(\mathbb{T})$ contains, for the class under scrutiny, the relevant information on K_0 and K_1. It would also be interesting to explore what would happen under tensoring instead with the unique classifiable simple unital C*-algebra B with a unique trace and $\mathrm{K}_0(B) \cong \mathrm{K}_1(B) \cong \mathbb{Z}$ with $[1_B]$ corresponding to 1. An advantage of this approach is that one would stay within the classifiable class for an already classifiable algebra A.

10. Additional axioms and properties

In this section, we introduce two important new axioms and a property for Cu-semigroups. As it turns out, the axioms are satisfied by the Cuntz semigroup of any C*-algebra, whereas the property, that measures cancellation in the Cuntz semigroup, holds for C*-algebras of stable rank one.

Definition 10.1. Let S be a Cu-semigroup. We say that S has *weak cancellation* provided $x + z \ll y + z$ in S implies $x \ll y$, for all $x, y, z \in S$.

Not every Cuntz semigroup is weakly cancellative. If, for example, A is a purely infinite simple C*-algebra, then $\mathrm{Cu}(A) = \{0, \infty\}$, where necessarily ∞ is compact. Hence $\infty + \infty \ll 0 + \infty$, but it is not true that $\infty \ll 0$. It is easy to show that weak cancellation passes to quotients, hence another example where weak cancellation fails in the non purely infinite setting may be obtained by taking a stably finite C*-algebra A with a purely infinite simple quotient, for example the cone over $\mathcal{O}_2$. An example with finite stable rank is the Toeplitz algebra $\mathcal{T}$, since it contains a non-unitary isometry s, so that $p := ss^*$ is a projection satisfying $p \oplus (1-p) = p + (1-p) = 1 \sim p$ with $p \neq 1$.

We record below an equivalent formulation for weak cancellation that is repeatedly used in the literature. The proof is left as an exercise; see [5, Lemma 2.5] for the argument.

Lemma 10.2. Let S be a Cu-semigroup. The following conditions are equivalent:

(i) S has weak cancellation.
(ii) If $x, y, z \in S$ safisfy $x + z \ll y + z$, then $x \leq y$.
(iii) If $x, y, z, z' \in S$ satisfy $x + z \leq y + z'$ and $z' \ll z$, then $x \leq y$.

The following theorem is due to Rørdam and Winter [82].

Theorem 10.3. Let A be a C*-algebra of stable rank one. Then $\mathrm{Cu}(A)$ has weak cancellation.

Proof. By part (iii) of Proposition 2.14, we may assume that A is stable. Let x, y, z in $\mathrm{Cu}(A)$ satisfy $x + z \ll y + z$. Assume first that z is compact, that is, $z \ll z$. Choose elements $a, b \in A_+$ with $x = [a]$, $y = [b]$, and use Proposition 4.14 to choose a projection $p \in A$ with $z = [p]$. We may assume that a, b are both orthogonal to p. Let $\varepsilon > 0$. Using Theorem 3.7, choose a unitary u (in the unitization of A) such that $u((a-\varepsilon)_+ + p)u^* \in A_{b+p}$. Then p and upu^* are projections in A_{b+p} whose Cuntz classes in A agree. Using that A_{b+p} is a hereditary subalgebra of A, one readily checks that their Cuntz classes in A_{b+p} also agree (this is almost identical to the proof of Lemma 5.2). Note that A_{b+p} has stable rank by part (ii) of Proposition 2.14, and hence is stably finite by part (i) of the same lemma. By Lemma 2.8 and the comments after it, we deduce that $p \sim_{\mathrm{MvN}} q$ in A_{b+p}. Using again that A_{b+p} has stable rank one, part (iv) of Proposition 2.14 gives us a unitary v in the unitization of A_{b+p} such that $vpv^* = upu^*$.

Observe now that $v^*u(a-\varepsilon)_+u^*v$ belongs to A_{b+p} and is orthogonal to p. Thus

$$v^*u(a-\varepsilon)_+u^*v \in (1-p)A_{p+b}(1-p) = A_b.$$

This shows that $(a-\varepsilon)_+ \precsim b$ and since ε is arbitrary, we conclude that $a \precsim b$, that is, $x \leq y$.

For the general case, choose $z' \ll z$ such that $x + z \leq y + z'$. Choose representatives as before $x = [a]$, $y = [b]$, $z = [c]$. We may assume that $z' = [(c-\varepsilon)_+]$ for some $\varepsilon > 0$, and so

$$a \oplus c \precsim b \oplus (c-\varepsilon)_+ \tag{10.1}$$

Upon rescaling c, which does not change Cuntz classes by Corollary 2.3, we may assume that $\|c\| = 1$. Define

$$h_\varepsilon(t) = \begin{cases} 1, & \text{if } t = 0 \\ \text{linear}, & \text{if } t \in [0, \varepsilon] \\ 0, & \text{if } t \geq \varepsilon. \end{cases}$$

Then $\varepsilon \leq c + h_\varepsilon(c) \leq 1$. Using part (ii) of Corollary 2.3 at the second step and using Lemma 3.2 at the third, we get

$$(c-\varepsilon)_+ + h_\varepsilon(c) \leq c + h_\varepsilon(c) \sim 1 \precsim c \oplus h_\varepsilon(c). \tag{10.2}$$

On the other hand, $(c-\varepsilon)_+ \perp h_\varepsilon(c)$ and thus Lemma 3.2 implies that

$$(c-\varepsilon)_+ \oplus h_\varepsilon(c) \sim (c-\varepsilon)_+ + h_\varepsilon(c) \tag{10.3}$$

Combining these things, we get

$$\begin{aligned} a \oplus 1 &\overset{(10.2)}{\precsim} a \oplus c \oplus h_\varepsilon(c) \\ &\overset{(10.1)}{\precsim} b \oplus (c-\varepsilon)_+ \oplus h_\varepsilon(c) \\ &\overset{(10.3)}{\precsim} b \oplus (c-\varepsilon)_+ + h_\varepsilon(c) \\ &\overset{(10.2)}{\precsim} b \oplus 1, \end{aligned}$$

and therefore $a \precsim b$ by the first part of the proof. □

Remark 10.4. Notice that if S is a Cu-semigroup with weak cancellation, then S has cancellation of compact elements. In particular, if A is a C*-algebra of stable rank one, then its Cuntz semigroup $\mathrm{Cu}(A)$ has cancellation of compact elements (in fact, a close inspection of the proof of Theorem 10.3 reveals one proves this first before concluding weak cancellation).

An abelian monoid S is said to be *algebraically ordered* provided $x \leq y$ in S exactly when there is $z \in S$ with $x + z = y$. The order in a Cu-semigroup is in general not algebraic (consider, for example, the Cuntz semigroup of $\mathcal{Z}$; see Theorem 7.3), and the axiom below measures how far this is from being the case.

Definition 10.5. Let S be a Cu-semigroup. We say that S satisfies (O5) (or has *almost algebraic order*) if, whenever $x + z \leq y$ and $x' \ll x$, $z' \ll z$, there is $w \in S$ such that

$$x' + w \leq y \leq x + w \quad \text{and} \quad z' \leq w.$$

The formulation above is taken from [8], and has the advantage that it behaves well with respect to inductive limits of Cu-semigroups; see [8, Theorem 4.5].

This axiom appears in a different form in other papers, namely by taking $z = 0$. In fact, this was the original formulation in [82], which therefore reads: whenever $x \leq y$ and $x' \ll x$, there is $w \in S$ such that $x' + w \leq y \leq x + w$. Let us temporarily refer to this version as axiom $\widetilde{\text{(O5)}}$.

Lemma 10.6. Let S be a Cu-semigroup. Then (O5) implies $\widetilde{\text{(O5)}}$ and, in the presence of weak cancellation, the converse holds.

Proof. The first part of the statement is trivial. Assume, for the converse, that S is a Cu-semigroup that has weak cancellation and satisfies $\widetilde{\text{(O5)}}$. We are to show that S satisfies (O5). To this end, let $x', x, z', z, y \in S$ satisfy $x + z \leq y$ and $x' \ll x$, $z' \ll z$. First choose $x' \ll x'' \ll x$, $z' \ll z'' \ll z$, and $y' \ll y$ such that $x'' + z'' \ll y'$. Since $x' \ll x'' \leq y$, by $\widetilde{\text{(O5)}}$ there is $w \in S$ such that $x' + w \leq y \leq x'' + w$. Using weak cancellation on the inequality

$$x'' + z'' \ll y' \ll y \leq x'' + w,$$

we obtain $z'' \ll w$. Thus $z' \leq w$. On the other hand $x' + w \leq y \leq x'' + w \leq x + w$, as desired. □

Remark 10.7. Suppose that S satisfies (O5), and let $x \in S_c$ and $y \in S$ satisfy $x \leq y$. Applying (O5) to $x \ll x \leq y$, we find $w \in S$ with $x + w = y$. This shows that compact elements can always be complemented and thus the subsemigroup S_c of compact elements is algebraically ordered. The same is true if one assumes the weaker $\widetilde{\text{(O5)}}$.

The reason why (O5) is significant is recorded below:

Proposition 10.8. Let A be a C*-algebra. Then $\mathrm{Cu}(A)$ satisfies (O5).

Proof. We prove the simpler version $\widetilde{\text{(O5)}}$, mostly following the original proof in [82]. (For the general argument, see [8, Proposition 4.6].) Thus assume that $x' \ll x \leq y$ in $\mathrm{Cu}(A)$, and set $x = [a]$, $y = [b]$. For simplicity we assume that $x' = [(a - 2\varepsilon)_+]$ for some $\varepsilon > 0$.

By Theorem 2.7, there is $v \in A$ such that $(a - \varepsilon)_+ = v^*v$ and $vv^* \in A_b$. Let h_ε be defined as in the proof of Theorem 10.3, so that $h_\varepsilon(vv^*) \perp (vv^* - \varepsilon)_+$ and $\varepsilon \leq vv^* + h_\varepsilon(vv^*)$. The latter inequality implies that $\varepsilon b \leq b^{\frac{1}{2}}(vv^* + h_\varepsilon(vv^*))b^{\frac{1}{2}}$ and thus

$$b \precsim b^{\frac{1}{2}}(vv^* + h_\varepsilon(vv^*))b^{\frac{1}{2}}. \tag{10.4}$$

Let $c = h_\varepsilon(vv^*)bh_\varepsilon(vv^*)$ and set $w = [c] \in \mathrm{Cu}(A)$. Note that

$$c \sim b^{\frac{1}{2}}h_\varepsilon(vv^*)^2 b^{\frac{1}{2}} \tag{10.5}$$

by part (iv) of Corollary 2.3. Since both $(vv^* - \varepsilon)_+$ and c belong to A_b, we have

$$(vv^* - \varepsilon)_+ + c \precsim b \tag{10.6}$$

by Proposition 2.4. Using Lemma 2.6 at the second step, and using Lemma 3.2 at the third step, we get

$$(a - 2\varepsilon)_+ \oplus c = (v^*v - \varepsilon)_+ \oplus c \sim (vv^* - \varepsilon)_+ \oplus c \sim (vv^* - \varepsilon)_+ + c \overset{(10.6)}{\precsim} b,$$

so that $x' + w \leq y$.

It remains to show that $b \precsim a \oplus c$, which gives $y \leq x + w$. Using Lemma 3.2 at the fourth step, and using $v^*v = (a - \varepsilon)_+$ and part (iv) of Corollary 2.3 at the fifth step, we get

$$\begin{aligned} b &\overset{(10.4)}{\precsim} b^{\frac{1}{2}}(vv^* + h_\varepsilon(vv^*)^2)b^{\frac{1}{2}} \\ &= b^{\frac{1}{2}}vv^*b^{\frac{1}{2}} + b^{\frac{1}{2}}h_\varepsilon(vv^*)^2 b^{\frac{1}{2}} \\ &\overset{(10.5)}{\precsim} vv^* + c \precsim vv^* \oplus c \\ &\sim (a - \varepsilon)_+ \oplus c \precsim a \oplus c. \end{aligned}$$

□

An abelian monoid S is said to have *Riesz decomposition* if whenever $x \leq y + z$ in S, there are elements $x_1, x_2 \in S$ with $x = x_1 + x_2$ and $x_1 \leq y$, $x_2 \leq z$. Again, Cu-semigroups do not in general have Riesz decomposition (and $\mathrm{Cu}(\mathcal{Z})$ is an example). The following axiom measures a certain degree of Riesz decomposition in a Cu-semigroup.

Definition 10.9. Let S be a Cu-semigroup. We say that S satisfies (O6), or that it has *almost Riesz decomposition*, if whenever $x', x, y, z \in S$ satisfy $x' \ll x \leq y + z$, then there are $s, t \in S$ with

$$x' \leq s + t, \quad s \leq x, y, \quad \text{and} \quad t \leq x, z.$$

Again, (O6) is satisfied by the Cuntz semigroup of every C*-algebra; see [74, Proposition 5.1.1].

Proposition 10.10. Let A be a C*-algebra. Then $\mathrm{Cu}(A)$ satisfies (O6).

Proof. We may assume that A is stable. Suppose that $x \leq y + z$ in $\mathrm{Cu}(A)$, and $x' \ll x$. Choose representatives $a, b, c \in A$ such that $x = [a]$, $y = [b]$, and $z = [c]$. We may assume that $b \perp c$ and for simplicity we also assume that $x' = [(a - \varepsilon)_+]$ for some $\varepsilon > 0$.

By Theorem 2.7, there are $v \in A$ and $\delta > 0$ such that $(a-\varepsilon)_+ = v^*v$ and $vv^* \in A_{(b+c-\delta)_+}$. Define a continuous function by

$$g_\delta(t) = \begin{cases} 0, & \text{if } t = 0 \\ \text{linear}, & \text{if } t \in [0,\delta] \\ 1, & \text{if } t > \delta. \end{cases}$$

Then $g_\delta(b+c)vv^* = vv^*$. Using at the first step that $g_\delta(b+c) = g_\delta(b) + g_\delta(c)$, which holds because $b \perp c$, and using Remark 5.3 at the second step, we have

$$\begin{aligned} & g_\delta(b+c)vv^*g_\delta(b+c) \\ &\qquad = g_\delta(b)vv^*g_\delta(b) + g_\delta(b)vv^*g_\delta(c) + g_\delta(c)vv^*g_\delta(b) + g_\delta(c)vv^*g_\delta(c) \\ &\qquad \leq 2(g_\delta(b)vv^*g_\delta(b) + g_\delta(c)vv^*g_\delta(c)). \end{aligned}$$

Using this at the third step and that $b \perp c$ again at the fourth step, we obtain

$$\begin{aligned} [(a-\varepsilon)_+] = [vv^*] &= [g_\delta(b+c)vv^*g_\delta(b+c)] \\ &\leq [g_\delta(b)vv^*g_\delta(b) + g_\delta(c)vv^*g_\delta(c)] \\ &= [g_\delta(b)vv^*g_\delta(b)] + [g_\delta(c)vv^*g_\delta(c)]. \end{aligned}$$

The proof is finished by setting $s := [g_\delta(b)vv^*g_\delta(b)] = [v^*g_\delta(b)^2v] \leq [(a-\varepsilon)_+], [b]$ and $t := [g_\delta(c)vv^*g_\delta(c)] \leq [(a-\varepsilon)_+], [c]$. □

11. C*-algebras with stable rank one

In this section we explore some properties satisfied by the Cuntz semigroups of C*-algebras with stable rank one. One obtains in particular solutions to three open problems on the Cuntz semigroup that, at the same time, reflect on the structure of such algebras. In these notes, we will discuss two of the three problems mentioned (see Sections 12 and 13). Large portions of this section are taken from [5].

We will need the following variation of the axiom (O6), introduced by Thiel in [85]:

Definition 11.1. Let S be a Cu-semigroup. We say that S satisfies (O6+) if whenever $x, y, z \in S$ satisfy $x \leq y+z$ and $u', u, v', v \in S$ are such that $u' \ll u \leq x, y$, and $v' \ll v \leq x, z$, then there are elements $s, t \in S$ such that

$$x \leq s + t, \quad u' \ll s \leq x, y, \quad \text{and} \quad v' \ll t \leq x, z.$$

It was shown in the proof of [85, Lemma 6.3] that (O6+) has a simpler equivalent formulation. We reproduce the argument below for convenience, since we shall use this later.

Lemma 11.2. Let S be a Cu-semigroup. Then S satisfies (O6+) if and only if the following (one-sided) property holds: whenever $x, y, z \in S$ satisfy $x \leq y + z$ and $u', u \in S$ are such that $u' \ll u \leq x, y$, then there is $s \in S$ such that $x \leq s + z$ and $u' \ll s \leq x, y$.

Proof. It is clear that (O6+) implies the property in the statement. Conversely, assume the above property and let $x, y, z, u', u, v', v \in S$ satisfy $x \leq y + z$, $u' \ll u \leq x$, and $v' \ll v \leq x, z$. Applying the property in the statement to $x \leq y + z$ and $u' \ll u \leq x, y$, we obtain $s \in S$ such that $x \leq s + z$ and $u' \ll s \leq x, y$. A second application of said property to $x \leq s + z$ and $v' \ll v \leq x, z$ yields an element $t \in S$ with $x \leq s + t$ and $v' \ll t \leq x, z$, as desired. □

The proof of the following is rather involved, hence we omit the details.

Theorem 11.3 ([85, Theorem 6.4])**.** Let A be a C*-algebra of stable rank one. Then $\mathrm{Cu}(A)$ satisfies (O6+).

Next, we observe that the assumption of stable rank one is necessary in Theorem 11.3. The following is [85, Example 6.7]

Example 11.4. The Cuntz semigroup of the C*-algebra $C(S^2)$ does not satisfy (O6+), and it is well-known that $\mathrm{sr}(C(S^2)) = 2$.

Proof. Denote by $\mathrm{Lsc}(S^2, \overline{\mathbb{N}})$ the monoid of lower semicontinuous functions from S^2 with values in $\overline{\mathbb{N}}$, equipped with pointwise order and addition. For any open subset $U \subseteq S^2$, we use $\mathbf{1}_U$ to denote the characteristic function of U, which is an element of $\mathrm{Lsc}(S^2, \overline{\mathbb{N}})$. Note that the non-compact elements of $\mathrm{Lsc}(S^2, \overline{\mathbb{N}})$ are given by

$$\mathrm{Lsc}(S^2, \overline{\mathbb{N}})_{\mathrm{nc}} = \mathrm{Lsc}(S^2, \overline{\mathbb{N}}) \setminus \{n\mathbf{1}_{S^2} : n \geq 1\}.$$

Using [74, Theorem 1.2] it is possible to show that

$$\mathrm{Cu}(C(S^2)) \cong (\mathbb{N}_{>0} \times \mathbb{Z}) \sqcup \mathrm{Lsc}(S^2, \overline{\mathbb{N}})_{\mathrm{nc}}.$$

Addition and order may be described as follows. Elements in each one of the components of the disjoint union are added as usual and ordered also as usual. If $(n, m) \in \mathbb{N}_{>0} \times \mathbb{Z}$ and $f \in \mathrm{Lsc}(S^2, \overline{\mathbb{N}})_{\mathrm{nc}}$, then $(n, m) + f = n\mathbf{1}_{S^2} + f$.

Next, $(n, m) \leq f$ if and only if $n\mathbf{1}_{S^2} \leq f$, and $f \leq (n, m)$ if and only if $f \leq n\mathbf{1}_{S^2}$.

Choose open subsets of $U, V \subseteq S^2$ with $\overline{U} \subseteq V$. Then $\mathbf{1}_U \ll \mathbf{1}_V$, and they are non-constant functions. We have

$$(1, 0) \leq (1, 1) + n\mathbf{1}_U \quad \text{and} \quad \mathbf{1}_U \ll \mathbf{1}_V \leq (1, 0), (1, 1).$$

To reach a contradiction, assume that $\mathrm{Cu}(C(S^2))$ satisfies (O6+). Then there is $s \in \mathrm{Cu}(C(S^2))$ such that $(1, 0) \leq s + \mathbf{1}_U$ and $\mathbf{1}_U \leq s \leq (1, 0), (1, 1)$.

Since $s \leq (1, 0), (1, 1)$ we see that $s \in \mathrm{Lsc}(S^2, \overline{\mathbb{N}})_{\mathrm{nc}}$ and thus there is $x \in S^2$ such that $s(x) = 0$. But since $\mathbf{1}_U \leq s$, we see that $x \notin U$, which implies that $(s+\mathbf{1}_U)(x) = 0$. Therefore $(1, 0)$ is not dominated by $s+\mathbf{1}_U$, which is impossible. □

Remark 11.5. It is natural to ask whether (O6+) follows from (O6) and weak cancellation (the latter, as we have shown in Theorem 10.3, holds for C*-algebras of stable rank one). That is however not the case. Again, $C(S^2)$ is a counterexample, since by the computation above one can show its Cuntz semigroup is weakly cancellative.

Our next goal is to show that Cuntz semigroups of C*-algebras of stable rank one admit infima which are compatible with addition, in the sense of the definition below. This will have important consequences later on.

Definition 11.6. A partially ordered set S is an *inf-semilattice* provided the order-theoretic infimum $x \wedge y$ exists for any elements $x, y \in S$. We say that an ordered semigroup is *inf-semilattice ordered* if it is an inf-semilattice and, for any $x, y, z \in S$, we have

$$(x + z) \wedge (y + z) = (x \wedge y) + z.$$

Lemma 11.7. Let S be a Cu semigroup which is an inf-semilattice and satisfies (O6+). If $x, y, z \in \mathrm{Cu}(A)$ satisfy $x \leq y + z$, then $x \leq (x \wedge y) + (x \wedge z)$.

Proof. Applying axiom (O6+) to $x \leq y + z$ with $u = u' = v = v' = 0$, we obtain $x \leq s + t$ for some $s \leq x, y$ and $t \leq x, z$. The existence of infima proves the lemma. □

The lemma below shows that axiom (O6+) is automatic for inf-semilattice *ordered* semigroups.

Lemma 11.8. An inf-semilattice ordered Cu-semigroup S satisfies (O6+).

Proof. Assume that $x, y, z \in S$ satisfy $x \leq y + z$ and $u', u, v', v \in S$ are such that $u' \ll u \leq x, y$, and $v' \ll v \leq x, z$. Let $s = x \wedge y$ and $t = x \wedge z$. Then clearly $u' \ll u \leq s$ and $v' \ll v \leq t$. Using the inf-semillatice condition repeatedly, we get

$$(x + x) \wedge (x + z) \wedge (x + y) \wedge (y + z) = (x + x \wedge y) \wedge (z + x \wedge y) = (x \wedge y) + (x \wedge z).$$

Using this at the second step, we conclude that

$$x \leq (2x) \wedge (x + z) \wedge (x + y) \wedge (y + z) = (x \wedge y) + (x \wedge z) = s + t. \qquad \square$$

As we outline in Theorem 11.17, the Cuntz semigroup of any C*-algebra of stable rank one is inf-semilattice ordered. Given the argument above, one would get a different proof that (O6+) is verified for the Cuntz semigorup of C*-algebras of stable rank one. However, (O6+) is used to show the inf-semilattice ordered property, so that raises the question of whether one can show that C*-algebras of stable rank one have inf-semilattice ordered Cuntz semigroups without using (O6+).

In preparation for Proposition 11.12 below, we need to briefly discuss a different picture of the Cuntz semigroup, as developed in [37].

Definition 11.9. A *(right) Hilbert module* over a C*-algebra A is a (right) A-module X together with a map $\langle \cdot, \cdot \rangle \colon X \times X \to A$ which is $\mathbb{C}$-linear on the second entry and satisfies

(i) $\langle x, ya \rangle = \langle x, y \rangle a$
(ii) $\langle x, y \rangle = \langle y, x \rangle^*$
(iii) $\langle x, x \rangle \geq 0$ and equals zero precisely when $x = 0$,

for all $x, y \in X$, $a \in A$, and such that X is complete with respect to the norm $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$.

The standard example of an A-Hilbert module is the C*-algebra A itself, with structure given by $\langle x, y \rangle = x^* y$. More generally, if $a \in A_+$, then $\overline{aA}$ is naturally a Hilbert A-module. We denote by $\ell^2(A)$ the so-called *standard Hilbert A-module*, which is given as

$$\ell^2(A) = \Big\{ (x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} A \colon \sum_{n \in \mathbb{N}} x_n^* x_n \text{ converges in the norm of } A \Big\},$$

with inner product defined as $\langle x, y \rangle = \sum_{n \in \mathbb{N}} x_n^* y_n$ for $x, y \in \bigoplus_{n \in \mathbb{N}} A$.

Definition 11.10. A Hilbert A-module X is said to be *countably generated* if there is a countable set $\{x_n \colon n \in \mathbb{N}\} \subseteq X$ such that $X = \overline{\sum_{n=1}^{\infty} x_n A}$.

A fundamental result in the theory of Hilbert modules is Kasparov's absorption theorem; see [65, Theorem 1.4.2].

Theorem 11.11 (Kasparov's theorem)**.** Let X be a countably generated Hilbert A-module. Then $X \oplus \ell^2(A) \cong \ell^2(A)$.

In [37], a notion of subequivalence between countably generated Hilbert A-modules, weaker than isomorphism, was introduced. By antisymmetrizing this subequivalence, a partially ordered semigroup was built, and it was shown that this semigroup is isomorphic to $\mathrm{Cu}(A)$. Under this isomorphism, a positive element $a \in (A \otimes \mathcal{K})_+$ corresponds to $\overline{a(A \otimes \mathcal{K})}$.

Further, in the stable rank one case, it was proved that equivalence of A-modules is the same as isomorphism, and a Hilbert A-module X is subequivalent to a Hilbert A-module Y precisely when $X \cong X' \subseteq Y$; see also [12, Section 4] for further details.

One of the key results in this and coming sections is the following result, proved in [5, Proposition 2.8]:

Proposition 11.12. Let A be a stable C*-algebra of stable rank one, and let $a \in A_+$. Then there are a C*-algebra B, also of stable rank one, that contains A as closed, two-sided ideal, and a projection $p_a \in B$ such that the following property holds:

$$\text{For } x \in \mathrm{Cu}(A), \text{ we have } x \leq [a] \text{ in } \mathrm{Cu}(A) \text{ if and only if } x \leq [p_a] \text{ in } \mathrm{Cu}(B).$$

Proof. We sketch the construction. Consider the Hilbert module $H = \overline{aA}$, which by Kasparov's theorem is isomorphic to a direct summand of $\ell^2(A)$, that is, $H \oplus H' \cong \ell^2(A)$ for some Hilbert module H'. Also, since A is stable, we have $\ell^2(A) \cong A$ as Hilbert modules.

Let $M(A)$ be the multiplier algebra of A, identified with the algebra of adjointable operators on A, and let $p_a \in M(A)$ be the projection that corresponds, under the previous identification, to the projection onto $\overline{aA}$, so that $\overline{aA} \cong p_aA$. Set $B = C^*(p_a, A) \subseteq M(A)$. By construction then, A is a closed, two-sided ideal of B and, since $B/A \cong \mathbb{C}$, we see that B also has stable rank one.

To prove the stated property, let $b \in A_+$ and set $x = [b] \in \mathrm{Cu}(A)$. If $b \precsim p_a$ in B and $\varepsilon > 0$ is given, then by Lemma 2.5 there is an element $d \in B$ such that $(b-\varepsilon)_+ = d^*p_ad$. Setting $v = p_ad \in p_aB$, we have $(b-\varepsilon)_+ = v^*v$.

Observe also that $v^*v \in A$, hence $v \in A$ by Remark 5.4, since A is an ideal of B. Thus $v \in p_aB \cap A = p_aA \cong \overline{aA}$. Altogether, we have that

$$\overline{(b-\varepsilon)_+A} = \overline{v^*vA} \cong \overline{vv^*A} \subseteq p_aA \cong \overline{aA}$$

as Hilbert A-modules. This implies, by using the Hilbert module picture of $\mathrm{Cu}(A)$, that $(b-\varepsilon)_+ \precsim a$ and, since $\varepsilon > 0$ is arbitrary, we obtain $b \precsim a$.

For the converse, it suffices to show that $[a] \leq [p_a]$ in $\mathrm{Cu}(B)$. This follows from $\overline{aB} = \overline{aA} \cong p_aA \subseteq p_aB$, as Hilbert B-modules. $\square$

Definition 11.13. We say that an ordered semigroup S satisfies the *Riesz interpolation property* if given $x, y, z, t \in S$ with $x, y \leq z, t$, there exists $w \in S$ such that $x, y \leq w \leq z, t$.

A partially ordered group is called an *interpolation group* if it satisfies the Riesz interpolation property.

If S is algebraically ordered and cancellative, it is well known that the Riesz interpolation property is equivalent to the Riesz decomposition property, and also to the Riesz refinement property. This is due to the Grothendieck group construction; see [53, Proposition 2.1].

It was shown in [68] that, for C*-algebras of real rank zero and stable rank one, $\mathrm{Cu}(A)$ enjoys the three Riesz properties. This is largely due to the fact that the Murray-von Neumann semigroup is cancellative and has Riesz decomposition; see [17] and [104]. However, this is no longer true if one drops the real rank zero assumption, even if one assumes stable rank one. An example of this is given by the Jiang-Su algebra $\mathcal{Z}$, as was observed in [85, Remark 6.9]. Indeed, we have seen in Theorem 7.3 that

$$\mathrm{Cu}(\mathcal{Z}) \cong \underbrace{\{c_n \colon n \in \mathbb{N}\}}_{\text{compacts}} \sqcup \{s_t \colon t \in (0,\infty]\} = \mathbb{N} \sqcup (0,\infty].$$

Let $x = c_1 \in \mathbb{N}$, and let $y = z = s_{\frac{2}{3}} \in (0,\infty]$. We clearly have that $x = c_1 \leq s_{\frac{4}{3}} = y + z$. Since c_1 is compact and there are no other compact elements below c_1 except c_0 and c_1, we see that if $x = u + v$, we must have $u = c_0$ and $v = c_1$, or reversed. Clearly $u, v \not\leq s_{\frac{2}{3}}$.

However, as we shall prove below, Riesz interpolation persists if one drops the assumption of real rank zero and keeps the stable rank one condition.

In the lemma below, we use axiom $\widetilde{\text{(O5)}}$, since the Cu-semigroup is assumed to be weakly cancellative; see Definition 10.5. We also use the one-sided version of (O6+); see Definition 11.1.

Lemma 11.14. Let S be a weakly cancellative Cu-semigroup satisfying (O5) and (O6+), and let $e, x \in S$. If e is compact, the set

$$\big\{z \in S : z \leq e, x\big\}$$

is upward directed.

Proof. We first notice that the set $\{z \in S : z \leq e, x\}$ is order-hereditary and closed under suprema of increasing sequences. To prove the lemma, it suffices to show that the set

$$\big\{z' \in S : \text{there is } z \in S \text{ such that } z' \ll z \leq e, x\big\}$$

is upward directed. (We leave the proof of this claim as an exercise.) Take elements $z_1, z_2, z_1', z_2' \in S$ that satisfy

$$z_1' \ll z_1 \leq e, x, \quad \text{and} \quad z_2' \ll z_2 \leq e, x.$$

Apply (O5) to $z_1' \ll z_1 \leq e$ to find $w \in S$ such that $z_1' + w \leq e \leq z_1 + w$. Since $z_1 \leq x$, we obtain $e \leq x + w$. We now apply (O6+) to this inequality together with $z_2' \ll z_2 \leq e, x$. Thus, we find $y \in S$ such that $e \leq y + w$ and $z_2' \ll y \leq e, x$. Putting together the left hand side of the inequality coming from (O5), the one coming from (O6+), and using that e is compact, we get

$$z_1' + w \leq e \ll e \leq y + w.$$

Weak cancellation in S implies $z_1' \ll y$. Hence, $z_1', z_2' \ll y \leq e, x$. Now choose $y' \in S$ with $z_1', z_2' \ll y' \ll y$, and check that y' has the desired properties. ☐

Theorem 11.15. Let A be a C*-algebra of stable rank one. Then $\mathrm{Cu}(A)$ has the Riesz interpolation property.

Proof. Let $x, y \in \mathrm{Cu}(A)$. We must show that the set $\{z \in \mathrm{Cu}(A) : z \leq x, y\}$ is upward directed. If x is compact, this follows from Lemma 11.14. We now use Proposition 11.12 to reduce the general case to this case.

By part (iii) of Proposition 2.14, we may assume that A is stable. Choose $a \in A_+$ such that $x = [a]$. Applying Proposition 11.12 to A and a, we obtain a C*-algebra B with stable rank one that contains A as a closed, two-sided ideal, and a projection $p_a \in B$ such that $z \in \mathrm{Cu}(A)$ satisfies $z \leq x$ if and only if $z \leq [p_a]$. Since $[p_a]$ is compact in $\mathrm{Cu}(B)$, and since B has stable rank one, it follows from Lemma 11.14 that the set $\{z \in \mathrm{Cu}(B) : z \leq [p_a], y\}$ is upward directed. The inclusion $A \subseteq B$ identifies $\mathrm{Cu}(A)$ with an ideal in $\mathrm{Cu}(B)$ by Lemma 5.2. The result follows once we show that

$$\big\{z \in \mathrm{Cu}(A) : z \leq x, y\big\} = \big\{z \in \mathrm{Cu}(B) : z \leq [p_a], y\big\}.$$

One inclusion follows using that $x \leq [p_a]$. For the converse inclusion, take $z \in \mathrm{Cu}(B)$ such that $z \leq [p_a], y$. Since $\mathrm{Cu}(A)$ is an ideal of $\mathrm{Cu}(B)$ and $y \in \mathrm{Cu}(A)$, we have $z \in \mathrm{Cu}(A)$. Since also $z \leq [p_a]$, Proposition 11.12 implies that $z \leq x$. ☐

We close this section showing that Theorem 11.15 can be used to prove that the Cuntz semigroup of a separable C*-algebra with stable rank one is inf-semilattice ordered. The existence of infima follows easily from the Riesz interpolation property.

Definition 11.16. A Cu-semigroup S is said to be *countably based* provided there is a countable set $\mathcal{B} \subseteq S$ such that for all $s, t \in S$ satisfying $s \ll t$, there is $u \in \mathcal{B}$ such that $s \ll u \ll t$.

It is known that the Cuntz semigroup $\mathrm{Cu}(A)$ of a separable C*-algebra A is always countably based. For example, one can take a countable dense subset F of A and then consider the set $\mathcal{B} = \big\{[(a-\frac{1}{n})_+]\colon a \in F, n \in \mathbb{N}\big\}$.

Theorem 11.17. Let A be a separable C*-algebra of stable rank one. Then $\mathrm{Cu}(A)$ is an inf-semilattice ordered semigroup.

Proof. (Outline) Without loss of generality, we may assume that A is stable.

Since A is separable, as observed above $\mathrm{Cu}(A)$ is countably based. It follows from this that every upward directed set has a supremum. Given $x, y \in \mathrm{Cu}(A)$, this applies in particular to the set $\{z \in \mathrm{Cu}(A) : z \leq x, y\}$, which is upward directed since $\mathrm{Cu}(A)$ has the Riesz interpolation property by Theorem 11.15. Notice that the supremum of the set $\{z \in \mathrm{Cu}(A) : z \leq x, y\}$ is precisely $x \wedge y$. Thus, $\mathrm{Cu}(A)$ is an inf-semilattice.

In order to prove the distributivity of $\wedge$ over addition, note that we always have $x \wedge y + z \leq x + z, y + z$. Thus we need to show that

$$(x+z) \wedge (y+z) \leq (x \wedge y) + z,$$

for all $x, y, z \in \mathrm{Cu}(A)$. To indicate the flavour of the argument, we give a proof in the case that both x and z are compact elements. The general case is obtained through successive generalizations.

Let $w = (x+z) \wedge (y+z)$. Choose $w' \in \mathrm{Cu}(A)$ such that $w' \ll w$. Applying (O5) (or rather, $\widetilde{(\mathrm{O5})}$) to the inequality $w' \ll w \leq x+z$, we find $v \in \mathrm{Cu}(A)$ such that $w' + v \leq x + z \leq w + v$. We get $x + z \leq y + z + v$. As A has stable rank one, $\mathrm{Cu}(A)$ has cancellation of compact elements (see Remark 10.4), and since z is compact by assumption, we obtain $x \leq y + v$. By Lemma 11.7, $x \leq (x \wedge y) + v$. Adding z on both sides we get $x + z \leq (x \wedge y) + v + z$. Since both x and z are compact, so is $x + z$, and thus

$$w' + v \leq x + z \ll x + z \leq (x \wedge y) + z + v\,.$$

Now weak cancellation implies $w' \leq (x \wedge y) + z$ and, since w' is arbitrary satisfying $w' \ll w$, the inequality $(x+z) \wedge (y+z) \leq (x \wedge y) + z$ holds. ☐

12. The classical Cuntz semigroup, its relation to $\mathrm{Cu}(A)$, and the Blackadar-Handelman conjecture

In the previous section we studied the basic structural properties of the Cuntz semigroup of any separable C*-algebra with stable rank one. We will obtain now an easy consequence of Theorem 11.15, which allows us to solve, in the stable rank one case, a conjecture due to Blackadar and Handelman on the structure of dimension functions.

Definition 12.1 ([38])**.** Let A be a unital C*-algebra. Set $M_\infty(A) = \bigcup_{n\in\mathbb{N}} M_n(A)$, identifying each $M_n(A)$ inside $M_{n+1}(A)$ as the upper left corner. Then $M_\infty(A)$ is a local C*-algebra, and the relation of Cuntz (sub)equivalence from Definition 2.1 can be restricted to it. Define

$$\mathrm{W}(A) = M_\infty(A)_+/{\sim},$$

the so-called *classical Cuntz semigroup* of A.

A *dimension function* on A is a map $d\colon M_\infty(A)_+ \to [0,\infty)$ satisfying the following properties:

- $d(a \oplus b) = d(a) + d(b)$ for all $a, b \in M_\infty(A)_+$;
- $d(a) \leq d(b)$ whenever $a \precsim b$;
- $d(1_A) = 1$.

Denote the set of dimension functions of A by $\mathrm{DF}(A)$. If we also denote by $\mathrm{K}_0^*(A)$ the Grothendieck group of $\mathrm{W}(A)$, it is not hard to verify that there is a bijection between $\mathrm{DF}(A)$ and $\mathrm{St}(\mathrm{K}_0^*(A),[1_A])$, the state space of the group $\mathrm{K}_0^*(A)$. Indeed, given a dimension function d on A, one sets $s_d([a]-[b])=d(a)-d(b)$, which defines a state on $\mathrm{K}_0^*(A)$.

The following first appeared in [17]:

Conjecture 12.2. (Blackadar-Handelman). Let A be a unital C*-algebra. Then $\mathrm{DF}(A)$ is a Choquet simplex.

The above conjecture was verified for C*-algebras with real rank zero and stable rank one ([68]), for certain C*-algebras of stable rank 2 ([2]), and for C*-algebras with finite radius of comparison and finitely many extremal quasitraces ([39]). It was also asked in [2, Problem 3.13] for which unital C*-algebras does it hold that $\mathrm{K}_0^*(A)$ is an interpolation group.

Note that we also have

$$\mathrm{W}(A)=\{x\in\mathrm{Cu}(A)\colon x=[a]\text{ for some }a\in M_\infty(A)_+\}.$$

In the case that A has stable rank one, we show below that $\mathrm{W}(A)$ is a hereditary subset of $\mathrm{Cu}(A)$; see [1, Lemma 3.4]. This means that, if $x\leq y$ in $\mathrm{Cu}(A)$ and $y\in\mathrm{W}(A)$, then $x\in\mathrm{W}(A)$.

Lemma 12.3. Let A be a C*-algebra of stable rank one. Then $\mathrm{W}(A)$ is hereditary in $\mathrm{Cu}(A)$. In particular,

$$\mathrm{W}(A)=\{x\in\mathrm{Cu}(A)\colon x\leq n[a]\text{ for some }a\in A_+,n\in\mathbb{N}\}.$$

Proof. Let $a\in(A\otimes\mathcal{K})_+$, $b\in M_\infty(A)_+$, and assume that $a\precsim b$. We need to show that there is $c\in M_\infty(A)_+$ such that $c\sim a$.

Since $A\otimes\mathcal{K}$ is the completion of $M_\infty(A)$ and $a\in(A\otimes\mathcal{K})_+$, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in $M_\infty(A)_+$ with $a=\lim\limits_{n\to\infty}a_n$ and $\|a-a_n\|\leq\frac{1}{n}$. By Lemma 2.5, for each $n\in\mathbb{N}$ there exists $d_n\in A\otimes\mathcal{K}$ such that $(a-\frac{1}{n})_+=d_na_nd_n^*$ and then

$$(a-\tfrac{1}{n})_+=d_na_nd_n^*\sim a_n^{\frac{1}{2}}d_n^*d_na_n^{\frac{1}{2}}\,.$$

Put $b_n:=a_n^{\frac{1}{2}}d_n^*d_na_n^{\frac{1}{2}}\in M_\infty(A)_+$. We have $[a]=\sup\limits_{n\in\mathbb{N}}[b_n]$ in $\mathrm{Cu}(A)$, and $([b_n])_{n\in\mathbb{N}}$ is $\ll$-increasing in $\mathrm{Cu}(A)$.

Now, the sequence $([b_n])_{n\in\mathbb{N}}$ is bounded above in $\mathrm{W}(A)$ by $[b]$. Therefore, it also has a supremum $[c]$ in $\mathrm{W}(A)$, by [26, Lemma 4.3]. In fact, the arguments in [26] show that for each n there exist m and $\delta_n>0$ with $(c-\frac{1}{n})_+\precsim(b_m-\delta_n)_+$, and such that $(\delta_n)_{n\in\mathbb{N}}$ strictly decreases to zero. Therefore

$$(c-\tfrac{1}{n})_+\precsim(b_m-\delta_n)_+\precsim b_m\precsim a$$

in $A\otimes\mathcal{K}$, and thus $c\precsim a$.

On the other hand, since also $b_n\precsim c$ for all n, and $[a]=\sup\limits_{n\in\mathbb{N}}[b_n]$ in $\mathrm{Cu}(A)$, we see that $a\precsim c$. Thus $c\sim a$, as desired. □

Lemma 12.4. Let S be a positively ordered semigroup that has the Riesz interpolation property. Then its Grothendieck group $G(S)$ is an interpolation group.

Proof. Let a_1,a_2,b_1,b_2 be elements in $G(S)$ such that $a_i\leq b_j$ for all $i,j=1,2$. There exist elements $z,x_i,y_j\in S$ such that $a_i=[x_i]-[z]$ and $b_j=[y_j]-[z]$. (If $a_i=[x_i]-[v_i]$ and $b_i=[y_i]-[w_i]$, then one may take $z=v_1+v_2+w_1+w_2$.) Therefore, by adding $[z]$ to the inequality we get $[x_i]\leq[y_j]$ for all $i,j=1,2$. Thus there exists $t\in S$ such that

$$x_i+t\leq y_j+t$$

for all $i, j = 1, 2$. By assumption, there exists $x \in S$ interpolating the above inequality. Consider the element $e = [x] - [z + t] \in G(S)$. Then it is easy to check that e satisfies $a_i \leq e \leq b_j$ for all $i, j = 1, 2$. □

Theorem 12.5. Let A be a C*-algebra of stable rank one. Then $\mathrm{K}_0^*(A)$ is an interpolation group and thus $\mathrm{DF}(A)$ is a Choquet simplex. In particular, Conjecture 12.2 holds if A has stable rank one.

Proof. We know from Lemma 12.3 that $\mathrm{W}(A)$ is hereditary. We use this to show that $\mathrm{W}(A)$ has the Riesz interpolation property. By Theorem 11.15, this is the case for $\mathrm{Cu}(A)$. Let $x, y, z, t \in \mathrm{W}(A)$ be such that $x, y \leq z, t$. Then this also holds in $\mathrm{Cu}(A)$ and thus there is $w \in \mathrm{Cu}(A)$ such that $x, y \leq w \leq z, t$. Since $\mathrm{W}(A)$ is hereditary, we have $w \in \mathrm{W}(A)$.

By Lemma 12.4, $\mathrm{K}_0^*(A)$ is an interpolation group, and using [53, Theorem 10.17], we obtain that its state space, that is, $\mathrm{DF}(A)$, is a Choquet simplex. □

One of the reasons for introducing $\mathrm{Cu}(A)$ was the need of a continuous invariant. Regarding W as a functor from C*-algebras to the category of positively ordered semigroups, it is clear that W is not continuous. (This already fails for $\mathcal{K} = \varinjlim M_n$.)

As it turns out, $\mathrm{Cu}(A)$ can be regarded as the completion of $W(A)$, just as $A \otimes \mathcal{K}$ is the completion of $M_\infty(A)$. In order to outline the exact relationship between these two semigroups, we need some additional concepts. The following is inspired by [49, Definition I-1.11, p.57].

Definition 12.6. Let $(X, \leq)$ be a partially ordered set. A binary relation $\prec$ on X is called an *auxiliary relation* if the following properties are satisfied:

(i) If $x \prec y$ then $x \leq y$, for all $x, y \in X$.
(ii) If $w \leq x \prec y \leq z$ then $w \prec z$, for all $w, x, y, z \in X$.

If, further, X is a monoid, then an auxiliary relation $\prec$ is said to be *additive* if it is compatible with addition and $0 \prec x$ for every $x \in X$.

Observe that an auxiliary relation as defined above is transitive. To see this, if $x \prec y$ and $y \prec z$, then $x \leq y$ by condition (i) and applying condition (ii) to $x \leq y \prec z \leq z$, we obtain $x \prec z$. In the case of a Cu-semigroup S, the compact containment relation $\ll$ is an example of an auxiliary relation on S. If A is a C*-algebra, then $\mathrm{W}(A)$ may be equipped with the following auxiliary relation: $[a] \prec [b]$ if and only if $[a] \leq [(b-\varepsilon)_+]$ for some $\varepsilon > 0$; see [8, Proposition 2.2.5]. Equivalently by Remark 4.2, $[a] \prec [b]$ in $\mathrm{W}(A)$ if and only if $[a] \ll [b]$ in $\mathrm{Cu}(A)$.

Definition 12.7. A W-semigroup is a positively ordered semigroup S together with an auxiliary relation $\prec$ such that the following axioms hold:

(W1) For each $a \in S$, the set $a^{\prec} = \{b \in S \colon b \prec a\}$ has a $\prec$-increasing countable cofinal subset (with respect to $\prec$).
(W3) $\prec$ is additive.
(W4) If $a \prec b + c$ in S then there are $b' \prec b$ and $c' \prec c$ such that $a \prec b' + c'$.

A positively ordered semigroup morphism $f \colon S \to T$ between two W-semigroups is a W-morphism provided it preserves $\prec$ and that is also continuous, in the sense that if $b \prec f(a)$ in T, then there is $a' \prec a$ in S such that $b \leq f(a')$. We will denote by $\mathbf{W}$ the category whose objects are the W-semigroups and whose morphisms are the W-morphisms. The set of W-morphisms between W-semigroups S and T will be denoted by $\mathbf{W}(S, T)$.

We remark that the terminology has evolved so that initially a W-semigroup was also required to satisfy (W2): for each $a \in S$, we have $a = \sup a^{\prec}$, but this is not relevant for the theory. It is not even relevant, for many purposes, to require that a W-semigroup is positively ordered (and only that it is equipped with a transitive relation $\prec$ satisfying the axioms above).

Again, if A is a C*-algebra, it was shown in [8, Proposition 2.2.5] that $\mathrm{W}(A)$ is a W-semigroup with the auxiliary relation defined above. Moreover, if $\varphi\colon A\to B$ is a homomorphism of C*-algebras, then the restriction $\mathrm{W}(\varphi)$ of $\mathrm{Cu}(A)$ to $\mathrm{W}(A)$ is a W-morphism $\mathrm{W}(\varphi)\colon \mathrm{W}(A)\to \mathrm{W}(B)$.

It is easy to verify that **Cu** is a full subcategory of **W** and that **W** has limits; see [8, Theorem 2.2.9]. It turns out it is also a reflective category, which follows from the theorem below; see [8].

Theorem 12.8. Given a W-semigroup $(S,\prec)$, there are a Cu-semigroup $\gamma(S)$ and a W-morphism $\alpha\colon S\to\gamma(S)$ such that:

(i) $a'\prec a$ in S whenever $\alpha(a')\ll\alpha(a)$.
(ii) If $b'\ll b$ in $\gamma(S)$, then there is $a\in S$ such that $b'\ll\alpha(a)\ll b$.

Proof. (Outline) We just show how to construct $\gamma(S)$. One considers the set $S_\prec$ of $\prec$-increasing sequences in S. Any two such sequences are added pointwise, and one declares $(a_n)_{n\in\mathbb{N}}\precsim(b_n)_{n\in\mathbb{N}}$ if for every $k\in\mathbb{N}$ there is $n\in\mathbb{N}$ such that $a_k\prec b_n$. This defines a translation invariant preorder that yields an equivalence relation by setting $(a_n)_{n\in\mathbb{N}}\sim(b_n)_{n\in\mathbb{N}}$ if and only if $(a_n)_{n\in\mathbb{N}}\precsim(b_n)_{n\in\mathbb{N}}$ and $(b_n)\precsim(a_n)_{n\in\mathbb{N}}$. We then define $\gamma(S)$ to be $S_\prec/\sim$. Addition is induced by addition of sequences and the order is induced by $\precsim$. It is possible to prove that $[(a_n)_{n\in\mathbb{N}}]\ll[(b_n)_{n\in\mathbb{N}}]$ precisely if there is $k\in\mathbb{N}$ such that $a_n\prec b_k$ for all $n\in\mathbb{N}$.

In order to define $\alpha\colon S\to\gamma(S)$, let $a\in S$ and apply (W1) to find $(a_n)_{n\in\mathbb{N}}\in S^\prec$ which is cofinal in $a^\prec$. Then set $\alpha(a)=[(a_n)_{n\in\mathbb{N}}]$. We omit the details. $\square$

The construction just outlined defines a functor $\gamma\colon\mathbf{W}\to\mathbf{Cu}$ which is a reflector for the inclusion. Applied to C*-algebras, this yields:

Theorem 12.9 ([8, Theorem 3.2.8])**.** The compositions $\gamma\circ\mathrm{W}$ and Cu are naturally isomorphic as functors from $\mathbf{C}^*$ to **Cu**. In other words, if A is a C*-algebra, then $\mathrm{Cu}(A)$ is naturally isomorphic to $\gamma(\mathrm{W}(A))$.

The result above is extremely useful when constructing objects in the category **Cu** as certain "completions" of objects in **W**. We already saw an example of this in Theorem 4.11, when we constructed inductive limits in **Cu**. Indeed, what we did there was to consider the inductive limit of Cu-semigroups in the category **W**, and then apply the functor γ. The same strategy can be used to construct other objects, such as (infinite) direct sums. On the other hand, many other constructions (such as products) will require a different treatment, since these constructions do not obviously exist in **W** either. This is done in Section 14, where we consider an even larger category **Q** and a natural functor $\tau\colon\mathbf{Q}\to\mathbf{Cu}$; see Theorem 14.5.

13. Functionals and the realization of ranks

In this section, we formulate the problem of realization of ranks and sketch the solution for C*-algebras of stable rank one. The inf-semilattice ordered structure of the Cuntz semigroup for such algebras is a key element for the solution. Another ingredient that is needed in this setting is an additional axiom for Cu-semigroups, called *Edward's condition*; see [4]. Since we will omit the proofs where this axiom is needed, we will also not discuss this condition here.

Recall from Definition 6.8 that a *functional* on a Cu-semigroup S is an additive function $\lambda\colon S\to[0,\infty]$ satisfying $\lambda(0)=0$, that preserves order and suprema of increasing sequences. We equip the set $F(S)$ of functionals on S with operations of addition and scalar multiplication by nonzero, positive real numbers, defined pointwise. Moreover, $F(S)$ is a topological cone with respect to the topology whose subbase is given by the collection of all sets

$$V_{x,r} = \{\lambda \in F(S) \colon \lambda(x) > r\} \quad \text{and} \quad W_{x,r} = \{\lambda \in F(S) \colon \lambda(x') < r \text{ for all } x' \ll x\},$$

for $x \in S$ and $r \in (0, \infty)$; see [60] and also [40]. With respect to this topology, given $\lambda \in F(S)$ and a net $(\lambda_i)_{i\in I}$ in $F(S)$, we have $\lambda_i \to \lambda$ if and only if

$$\limsup \lambda_i(x') \leq \lambda(x) \leq \liminf \lambda_i(x) \text{ for all } x', x \in S \text{ such that } x' \ll x.$$

It was shown in [60, Theorem 3.17] (see also [40, Theorem 4.8]) that, with this topology, $F(S)$ is a compact Hausdorff ordered topological cone.

By Theorem 6.9 and the comments after it, for any C*-algebra A there is a natural bijection between $F(\mathrm{Cu}(A))$ and the set of $[0, \infty]$-valued, lower semicontinuous 2-quasitraces on A.

A significant difference when considering *non-normalized* functionals on Cu-semigroups, is that these naturally arise from the ideal structure of the semigroup, as follows:

Lemma 13.1. Let S be a Cu-semigroup, let $I \subseteq S$ be an ideal, and let $\lambda\colon I \to [0, \infty]$ be a functional. Define $\tilde{\lambda}\colon S \to [0, \infty]$ by

$$\tilde{\lambda}(x) = \begin{cases} \lambda(x), & \text{if } x \in I; \\ \infty, & \text{otherwise.} \end{cases}$$

Then $\tilde{\lambda}$ is a functional on S.

Proof. Let us show that $\tilde{\lambda}$ is order-preserving. If $x \leq y$ in S and $y \notin I$, then $\tilde{\lambda}(y) = \infty$, and clearly $\tilde{\lambda}(x) \leq \tilde{\lambda}(y)$. If $y \in I$, then $x \in I$ as well, since I is an ideal of S, and thus $\tilde{\lambda}(x) = \lambda(x) \leq \lambda(y) = \tilde{\lambda}(y)$.

Next, let $x, y \in S$. Clearly $x + y \in I$ if and only if both $x, y \in I$. If $x, y \in I$, then

$$\tilde{\lambda}(x + y) = \lambda(x + y) = \lambda(x) + \lambda(y) = \tilde{\lambda}(x) + \tilde{\lambda}(y).$$

If either $x \notin I$ or $y \notin I$, then $x + y \notin I$, hence $\tilde{\lambda}(x + y) = \infty = \tilde{\lambda}(x) + \tilde{\lambda}(y)$. That $\tilde{\lambda}$ preserves suprema of increasing sequences follows in a similar manner. $\square$

For a Cu-semigroup S that satisfies (O5), we give below the appropriate notion of dual for the cone $F(S)$. Denote by $\mathrm{Lsc}(F(S))$ the set of functions $f\colon F(S) \to [0, \infty]$ that are additive, order-preserving, homogeneous (with respect to nonzero, positive scalars), lower semicontinuous, and satisfy $f(0) = 0$. This set is equipped with pointwise order, addition, and scalar multiplication by nonzero positive scalars.

Definition 13.2. Let S be a Cu-semigroup and let $x \in S$. Given $x \in S$, the *rank* of x is the function $\widehat{x}\colon F(S) \to [0, \infty]$ given by evaluation, namely:

$$\widehat{x}(\lambda) = \lambda(x)$$

for all $\lambda \in F(S)$. One can check that $\widehat{x}$ belongs to $\mathrm{Lsc}(F(S))$.

The *rank map* $\mathrm{rk}\colon S \to \mathrm{Lsc}(F(S))$ of S is defined by $\mathrm{rk}(x) = \widehat{x}$ for all $x \in S$.

It is easy to check that the rank map preserves addition, order, and suprema of increasing sequences.

The *realification* of S, denoted by S_R, was introduced in [74] as the smallest subsemigroup of $\mathrm{Lsc}(F(S))$ that is closed under suprema of increasing sequences and contains all elements of the form $\frac{1}{n}\widehat{x}$ for $x \in S$ and $n \geq 1$. It can be shown that $S_R \cong S \otimes_{\mathrm{Cu}} [0, \infty]$, thus justifying the term "realification"; see the proof of Theorem 14.7 for the definition of tensor products in **Cu**. Moreover, it was proved in [74, Proposition 3.1.1] that S_R is a Cu-semigroup satisfying (O5); see also [8, Proposition 7.5.6].

Given $f, g \in \mathrm{Lsc}(F(S))$, we write $f \lhd g$ if $f \leq (1-\varepsilon)g$ for some $\varepsilon > 0$ and if f is continuous at each $\lambda \in F(S)$ satisfying $g(\lambda) < \infty$. We denote by $L(F(S))$ the subsemigroup of $\mathrm{Lsc}(F(S))$ consisting of those $f \in \mathrm{Lsc}(F(S))$ that can be written as the pointwise supremum of a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{Lsc}(F(S))$ such that $f_n \lhd f_{n+1}$ for all $n \in \mathbb{N}$.

One has that in fact $S_R = L(F(S))$, as was shown in [74, Theorem 3.2.1]. It was also proved in [74, Theorem 4.2.2] that $L(F(S))$ is inf-semilattice ordered. The semigroup $L(F(S))$ is thought of as the *dual* of $F(S)$, since $F(L(F(S)) = F(S)$, although it is not known whether $L(F(S)) = \mathrm{Lsc}(F(S))$.

Problem 13.3. *The problem of realizing functions as ranks.* Let S be a Cu-semigroup satisfying (O5). The problem of realizing functions on $F(S)$ as ranks of elements in S consists of finding necessary and sufficient conditions for the map $x \mapsto \widehat{x}$ to be a surjection from S to $L(F(S))$.

The following notion is crucial to solve the problem of realization of ranks in the stable rank one setting. The motivation for the terminology can be found in [87]. Recall that by an *ideal* in a C*-algebra we always mean a closed, two-sided ideal.

Definition 13.4. An *ideal-quotient* in a C*-algebra A is a quotient of the form I/J, where $J \subseteq I$ are ideals of A. A C*-algebra is *nowhere scattered* if it has no non-zero elementary[12] ideal quotients.

Nowehere scatteredness can be nicely characterized in terms of Cuntz semigroups and functionals. To this end we need the lemma below, which is a nice application of the axioms (O5) and (O6) of independent interest. Recall the notation ∞_s from Notation 5.6.

Lemma 13.5. Let S be a Cu-semigroup satisfying axioms (O5) and (O6), let $\lambda \in F(S)$ satisfying

(i) $\lambda(S) = \overline{\mathbb{N}}$, and
(ii) $\lambda(s) = 0$ if and only if $s = 0$,

and fix $s_0 \in S$ with $\lambda(s_0) = 1$. Then $I = \{s \in S \colon s \leq \infty_{s_0}\}$ is an ideal in S, and λ restricts to an isomorphism $I \cong \overline{\mathbb{N}}$ of Cu-semigroups.

Proof. Let $s \in S$ satisfy $\lambda(s) = k < \infty$. We claim that s is compact. To see this, write $s = \sup_{n\in\mathbb{N}} t_n$ with $t_n \ll t_{n+1}$. Since $k = \lambda(s) = \sup_{n\in\mathbb{N}} \lambda(t_n)$ and $\lambda(t_n) \in \mathbb{N}$ for all n, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\lambda(t_n) = k$. For such $n \geq n_0$, apply (O5) to $t_n \ll t_{n+1} \leq s$ to find $c \in S$ such that $t_n + c \leq s \leq t_{n+1} + c$. Applying λ yields $\lambda(c) = 0$, which implies $c = 0$ by (i). This shows that s is compact.

Fix $m \in \mathbb{N}$ and $t \in S$. We claim that $t \leq ms_0$ if and only if there exists $k \leq m$ with $t = ks_0$. One implication is obvious, so we prove the other one. The inequality $t \leq ms_0$ implies that $\lambda(t) \leq m\lambda(s_0) < \infty$, and thus t is compact by the previous claim. In order to establish the claim, we may clearly assume that $t \neq 0$ and proceed by induction on m. Suppose that $m = 1$. Use Remark 10.7 to find $t' \in S$ with $t + t' = s_0$. Applying λ we get $\lambda(t') = 0$, which again by (i) implies that $t' = 0$, showing that $t = s_0$. This proves the case $m = 1$ of the induction.

Assume now that $t \leq ms_0 = (m-1)s_0 + s_0$. Apply (O6) to find elements $t_1, t_2 \in S$ such that $t_1 \leq (m-1)s_0, t$ and $t_2 \leq t, s_0$. Notice that t_1 and t_2 are compact elements as well, since they are dominated by the compact element t. By the induction assumption, there is $k \leq m-1$ such that $t_1 = ks_0$. If $t_2 = 0$, then $ks_0 = t_1 \leq t \leq t_1 + t_2 = ks_0$, and hence $t = ks_0$. If $t_2 \neq 0$, then $t_2 = s_0$ since

[12]Recall that a C*-algebra is said to be *elementary* if it is isomorphic to the compact operators on some Hilbert space. Equivalently, a C*-algebra A is elementary if it is simple and there exists a projection $p \in A$ satisfying $pAp \cong \mathbb{C}$.

$t_2 \leq s_0$. Now $t_1 = ks_0 \leq t \leq t_1 + t_2 = (k+1)s_0$, and we may find elements $c, d \in S$ such that $ks_0 + c = t$ and $t + d = (k+1)s_0$. Putting these equalities together we obtain $ks_0 + c + d = (k+1)s_0$, and applying λ we get $\lambda(c+d) = 1$. Thus one of c or d must be zero, and hence t equals either ks_0 or $(k+1)s_0$, as desired. This proves the claim.

Let $t \in S \setminus \{0\}$ satisfy $t \leq \infty_{s_0}$. Find a $\ll$-increasing sequence $(t_n)_{n\in\mathbb{N}}$ in S with $t = \sup\limits_{n\in\mathbb{N}} t_n$. Fix $n \in \mathbb{N}$. Since

$$t_n \ll t = \sup_{m\in\mathbb{N}} ms_0,$$

there exists $k \in \mathbb{N}$ with $t_n \leq m_n s_0$. By the second claim, there exists $k_n \leq m_n$ such that $t_n = k_n s_0$. Thus we must have either $t = ks_0$ for some $k \in \mathbb{N}$, or else $t = \infty_{s_0}$.

Set $I = \{s \in S\colon s \leq \infty_{s_0}\}$, and observe that I is an ideal in S. In particular, I is a Cu-semigroup. The paragraph above shows that $I = \{ms_0 \in S\colon m \in \overline{\mathbb{N}}\}$. Furthermore, by applying λ we see that $ks_0 \leq ms_0$ precisely when $k \leq m$. In other words, this shows that λ restricts to a Cu-semigroup isomorphism $I \cong \overline{\mathbb{N}}$. ☐

Proposition 13.6. Let A be a C*-algebra. Then the following are equivalent:

(i) A is *not* nowhere scattered,
(ii) there exists a functional $\lambda \in F(\mathrm{Cu}(A))$ such that $\lambda(\mathrm{Cu}(A)) \cong \overline{\mathbb{N}}$.

Proof. Assume that (i) holds, and find closed, two sided ideals I, J in A with $J \subseteq I$ such that I/J is elementary. Then $\mathrm{Cu}(I/J) \cong \overline{\mathbb{N}}$ by Example 3.3. Moreover, by Theorem 5.9 the quotient map $\pi\colon I \to I/J$ induces a surjective Cu-morphism $\mathrm{Cu}(\pi)\colon \mathrm{Cu}(I) \to \mathrm{Cu}(I/J) \cong \overline{\mathbb{N}}$; in particular, $\mathrm{Cu}(\pi)$ is a functional on $\mathrm{Cu}(I)$. Let $\lambda\colon \mathrm{Cu}(A) \to [0,\infty]$ be the functional obtained by extending $\mathrm{Cu}(\pi)$ to $\mathrm{Cu}(A)$ as in Lemma 13.1. It is then clear from the definition of λ that $\lambda(\mathrm{Cu}(A)) = \overline{\mathbb{N}}$.

Conversely, write $S = \mathrm{Cu}(A)$ and let $\lambda\colon S \to [0,\infty]$ be a functional satisfying $\lambda(S) = \overline{\mathbb{N}}$. Consider the ideal $K = \lambda^{-1}(0)$. It is easy to check that λ induces a functional $\overline{\lambda}\colon S/K \to [0,\infty]$ such that $\overline{\lambda}(S/K) = \overline{\mathbb{N}}$. We may now apply Lemma 13.5 to obtain an ideal $\overline{L}$ of S/K which is isomorphic to $\overline{\mathbb{N}}$. Thus, there is an ideal L of S that contains K such that $L/K \cong \overline{\mathbb{N}}$. This implies that A has an ideal-quotient whose Cuntz semigroup is isomorphic to $\overline{\mathbb{N}}$, and thus this ideal-quotient is elementary by [87, Lemma 8.2].[13] Therefore A is not nowhere scattered. ☐

The key to solve the problem posed in Problem 13.3 in the stable rank one case relies on the following:

Definition 13.7 (The map α)**.** Let A be a separable C*-algebra with stable rank one. Define

$$\alpha\colon L(F(\mathrm{Cu}(A)) \to \mathrm{Cu}(A)$$

by setting $\alpha(f) = \sup\{x \in \mathrm{Cu}(A)\colon \widehat{x} \ll f\}$.

It is not at all obvious that the set $\{x \in \mathrm{Cu}(A)\colon \widehat{x} \ll f\}$ has a supremum – this follows from the fact that A is assumed to have stable rank one, and is shown in [5, Theorem 7.2, Proposition 7.3].

Lemma 13.8. The map α from Definition 13.7 preserves order, suprema of increasing sequences, and infima of pairs of elements.

[13] Indeed, if a C*-algebra B satisfies $\mathrm{Cu}(B) \cong \overline{\mathbb{N}}$, then B must be elementary. The basic idea is this: Since $\mathrm{Cu}(B)$ is simple, so is B. Now, using [25, Theorem 5.8], choose a projection $q \in B \otimes \mathcal{K}$ that corresponds to 1. If $a \in B_+ \setminus \{0\}\}$, then $q \precsim a$, hence there is a projection $p \in B$ with $p \sim q$. Now p is minimal, so $pBp = \mathbb{C}p$, which is known to imply that B is elementary.

Proof. The core of the argument consists of proving that in fact $\alpha(f) = \sup I_f$, where

$$I_f = \{x \in \mathrm{Cu}(A) \colon \widehat{y} \ll f \text{ for all } y \ll x\}.$$

We will not prove this, and isntead we will only explain how to to prove the lemma using it. To see that α preserves the order, let $f \leq g$ in $L(F(\mathrm{Cu}(A)))$. Then $I_f \subseteq I_g$, and thus $\alpha(f) \leq \alpha(g)$.

To check that α preserves suprema, let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence of elements in $L(F(\mathrm{Cu}(A)))$, and set $f = \sup\limits_{n\in\mathbb{N}} f_n$. Since, as observed, α is order-preserving, the sequence $(\alpha(f_n))_{n\in\mathbb{N}}$ is increasing in $\mathrm{Cu}(A)$ and thus it has a supremum $x = \sup\limits_{n\in\mathbb{N}} \alpha(f_n)$. Since $\alpha(f_n) \leq \alpha(f)$ for all $n \in \mathbb{N}$, we have $x \leq \alpha(f)$. Now let $z \in \mathrm{Cu}(A)$ satisfy $\widehat{z} \ll f$. Using that, by definition, $\alpha(f)$ is the supremum of all such z, it suffices to show that $z \leq x$. As $\widehat{z} \ll f$, we have $\widehat{z} \ll f_n$ for some $n \in \mathbb{N}$, and then $z \in I_{f_n}$. Therefore $z \leq \alpha(f_n) \leq x$.

Finally, to prove that α preserves infima, let $f, g \in L(F(\mathrm{Cu}(A)))$. Since α is order-preserving, we get $\alpha(f \wedge g) \leq \alpha(f) \wedge \alpha(g)$. To show the converse inequality, let $0 < \varepsilon < 1$ and suppose that $z \leq \alpha((1-\varepsilon)f) \wedge \alpha((1-\varepsilon)g)$. Then

$$\widehat{z} \leq (1-\varepsilon)f \wedge (1-\varepsilon)g = (1-\varepsilon)(f \wedge g),$$

whence $z \leq \alpha(f \wedge g)$. This implies $\alpha((1-\varepsilon)f) \wedge \alpha((1-\varepsilon)g) \leq \alpha(f \wedge g)$. Finally, let $\varepsilon \to 0$ and use that α preserves suprema of increasing sequences to obtain $\alpha(f) \wedge \alpha(g) \leq \alpha(f \wedge g)$. □

In general, the map α defined above is superadditive, in the sense that $\alpha(f) + \alpha(g) \leq \alpha(f+g)$. Furthermore, if A is separable and nowhere scattered, then it is additive; see [5, Proposition 7.4(iv), Corollary 8.5(ii)]. In general, however, α is not additive. The simplest (not nowhere scattered) example is $A = \mathbb{C}$. In this case, $\mathrm{Cu}(A) \cong \overline{\mathbb{N}}$, and, identifying $\lambda \in F(\mathrm{Cu}(\mathbb{C}))$ with $\lambda(1)$, we have $F(\mathrm{Cu}(\mathbb{C})) \cong [0,\infty]$. Thus $(L(F(\mathrm{Cu}(A))), \ll) \cong ([0,\infty], <)$, again identifying a function f with $f(1)$. Now the map

$$\alpha \colon [0,\infty] \cong L(F(\mathrm{Cu}(A))) \to \mathrm{Cu}(A) \cong \overline{\mathbb{N}}$$

is given by $\alpha(z) = \sup\{n \in \overline{\mathbb{N}} \colon n < z\} = \lfloor z \rfloor$ for all $z \in [0,\infty]$, which is clearly not additive.

Recall the definition of ∞_a from Notation 5.6.

Theorem 13.9. Let A be a separable, nowhere scattered C*-algebra of stable rank one. Then, for all $f \in L(F(\mathrm{Cu}(A)))$ we have

$$f = \widehat{\alpha(f)}.$$

Proof. (Outline) The set $\{x \in \mathrm{Cu}(A) \colon \widehat{x} \leq \infty_f\}$ is an ideal of $\mathrm{Cu}(A)$, and thus has the form $\mathrm{Cu}(I)$ for a closed two-sided ideal I of A. Note that I is automatically separable, nowhere scattered, and has stable rank one.

Using that $L(F(\mathrm{Cu}(A))) = \mathrm{Cu}(A)_R$, one can choose a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathrm{Cu}(A)$ and $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $f = \sup\limits_{n\in\mathbb{N}} \frac{\widehat{x_n}}{k_n}$. Notice that $x_n \in I$ for all $n \in \mathbb{N}$. Considering $\frac{\widehat{x_n}}{k_n} \in L(F(\mathrm{Cu}(I)))$, denote by f_0 its supremum, so that $f(\lambda) = f_0(\lambda_{|\mathrm{Cu}(I)})$.

One now checks that f_0 is full in $L(F(\mathrm{Cu}(I)))$. By letting $\alpha_I \colon L(F(\mathrm{Cu}(I))) \to \mathrm{Cu}(I)$ be the map as in Definition 13.7 it is possible to show, with considerable effort, that $f_0 = \widehat{\alpha_I(f_0)}$; see [5, Theorem 7.10].

Next, we claim that $\alpha(f) = \alpha(f_0)$. To see this, let

$$L = \big\{x \in \mathrm{Cu}(A) : \widehat{x} \leq (1-\varepsilon)f \text{ in } L(F(\mathrm{Cu}(A))) \text{ for some } \varepsilon > 0\big\}.$$

Using the definition of α, we have that $\alpha(f)$ is the supremum of L in $\mathrm{Cu}(A)$. If $x \in \mathrm{Cu}(A)$ and $\varepsilon > 0$ satisfy $\widehat{x} \leq (1-\varepsilon)f$ in $L(F(\mathrm{Cu}(A)))$, then x belongs to $\mathrm{Cu}(I)$

and hence $\widehat{x} \leq (1-\varepsilon)f_0$ in $L(F(\mathrm{Cu}(I)))$. This implies that $L \subseteq \mathrm{Cu}(I)$ and so $\alpha(f_0)$ is the supremum of L in $\mathrm{Cu}(I)$. Using that $\mathrm{Cu}(I) \subseteq \mathrm{Cu}(A)$ is hereditary, we see that the supremum of L in $\mathrm{Cu}(I)$ and in $\mathrm{Cu}(A)$ agree. Therefore $\alpha(f) = \alpha(f_0)$, as was claimed.

Finally, if $\lambda \in F(\mathrm{Cu}(A))$, we have

$$\widehat{\alpha(f)}(\lambda) = \lambda(\alpha(f)) = \lambda|_{\mathrm{Cu}(I)}(\alpha(f_0)) = f_0(\lambda|_{\mathrm{Cu}(I)}) = f(\lambda),$$

and thus $\widehat{\alpha(f)} = f$ in $L(F(\mathrm{Cu}(A)))$. □

14. Structure of the category **Cu**

This section will be devoted to discussing the following result, which combines results from a number of papers; see [8, 9, 11].

Theorem 14.1. The category **Cu** of abstract Cuntz semigroups is a closed, symmetric, monoidal, bicomplete category.

Below we shall present the constructions that are used in order to prove the above theorem; see Theorem 14.7, Theorem 14.9, and Theorem 14.11. We will also give the relevant definitions of the concepts that appear in its statement.

Definition 14.2. A *$\mathcal{Q}$-semigroup* is a positively ordered monoid satisfying axioms (O1) and (O4) from Definition 4.5, together with an additive auxiliary relation $\prec$ as in Definition 12.6. A *$\mathcal{Q}$-morphism* is a morphism of positively ordered monoids that preserves the auxiliary relation and suprema of increasing sequences. Given $\mathcal{Q}$-semigroups S and T, we denote by $\mathbf{Q}(S,T)$ the set of all $\mathcal{Q}$-morphisms from S to T.

Given a Cu-semigroup S, then S together with the relation $\ll$ is a $\mathcal{Q}$-semigroup. Moreover, given two Cu-semigroups S and T, a map $\varphi\colon S \to T$ is a Cu-morphism if and only if it is a $\mathcal{Q}$-morphism when considered as a map from $(S,\ll)$ to $(T,\ll)$. We obtain a functor $\iota\colon \mathbf{Cu} \to \mathbf{Q}$ that sends a Cu-semigroup S to the $\mathcal{Q}$-semigroup $(S,\ll)$, and embeds **Cu** as a full subcategory of $\mathbf{Q}$.

Definition 14.3 (The τ-construction)**.** Let $S = (S,\prec)$ be a $\mathcal{Q}$-semigroup. A *path* in S is an order-preserving map $f\colon (-\infty,0] \to S$ such that $f(t) = \sup_{t'<t} f(t')$ for all $t \in (-\infty,0]$, and such that $f(t') \prec f(t)$ whenever $t' < t$. We denote the set of paths in S by $\mathrm{Paths}(S)$.

Pointwise addition, together with the constant zero path, give $\mathrm{Paths}(S)$ the structure of a commutative monoid. Given $f,g \in \mathrm{Paths}(S)$, we write $f \precsim g$ if, for every $t < 0$, there is $t' < 0$ such that $f(t) \prec g(t')$. Set $f \sim g$ if $f \precsim g$ and $g \precsim f$. It follows from [9, Lemma 3.4] that the relation $\precsim$ is reflexive, transitive, and compatible with addition of paths. We set

$$\tau(S) := \mathrm{Paths}(S)/\!\sim.$$

Given $f \in \mathrm{Paths}(S)$, its equivalence class in $\tau(S)$ is denoted by $[f]$. Equip $\tau(S)$ with an addition and order by setting $[f]+[g] := [f+g]$ and $[f] \leq [g]$ if $f \precsim g$.

The construction just outlined will be referred to as the *τ-construction*, and it has a number of features that we now describe.

Theorem 14.4. ([9, Theorem 3.15]). Retaining the notation in Definition 14.2, if S is a $\mathcal{Q}$-semigroup, then $\tau(S)$ is a Cu-semigroup.

Proof. (Outline) Let $([f_n])_{n\in\mathbb{N}}$ be an increasing sequence of paths in S. An inductive process allows one to construct a strictly increasing sequence $(t_m)_{m\in\mathbb{N}}$ in $(-\infty,0]$ and a path f in S satisfying

(i) $\sup_{m\in\mathbb{N}} t_m = 0$
(ii) $f_n(t_m) \prec f_l(t_l)$ whenever $n, m < l$.
(iii) $f_n(t_n) = f(\frac{-1}{n+1})$ for all $n \geq 1$.

Then one can verify that $[f] = \sup_{n\in\mathbb{N}}[f_n]$ and thus $\tau(S)$ satisfies (O1). In order to verify (O2), given $f \in \mathrm{Paths}(S)$ and $\varepsilon > 0$, define $f_\varepsilon \colon (-\infty, 0] \to S$ by

$$f_\varepsilon(t) = \begin{cases} f(t), & \text{if } t < -\varepsilon \\ 0, & \text{otherwise.} \end{cases}$$

If follows by construction that $[f] = \sup_{\varepsilon>0}[f_\varepsilon]$. We need to check that $[f_\varepsilon] \ll [f]$. To this end, let $([g_n])_{n\in\mathbb{N}}$ be an increasing sequence in $\tau(S)$ such that $[f] \leq \sup_{n\in\mathbb{N}}[g_n]$. By the construction outlined at the beginning of the proof, there are a path g in S and an increasing sequence $(t_m)_{m\in\mathbb{N}}$ in $(-\infty, 0]$ with supremum 0 such that $[g] = \sup_{n\in\mathbb{N}}[g_n]$ and $g(\frac{-1}{m}) = g_m(t_m)$ for all $m \in \mathbb{N}$.

Choose $m_0 \geq 1$ such that $-\varepsilon < -\frac{1}{m_0}$. Since $f \precsim g$, there is $r > 0$ such that $f(-\frac{1}{m_0}) \prec g(r)$. Now choose m_1 such that $r < -\frac{1}{m_1+1}$. Then, if $t < -\varepsilon$, we have

$$f_\varepsilon(t) = f(t) \prec f\big(\tfrac{-1}{m_0}\big) \prec g(r) \prec g\big(\tfrac{-1}{m_1+1}\big) = g_{m_1}(t_{m_1}),$$

which shows that $f_\varepsilon \prec g_{m_1}$. This implies that $\tau(S)$ satisfies (O2). Axioms (O3) and (O4) are more routine to verify. □

We define $\varepsilon_S \colon \tau(S) \to S$ by $\varepsilon_S([f]) = f(0)$ for $f \in \mathrm{Paths}(S)$. One can check that ε_S is a well-defined $\mathcal{Q}$-morphism. Since for a path $f \in \mathrm{Paths}(S)$, we think of $f(0)$ as the endpoint of f, we call ε_S the *endpoint map*.

Given a $\mathcal{Q}$-morphism $\varphi \colon S \to T$, we define $\tau(\varphi) \colon \tau(S) \to \tau(T)$ by $\tau(\varphi)([f]) = [\varphi \circ f]$ for $f \in \mathrm{Paths}(S)$. One can show that $\tau(\varphi)$ is a Cu-morphism. This defines a covariant functor $\tau \colon \mathbf{Q} \to \mathbf{Cu}$. We omit the proof of the following result.

Theorem 14.5 ([9, Theorem 4.12])**.** The category $\mathbf{Cu}$ is a full, coreflective subcategory of $\mathbf{Q}$. The functor $\tau \colon \mathbf{Q} \to \mathbf{Cu}$ is a right adjoint to the inclusion functor $\iota \colon \mathbf{Cu} \to \mathbf{Q}$. Moreover, given a $\mathcal{Q}$-semigroup S, the endpoint map $\varepsilon_S \colon \tau(S) \to S$ is a universal $\mathcal{Q}$-morphism, in the sense that for every Cu-semigroup T, there is a natural bijection

$$\mathbf{Cu}\big(T, \tau(S)\big) \cong \mathbf{Q}\big(\iota(T), S\big),$$

implemented by sending a Cu-morphism $\psi \colon T \to \tau(S)$ to $\varepsilon_S \circ \psi$.

We now define what is meant by a monoidal category; see also [64].

Definition 14.6. A *monoidal category* is a category $\mathbf{C}$ together with a bifunctor $\otimes \colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, a unit object I, and natural isomorphisms

$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z) \quad \text{and} \quad I \otimes X \cong X \cong X \otimes I,$$

whenever X, Y, Z are objects in $\mathbf{C}$. (Other coherence axioms are also required.) We say that $\mathbf{C}$ is moreover *symmetric* if $X \otimes Y \cong Y \otimes X$ for any $X, Y \in \mathbf{C}$.

Theorem 14.7. The category $\mathbf{Cu}$ is monoidal.

Proof. To show that $\mathbf{Cu}$ is monoidal we need to construct the tensor product of any two Cu-semigroups S and T. We only sketch here how to proceed and refer the reader to the material in [8, Chapter 6].

Given Cu-semigroups S, T, we first form the algebraic tensor product $S \odot T$ as positively ordered monoids, based on expressions on the free abelian monoid $\mathbb{N}[S^\times \times T^\times]$ so that if $a' \leq a$ in S and $b' \leq b$ in T, one has $a' \odot b' \leq a \odot b$. For

$f, g \in \mathbb{N}[S^\times \times T^\times]$, we set $f \preceq g$ provided $g = \sum_{j \in J} a_j \odot b_j$ and $f \leq \sum_{j \in J'} a'_j \odot b'_j$, where $J' \subseteq J$ and $a'_j \ll a_j$, $b'_j \ll b_j$ for each $j \in J'$.

With this structure one can show, after considerable effort, that $(S \odot T, \preceq)$ is a W-semigroup and it follows that $\gamma(S \odot T)$ is the tensor product $S \otimes_{\mathrm{Cu}} T$ of S and T in the category **Cu**, where γ is as in the proof of Theorem 12.8. The unit for the tensor product is $\overline{\mathbb{N}}$. □

We now focus on the adjoint of the tensor product.

Definition 14.8. We say that a monoidal category $\mathbf{C}$ is *closed* if, for each object $Y \in \mathbf{C}$, the functor $- \otimes Y \colon \mathbf{C} \to \mathbf{C}$ has a right adjoint, that we denote by $[\![Y, -]\!]$.

In a closed, monoidal category $\mathbf{C}$, we thus have a natural bijection (in X and Z)

$$\mathbf{C}(X \otimes Y, Z) \cong \mathbf{C}(X, [\![Y, Z]\!]).$$

Unlike in the category of abelian groups, the adjoint of the tensor product in **Cu** does not merely consist of the usual hom-set in the category. The reason for this is that, for Cu-semigroups S and T, the set $\mathbf{Cu}(S, T)$ may be too small, and not even a Cu-semigroup. Indeed, for any Cu-semigroup S we have $\mathbf{Cu}(\overline{\mathbb{N}}, S) \cong S_c$, so in particular $\mathbf{Cu}(\overline{\mathbb{N}}, \overline{\mathbb{N}}) \cong \mathbb{N}$ is not a Cu-semigroup. Instead, one has to consider the set of generalized Cu-semigroups, as in Definition 4.5. Notice that, given Cu-semigroups S and T, the set $\mathbf{Cu}[S, T]$ is a positively ordered monoid, when equipped with pointwise order and addition. It also satisfies axioms (O1) and (O4): the supremum of a sequence of generalized Cu-morphisms is just the pointwise supremum.

Define a binary relation on $\mathbf{Cu}[S, T]$ by setting $f \prec g$ if, whenever $s' \ll s$ in S, we have $f(s') \ll g(s)$ in T. With this relation, a generalized Cu-morphism $f \colon S \to T$ is a Cu-morphism precisely when $f \prec f$. It is an easy exercise to verify that $\prec$ is an auxiliary relation for the pointwise order in $\mathbf{Cu}[S, T]$ which makes the latter into a $\mathcal{Q}$-semigroup, although not necessarily a Cu-semigroup.

For example, write $\mathbb{P} = [0, \infty]$, which is known to be the Cuntz semigroup of the Jacelon-Razak algebra $\mathcal{W}$ (see [58] and [71]). One can show that $(\mathbf{Cu}[\mathbb{P}, \mathbb{P}], \prec)$ is isomorphic to $(\mathbb{P}, \prec_1)$, where $\prec_1$ is defined as follows: $a \prec_1 b$ if and only if $a \leq \infty$ and $a \leq b$; see [9, Examples 4.14 and 5.13]. With this structure, $(\mathbb{P}, \prec_1)$ is a $\mathcal{Q}$-semigroup, and the τ-construction applied to it yields $[0, \infty) \sqcup (0, \infty]$ where the compact elements are the ones in the first component. Let us denote these elements by c_a, for $a \in [0, \infty)$, and the elements in the second component by s_a, for $a \in (0, \infty]$. Addition and order are given by:

- $c_a + c_b = c_{a+b}$, $s_a + s_b = s_{a+b}$, and $c_a + s_b = s_{a+b}$.
- $s_a \leq c_b$ if and only if $a \leq b$; $c_a \leq s_b$ if and only if $a < b$.

Theorem 14.9. The category **Cu** is closed.

Proof. We define $[\![S, T]\!] = \tau(\mathbf{Cu}[S, T], \preceq)$, which by Theorem 14.4 is a Cu-semigroup, and refer to this as the *internal-hom functor* for S and T.

The internal-hom functor is the adjoint of the tensor product as defined in Theorem 14.7. In fact, it was shown in [9, Theorem 5.10] that there is a natural isomorphism of positively ordered monoids

$$\mathbf{Cu}(S, [\![T, P]\!]) \cong \mathbf{Cu}(S \otimes_{\mathrm{Cu}} T, P).$$ □

In these notes we will only discuss completeness (and not cocompleteness), as it is what will be used below in the construction of ultraproducts. We now recall the basic notions.

Definition 14.10. Let $\mathbf{C}$ be a category, let $\mathbf{I}$ be a small category, and let $\mathrm{F} \colon \mathbf{I} \to \mathbf{C}$ be a functor. A *cone* to F is a pair (L, φ), where L is an object in $\mathbf{C}$ and $\varphi = (\varphi_i)_{i \in \mathbf{I}}$

is a collection of morphisms $\varphi_i \colon L \to \mathrm{F}(i)$ in $\mathbf{C}$ such that $\mathrm{F}(f) \circ \varphi_i = \varphi_j$ for any morphism $f \colon i \to j$ in $\mathbf{I}$.

A *small limit* of F is a universal cone (L, φ), that is, if (L', φ') is another cone, there exists a unique $\mathbf{C}$-morphism $\alpha \colon L' \to L$ such that $\varphi'_i = \varphi_i \circ \alpha$ for every $i \in \mathbf{I}$. This is summarized in the diagrams below:

$$
\begin{tikzcd}
 & \mathrm{F}(i) \arrow[d, "\mathrm{F}(f)"] \\
L \arrow[ur, "\varphi_i"] \arrow[dr, "\varphi_j"'] & \\
 & \mathrm{F}(j),
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \mathrm{F}(i) \arrow[d, "\mathrm{F}(f)"] \\
L' \arrow[r, "\alpha"] \arrow[urr, bend left, "\varphi'_i"] \arrow[drr, bend right, "\varphi'_j"'] & L \arrow[ur, "\varphi_i"] \arrow[dr, "\varphi_j"'] & \\
 & & \mathrm{F}(j).
\end{tikzcd}
$$

If a limit of F exists, then it is unique up to natural isomorphism, and we denote it by $(\mathbf{C}\text{-}\varprojlim \mathrm{F}, \pi)$ or just by $\mathbf{C}\text{-}\varprojlim \mathrm{F}$.

The category $\mathbf{C}$ is said to be *complete* if all functors from small categories into it have limits. (Cocompleteness is defined dually.)

The following is proved in [11, Theorem 3.8].

Theorem 14.11. The category $\mathbf{Cu}$ is complete.

Proof. Let $\mathrm{F} \colon \mathbf{I} \to \mathbf{Cu}$ be a functor from a small category $\mathbf{I}$. The basic strategy consists of completing, via the τ-construction, the limit of the composition $\mathbf{I} \to \mathbf{Cu} \hookrightarrow \mathbf{Q}$. We outline the procedure below.

Denote by $\mathbf{PoM}$ the category of positively ordered monoids. Let $(S_i)_{i \in \mathbf{I}}$ be a collection of objects in $\mathbf{PoM}$. The product of this family in $\mathbf{PoM}$ is given by

$$\mathbf{PoM} - \prod_{i \in \mathbf{I}} S_i = \{(s_i)_{i \in \mathbf{I}} \colon s_i \in S_i \text{ for all } i \in \mathbf{I}\},$$

with componentwise addition and order. Regarding F as a functor $\mathrm{F} \colon \mathbf{I} \to \mathbf{PoM}$, set

$$S := \Big\{ (s_i)_{i \in \mathbf{I}} \in \mathbf{PoM}\text{-}\prod_{i \in \mathbf{I}} \mathrm{F}(i) \colon \mathrm{F}(f)(s_i) = s_j \text{ for all } f \colon i \to j \text{ in } \mathbf{I} \Big\}.$$

It is straightforward to verify that $0 \in S$ and that S is closed under addition in $\mathbf{PoM}\text{-}\prod_{i \in \mathbf{I}} \mathrm{F}(i)$, hence S is also a positively ordered monoid.

For each $i \in \mathbf{I}$, the projection map $\pi_i \colon \mathbf{PoM}\text{-}\prod_{j \in J} \mathrm{F}(j) \to \mathrm{F}(i)$ restricts to a $\mathbf{PoM}$-morphism $\pi_i \colon S \to \mathrm{F}(i)$. Set $\pi = (\pi_i)_{i \in \mathbf{I}}$. It is also straightforward to verify that (S, π) is the limit of F in $\mathbf{PoM}$. If further $\mathrm{F}(i)$ satisfies (O1) and (O4) for each $i \in \mathbf{I}$, then this is also the case for S.

Also, since the range of F is contained in $\mathbf{Q}$ (in fact, in $\mathbf{Cu}$), we may define an auxiliary relation $\prec_{\mathrm{pw}}$ on S by stating $(s_i) \prec_{\mathrm{pw}} (t_i)$ precisely when $s_i \prec t_i$ in $\mathrm{F}(i)$ for each $i \in \mathbf{I}$. This construction shows that $(S, \prec_{\mathrm{pw}})$ is the limit of F in the category $\mathbf{Q}$. In fact, the argument just outlined shows that $\mathbf{Q}$ is complete.

Let $(S, (\pi_i)_{i \in \mathbf{I}})$ be the limit of F in $\mathbf{Q}$ as outlined, and let $\tau(S)$ be the τ-completion of S. Set $S_i := \tau(\mathrm{F}(i))$, which we identify with $\mathrm{F}(i)$, and set

$$\psi_i = \tau(\pi_i) \colon \tau(S) \to \tau(S_i) \cong S_i.$$

Then $\tau(S)$ together with $(\psi_i)_{i \in \mathbf{I}}$ is the limit of F in $\mathbf{Cu}$, that is:

$$\mathbf{Cu}\text{-} \varprojlim \mathrm{F} = \tau\left(\mathbf{Q}\text{-} \varprojlim \mathrm{F}\right) = \tau\left(\mathbf{PoM}\text{-} \varprojlim \mathrm{F}, \ll_{\mathrm{pw}}\right). \qquad \square$$

As an immediate consequence, we have:

Corollary 14.12. The category **Cu** has arbitrary products, arbitrary inverse limits, and finite pullbacks. In particular, if $(S_i)_{i\in I}$ is a family of Cu-semigroups, then

$$\textbf{Cu-}\prod_{i\in I} S_i = \tau\left(\textbf{Q-}\prod_{i\in I} S_i\right) = \tau\left(\textbf{PoM-}\prod_{i\in I} S_i, \ll_{\mathrm{pw}}\right).$$

Since we are mostly interested in the category **Cu**, we will from now on denote the product in **Cu** simply by $\prod$, instead of $\textbf{Cu-}\prod$. (The notation should not be confused with the product of C*-algebras.)

15. Applications to products and ultraproducts of C*-algebras

In this section we explore the extent to which the Cuntz semigroup functor preserves products and ultraproducts of C*-algebras. We begin with a simple observation, which is a consequence of the universal property of the product.

Remark 15.1. Let $(A_j)_{j\in J}$ be a family of C*-algebras, and set $A = \prod_{j\in J} A_j$. For each $j \in J$, the natural projection $\pi_j\colon A \to A_j$ induces a Cu-morphism $\tilde{\pi}_i\colon \mathrm{Cu}(A) \to \mathrm{Cu}(A_j)$. Then, by the universal property of the product, there is a unique Cu-morphism

$$\Phi\colon \mathrm{Cu}(A) \to \prod_{j\in J} \mathrm{Cu}(A_j)$$

such that $\tilde{\pi}_i = \sigma_i \circ \Phi$ for all $j \in J$, where $\sigma_i\colon \prod_{j\in J} \mathrm{Cu}(A_j) \to \mathrm{Cu}(A_i)$ denotes the natural Cu-morphism associated to the product in the category **Cu**.

Lemma 15.2. Let $(A_j)_{j\in J}$ be a family of C*-algebras, and set $A = \prod_{j\in J} A_j$. Let $\underline{a} = (a_j)_{j\in J}$ and $\underline{b} = (b_j)_{j\in J} \in A_+$. Then $\underline{a} \precsim \underline{b}$ in A if, and only if, $\Phi([\underline{a}]) \leq \Phi([\underline{b}])$ in $\prod_{j\in J} \mathrm{Cu}(A_j)$.

Proof. We show that the conditions in the statement are equivalent to:

(∗) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $(a_j - \varepsilon)_+ \precsim (b_j - \delta)_+$ in A_j for every $j \in J$.

To see this, set $\varphi_t(\underline{a}) = ([(a_j + t)_+])_j$, and $\varphi_t(\underline{b}) = ([(b_j + t)_+])_j$, for any $t \in (-\infty, 0]$, whence $\Phi([\underline{a}]) = [\varphi_t(\underline{a})_{t\in(-\infty,0]}]$, and $\Phi([\underline{b}]) = [\varphi_t(\underline{b})_{t\in(-\infty,0]}]$.

As shown in Theorem 14.11, we have

$$\prod_{j\in J} \mathrm{Cu}(A_j) = \tau\Big(\textbf{PoM-}\prod_{j\in J} \mathrm{Cu}(A_j), \ll_{\mathrm{pw}}\Big),$$

and thus $\Phi([\underline{a}]) \leq \Phi([\underline{b}])$ if and only if for every $t < 0$ there exists $t' < 0$ such that $\varphi_t(\underline{a}) \ll_{\mathrm{pw}} \varphi_{t'}(\underline{b})$. That is, given $\varepsilon > 0$ there exists $\delta > 0$ such that $\varphi_{-\varepsilon}(\underline{a}) \ll_{\mathrm{pw}} \varphi_{-\delta}(\underline{b})$. Thus the second condition in the statement is equivalent to (*).

Rørdam's lemma (Theorem 2.7) shows that the first condition of the statement implies (∗). For the converse, assume (∗) and let $\varepsilon > 0$. Again by Rørdam's lemma, we need to find $\underline{s}$ in A such that $\underline{s}\underline{s}^* = (\underline{a} - \varepsilon)_+$ and $\underline{s}^*\underline{s}A_{\underline{b}}$. By assumption, for each $j \in J$ there is $\delta > 0$ such that $(a_j - \frac{\varepsilon}{2})_+ \precsim (b_j - \delta)_+$ in A_j. By Rørdam's lemma applied to $(a_j - \frac{\varepsilon}{2})_+ \precsim (b_j - \delta)_+$ and $\frac{\varepsilon}{2}$, we obtain $s_j \in A_j$ such that

$$(a_j - \varepsilon)_+ = \big((a_j - \tfrac{\varepsilon}{2})_+ - \tfrac{\varepsilon}{2}\big)_+ = s_j s_j^*, \text{ and } s_j^* s_j \in (A_j)_{(b_j-\delta)_+}.$$

Since $\|s_j\| \leq \|a_j\|^{\frac{1}{2}}$ for all $j \in J$, we see that $(s_j)_{j\in J}$ is bounded, and thus $\underline{s} := (s_j)_{j\in J} \in A$ satisfies $\underline{s}\underline{s}^* = (\underline{a} - \epsilon)_+$. Now let $f_\delta\colon \mathbb{R} \to [0, 1]$ be continuous with $f(t) = 0$ for $t \leq 0$ and $f(t) = 1$ for $t \geq \delta$. Then $f_\delta(b_j)s_j^*s_j f_\delta(b_j) = s_j^*s_j$ for each j, and therefore $f_\delta(\underline{b})\underline{s}^*\underline{s}f_\delta(\underline{b}) = \underline{s}^*\underline{s}$, which implies that $\underline{s}^*\underline{s}$ belongs to the hereditary algebra generated by $\underline{b}$. □

Proposition 15.3. Let $(A_j)_{j\in J}$ be a family of *stable* C*-algebras. Then the canonical map Φ from Remark 15.1 is a Cu-isomorphism.

Proof. Recall first that a C*-algebra A is said to have property (S) if if for every $a \in A_+$ and every $\varepsilon > 0$ there exist $b \in A_+$ and $x \in A$ such that $a = x^*x$, $b = xx^*$ and $\|ab\| \leq \varepsilon$. It is known that any stable C*-algebra has property (S), and the converse holds in the separable case; see [57].

Set $A = \prod_{j\in J} A_j$. We claim that if all A_j are stable, then A has property (S). To show this, let $a = (a_j)_{j\in J} \in A_+$ and $\varepsilon > 0$, and use that each A_j has property (S) to find $b_j \in (A_j)_+$ and $x_j \in A_j$ such that

$$a_j = x_j^*x_j, \quad b_j = x_jx_j^*, \quad \text{and} \quad \|a_jb_j\| \leq \varepsilon.$$

Set $b := (b_j)_{j\in J}$ and $x := (x_j)_{j\in J}$. Note that $\|x\| = \sup_i \|x_i\| < \infty$ since $\|x_j\| = \|a_j\|^{\frac{1}{2}} \leq \|a\|^{\frac{1}{2}}$ for each j. Similarly, $\|b\| < \infty$. Hence b, x belong to A, so $a = x^*x$, $b = xx^*$ and $\|ab\| \leq \varepsilon$.

Since A has property (S), it is possible to compute its Cuntz semigroup by just looking at Cuntz classes of its positive elements (withouth need to go to matrices). The basic idea is that if a is a positive element in the stabilisation of A, then one may choose a separable subalgebra of A with property (S) that contains a, and such subalgebra will then be stable. This is used to show that Cuntz equivalence on A and on $A \otimes \mathcal{K}$ agree.

We now prove that the natural map Φ from Remark 15.1 is an isomorphism. We already know from Lemma 15.2 that Φ is an order-embedding. Therefore, to see it is surjective it is enough to verify, using that $\mathrm{Cu}(A)$ has suprema of increasing sequences preserved by Φ, that the image of Φ is order-dense.

Let $x, y \in \prod_{j\in J} \mathrm{Cu}(A_j)$ satisfy $x \ll y$. We will find $\underline{b} \in A_+$ such that $x \ll \Phi([\underline{b}]) \ll y$. Choose $\ll_{\mathrm{pw}}$-increasing paths

$$(\mathbf{x}_t)_{t\in(-\infty,0]}, (\mathbf{y}_t)_{t\in(-\infty,0]} \in \textbf{PoM-}\prod_{j\in J} \mathrm{Cu}(A_j)$$

representing x and y, respectively. By [9, Lemma 3.16], we may choose $t < 0$ such that $\mathbf{x}_0 \ll_{\mathrm{pw}} \mathbf{y}_t$.

For each $j \in J$, choose $x_{0,j} \in \mathrm{Cu}(A_j)$ such that $\mathbf{x}_0 = (x_{0,j})_{j\in J}$, and choose $a_j \in (A_j)_+$ such that $\mathbf{y}_t = ([a_j])_{j\in J}$. Then $x_{0,j} \ll [a_j]$. Using functional calculus (see [11, Lemma 5.8]), one can find a contraction $b_j \in (A_j)_+$ such that $x_{0,j} \ll [(b_j - \frac{1}{2})_+]$, and $[b_j] = [a_j]$. Set $\underline{b} := (b_j)_{j\in J}$, which is a contraction in A_+. We have $\mathbf{x}_0 \ll_{\mathrm{pw}} ([(b_j - \frac{1}{2})_+])_{j\in J} = \varphi_{-\frac{1}{2}}(\underline{b})$, and

$$\varphi_0(\underline{b}) = ([b_j])_{j\in J} = ([a_j])_{j\in J} = \mathbf{y}_t \ll_{\mathrm{pw}} \mathbf{y}_{\frac{t}{2}}.$$

Therefore $x = [(\mathbf{x}_t)_{t\leq 0}] \ll [(\varphi_t(\underline{b}))_{t\leq 0}] = \Phi([\underline{b}]) \ll [(\mathbf{y}_t)_{t\leq 0}] = y$. This shows that $\underline{b}$ has the desired properties. $\square$

Example 15.4. The above result does not hold if the algebras are not stable. Set $A_j := \mathbb{C}$ for each $j \in \mathbb{N}$, and set $A := \prod_{j\in\mathbb{N}} A_j \cong \ell^\infty(\mathbb{N})$. We have $\mathrm{Cu}(A_j) \cong \overline{\mathbb{N}}$ for each j, and the product $\prod_{j\in\mathbb{N}} \overline{\mathbb{N}}$ is defined as the equivalence classes of $\ll_{\mathrm{pw}}$-increasing paths $(-\infty, 0] \to \textbf{PoM-}\prod_{j\in\mathbb{N}} \overline{\mathbb{N}}$, so that $\prod_{j\in\mathbb{N}} \overline{\mathbb{N}}$ may be identified with equivalence classes of componentwise increasing paths $(-\infty, 0) \to \textbf{PoM-}\prod_{j\in\mathbb{N}} \overline{\mathbb{N}}$. In particular, compact elements in $\prod_{j\in\mathbb{N}} \overline{\mathbb{N}}$ naturally corresponds to functions $\mathbb{N} \to \mathbb{N}$.

We claim that the natural order-embedding $\Phi\colon \mathrm{Cu}(A) \to \prod_{j\in\mathbb{N}} \overline{\mathbb{N}}$ is not surjective. To see this, note that Φ maps the Cuntz class of the unit of A to the compact element in $\prod_{j\in\mathbb{N}} \overline{\mathbb{N}}$ corresponding to the function $f\colon \mathbb{N} \to \mathbb{N}$ with $f(j) = 1$

for all j. Let x be the compact element in $\prod_{j\in\mathbb{N}}\overline{\mathbb{N}}$ that corresponds to the function $g\colon\mathbb{N}\to\mathbb{N}$ with $g(j)=j$ for all j. Since $g\not\leq nf$ for every $n\in\mathbb{N}$, we have $x\not\leq\infty_{\Phi([1_A])}$. In particular, $\Phi([1_A])$ is not full, and $\mathrm{Cu}(A)$ is isomorphic to a proper ideal of $\prod_{j\in\mathbb{N}}\mathrm{Cu}(A_j)$; see Notation 5.6 and the comments after it.

In order to capture the Cuntz semigroup of the product of C*-algebras in the non-stable case, we need to keep track of the position of the algebra and to this end the invariant needs to be modified.

Definition 15.5. A *scale* for a Cu-semigroup S is a downward hereditary subset $\Sigma\subseteq S$ that is closed under suprema of increasing sequences in S, and that generates S as an ideal. The pair (S,Σ) will be referred to as a *scaled* Cu*-semigroup*.

Given scaled Cu-semigroups (S,Σ) and (T,Θ), a *scaled* Cu*-morphism* is a Cu-morphism $\alpha\colon S\to T$ satisfying $\alpha(\Sigma)\subseteq\Theta$. We let $\mathbf{Cu}_{\mathrm{sc}}$ denote the category of scaled Cu-semigroups and scaled Cu-morphisms.

For a C*-algebra A, note that

$$\Sigma_A := \big\{x\in\mathrm{Cu}(A)\colon \text{there exists } a\in A_+ \text{ such that } x\leq[a]\big\}$$

is a scale for $\mathrm{Cu}(A)$. We call $\mathrm{Cu}_{\mathrm{sc}}(A):=(\mathrm{Cu}(A),\Sigma_A)$ the *scaled Cuntz semigroup* of A. Given a homomorphism $\varphi\colon A\to B$ of C*-algebras, the induced Cu-morphism $\mathrm{Cu}(\varphi)\colon\mathrm{Cu}(A)\to\mathrm{Cu}(B)$ maps Σ_A into Σ_B, and thus $\mathrm{Cu}_{\mathrm{sc}}$ defines a functor from the category $\mathbf{C}^*$ of C*-algebras to $\mathbf{Cu}_{\mathrm{sc}}$. The following is [11, Theorem 4.6].

Theorem 15.6. The category $\mathbf{Cu}_{\mathrm{sc}}$ is complete.

Proof. Let $\mathbf{I}$ be a small category, and let $\mathrm{F}\colon\mathbf{I}\to\mathbf{Cu}_{\mathrm{sc}}$ be a functor, written $i\mapsto\mathrm{F}(i)=(S_i,\Sigma_i)$. Considering the underlying functor $\mathbf{I}\to\mathbf{PoM}$ given by $i\mapsto S_i$, we let $(S,(\pi_i)_{i\in\mathbf{I}})$ be the limit of F in $\mathbf{PoM}$ with S as in the proof of Theorem 14.11. We have that

$$\Sigma_0 := S\cap\prod_{i\in\mathbf{I}}\Sigma_i = \Big\{(s_i)_{i\in\mathbf{I}}\in\prod_{i\in\mathbf{I}}\Sigma_i\colon \mathrm{F}(f)(s_i)=s_j \text{ for all } f\colon i\to j \text{ in } \mathbf{I}\Big\}.$$

Then Σ_0 is a downward hereditary subset of S satisfying $\pi_i(\Sigma_0)\subseteq\Sigma_i$ for all $i\in\mathbf{I}$.

Composing with the forgetful functor (that forgets the scaled structure), we get as limit $\tau(S,\ll_{\mathrm{pw}})$ together with maps

$$\psi_i := \tau(\pi_i)\colon \tau(S,\ll_{\mathrm{pw}})\to\tau(S_i,\ll)\cong S_i$$

for all $i\in\mathbf{I}$; see the proof of Theorem 14.11. Set

$$\Sigma := \Big\{[(\mathbf{x}_t)_{t\leq 0}]\in\tau(S,\ll_{\mathrm{pw}}) : \mathbf{x}_t\in\Sigma_0 \text{ for all } t<0\Big\}.$$

Then Σ is a downward hereditary subset of $\tau(S,\ll_{\mathrm{pw}})$ that is closed under passing to suprema of increasing sequences. Let $\langle\Sigma\rangle$ denote the ideal of $\tau(S,\ll_{\mathrm{pw}})$ generated by Σ. Then $(\langle\Sigma\rangle,\Sigma)$ is a scaled Cu-semigroup. Moreover, for each $i\in\mathbf{I}$ we have $\psi_i(\Sigma)\subseteq\Sigma_i$, which shows that $\psi_i\colon(\langle\Sigma\rangle,\Sigma)\to(S_i,\Sigma_i)$ is a scaled Cu-morphism. One can show that this defines a limit for F in $\mathbf{Cu}_{\mathrm{sc}}$. We omit the details. $\square$

Theorem 15.7. The scaled Cuntz semigroup functor preserves products.

Proof. We first show how to construct the product in the category $\mathbf{Cu}_{\mathrm{sc}}$. This uses as an ingredient the proof of Theorem 15.6.

Let $(S_j)_{j\in J}$ be a family of Cu-semigroups and let (S,Σ) be their scaled product in $\mathbf{Cu}_{\mathrm{sc}}$. To get a concrete description of this object, we first take the set-theoretic product $\prod_{j\in J}\Sigma_j$, which is a downward hereditary subset of $\mathbf{PoM}$-$\prod_{j\in J}S_j$, and set

$$\Sigma = \Big\{[(\mathbf{x}_t)_{t\leq 0}]\in\prod_{j\in J}S_j : \mathbf{x}_t\in\prod_{j\in J}\Sigma_j \text{ for every } t<0\Big\}.$$

Then S is the ideal of $\prod_{j\in J} S_j$ generated by Σ. Given $[(\mathbf{x}_t)_{t\leq 0}] \in \prod_{j\in J} S_j$, we have $[(\mathbf{x}_t)_{t\leq 0}] \in S$ if and only if for every $t<0$ there exist $\sigma^{(1)},\ldots,\sigma^{(N)} \in \prod_{j\in J} \Sigma_j$ such that $\mathbf{x}_t \ll_{\mathrm{pw}} \sigma^{(1)}+\ldots+\sigma^{(N)}$.

If now $(A_j)_{j\in J}$ is a family of C*-algebras, we use again (S,Σ) to denote the scaled product of $(\mathrm{Cu}_{\mathrm{sc}}(A_j))_{j\in J}$ with $\Sigma \subseteq S \subseteq \prod_{j\in J} \mathrm{Cu}(A_j)$ as defined above. Set $A=\prod_{j\in J} A_j$. Then the map $\Phi\colon \mathrm{Cu}(A)\to \prod_{j\in J}\mathrm{Cu}(A_j)$ defined in Remark 15.1 is an order-embedding by Lemma 15.2. Using the strategy in the proof of Proposition 15.3 with additional care, one can show that the image of Φ is S and moreover it identifies the scale of $\mathrm{Cu}(A)$ with Σ:

$$\mathrm{Cu}_{\mathrm{sc}}(A) = (\mathrm{Cu}(A),\Sigma_A) \cong \prod_{j\in J} (\mathrm{Cu}(A_j),\Sigma_{A_j}) = (S,\Sigma). \qquad \square$$

To close this section, we turn our attention to ultraproducts. First, we give a categorical definition.

Definition 15.8. Let $\mathbf{C}$ be a category that has products and inductive limits, let J be a set, let $\mathcal{U}$ be an ultrafilter on J, and let $(X_j)_{j\in J}$ be a family of objects in $\mathbf{C}$. Given a subset $K\subseteq J$ and $j\in K$, we write $\pi_{j,K}\colon \prod_{k\in K} X_k \to X_j$ for the canonical quotient map. Given subsets $G\subseteq K\subseteq J$, the universal property of the product gives a morphism

$$\varphi_{G,K}\colon \prod_{j\in K} X_j \to \prod_{j\in G} X_j,$$

such that $\pi_{j,K}=\pi_{j,G}\circ\varphi_{G,K}$ for each $j\in G$.

Ordering the elements of $\mathcal{U}$ by reversed inclusion, we have that $\mathcal{U}$ is upward directed, and thus the objects $\prod_{j\in K} X_j$ for $K\in\mathcal{U}$, and morphisms $\varphi_{G,K}$ for $K,G\in\mathcal{U}$ with $K\supseteq G$, define an inductive system indexed over $\mathcal{U}$. The inductive limit of this system is called the (categorical) *ultraproduct* of $(X_j)_{j\in J}$ along $\mathcal{U}$:

$$\prod_{\mathcal{U}} X_j := \varinjlim_{K\in\mathcal{U}} \prod_{j\in K} X_j.$$

We let $\pi_{\mathcal{U}}\colon \prod_{j\in J} X_j \to \prod_{\mathcal{U}} X_j$ denote the natural morphism to the inductive limit.

Let $\mathbf{C}$ and $\mathbf{D}$ be categories with products and inductive limits, and let $\mathrm{F}\colon \mathbf{C}\to\mathbf{D}$ be a functor that preserves inductive limits and products. Then for any set J, any ultrafilter $\mathcal{U}$ on J, and any family $(X_j)_{j\in J}$ of objects in $\mathbf{C}$, there is a natural isomorphism

$$\Phi_{\mathcal{U}}\colon \mathrm{F}\Big(\prod_{\mathcal{U}} X_j\Big) \to \prod_{\mathcal{U}} \mathrm{F}(X_j).$$

Let $(A_j)_{j\in J}$ be a family of C*-algebras and let $\mathcal{U}$ be an ultrafilter on J. Set $A=\prod_{\mathcal{U}} A_j$. The discussion above, combined with the properties of the categories $\mathbf{Cu}$ and $\mathbf{Cu}_{\mathrm{sc}}$, as well as the functors Cu and $\mathrm{Cu}_{\mathrm{sc}}$, yield natural (scaled) **Cu**-morphisms

$$\Phi_{\mathcal{U}}\colon \mathrm{Cu}(A)\to \prod_{\mathcal{U}} \mathrm{Cu}(A_j) \text{ and } \Phi_{\mathcal{U},\mathrm{sc}}\colon \mathrm{Cu}_{\mathrm{sc}}(A)\to \prod_{\mathcal{U}} \mathrm{Cu}_{\mathrm{sc}}(A_j).$$

Applying Proposition 15.3 and Theorem 15.7, we obtain the following results:

Proposition 15.9. Given an ultrafilter $\mathcal{U}$ on a set J and a family $(A_j)_{j\in J}$ of stable C*-algebras, the map $\Phi_{\mathcal{U}}\colon \mathrm{Cu}\big(\prod_{\mathcal{U}} A_j\big)\to \prod_{\mathcal{U}} \mathrm{Cu}(A_j)$ is an isomorphism.

Theorem 15.10. The scaled Cuntz semigroup functor preserves ultraproducts. In other words, given an ultrafilter $\mathcal{U}$ on a set J and a family $(A_j)_{j\in J}$ of C*-algebras, the map $\Phi_{\mathcal{U},\mathrm{sc}}\colon \mathrm{Cu}_{\mathrm{sc}}\big(\prod_{\mathcal{U}} A_j\big)\to \prod_{\mathcal{U}} \mathrm{Cu}_{\mathrm{sc}}(A_j)$ is an isomorphism.

A different view on ultraproducts, more akin to the usual construction for C*-algebras, can also be given in this setting. We give the statement without proof, and refer the reader to [11, Section 7] for further details.

Theorem 15.11. Let $\mathcal{U}$ be an ultrafilter on a set J, and let $(S_j)_{j\in J}$ be a family of Cu-semigroups. Set

$$\mathrm{c}_{\mathcal{U}}\big((S_j)_j\big) := \Big\{[(\mathbf{x}_t)_{t\leq 0}] \in \prod_{j\in J} S_j : \mathrm{supp}(\mathbf{x}_t) \notin \mathcal{U} \text{ for each } t<0\Big\}.$$

Then $\mathrm{c}_{\mathcal{U}}\big((S_j)_{j\in J}\big)$ is an ideal in $\prod_{j\in J} S_j$, and there is a natural isomorphism

$$\prod_{\mathcal{U}} S_j \cong \Big(\prod_{j\in J} S_j\Big)/\mathrm{c}_{\mathcal{U}}((S_j)_{j\in J}).$$

If now $(A_j)_{j\in J}$ is a family of C*-algebras and (S,Σ) is the scaled product of $(\mathrm{Cu}_{\mathrm{sc}}(A_j))_{j\in J}$ with $\Sigma \subseteq S \subseteq \prod_{j\in J}\mathrm{Cu}(A_j)$ as in Theorem 15.6, we have a natural isomorphism

$$\mathrm{Cu}\Big(\prod_{\mathcal{U}} A_j\Big) \cong S/\left(S \cap \mathrm{c}_{\mathcal{U}}\big((\mathrm{Cu}(A_j))_{j\in J}\big)\right).$$

In particular, the map $\Phi_{\mathcal{U}}$ from Definition 15.8 is an order-embedding that identifies $\mathrm{Cu}\big(\prod_{\mathcal{U}} A_j\big)$ with the image of S under the map $\pi_{\mathcal{U}}$ from Theorem 15.11.

The computation of the Cuntz semigroup of ultraproducts of C*-algebras in terms of the semigroups of the individual algebras (by means of the scaled product as described above) will be important in studying when the completion of the so-called limit traces is dense in the trace simplex of an ultraproduct; see [6].

16. Outlook

In this final section, we give a sample of problems in the area of Cuntz semigroups which we think are likely to guide the research in this field in the upcoming years. Some of the problems are very difficult and should be regarded as long-term goals, while other ones are more tangible. Also, some of the problems are stated in rather vague terms, while other ones are fairly concrete. Some of the problems below were suggested in the Cuntz semigroup workshop which took place in September 2022 in Kiel, Germany.

As explained in Theorem 9.6, Toms used the Cuntz semigroup to distinguish two simple, separable, unital, nuclear C*-algebras with identical Elliott invariants, thus providing a counterexample to the original formulation of the Elliott conjecture. The following is thus a natural task:

Problem 16.1. For the non $\mathcal{Z}$-stable C*-algebra A constructed by Toms (see Theorem 9.6), compute $\mathrm{Cu}(A)$.

Since Toms' construction is based on that of Villadsen, one should also attempt the following:

Problem 16.2. For a Villadsen algebra A of the first type as in [97], compute $\mathrm{Cu}(A)$.

Some Villadsen algebras of the first type are $\mathcal{Z}$-stable, and in those cases the computation of the Cuntz semigroup is given by Theorem 9.7. Both Problem 16.1 and Problem 16.2 are rather vague, and one concrete problem one should attempt is the computation of the dimensions of these Cuntz semigroups, in the sense of [88, 86].

It was shown in [93] that any two unital homomorphisms from $\mathcal{Z}$ into a $\mathcal{Z}$-stable C*-algebra are approximately unitarily equivalent, and in particular they must agree at the level of the functor Cu. It would be interesting to find an explicit example showing that this fails in the non $\mathcal{Z}$-stable case. Therefore, we suggest:

Problem 16.3. Let A be a non $\mathcal{Z}$-stable C*-algebra as in either Problem 16.1 or Problem 16.2. Are there two distinct Cu-morphisms $\mathrm{Cu}(\mathcal{Z}) \to \mathrm{Cu}(A)$ preserving unit classes?

Another class of C*-algebras for which it would be very interesting to compute the Cuntz semigroup is that of reduced group C*-algebras, particularly for C*-simple groups. While obtaining an explicit computation may be out of reach, it should be possible to obtain some structural information. As a first step, we propose the following problem, which is a Cuntz semigroup version of Blackadar's fundamental comparison property [16] for projections in $C^*_{\mathrm{r}}(\mathbb{F}_n)$:

Problem 16.4. Compute $\mathrm{Cu}(C^*_{\mathrm{r}}(\mathbb{F}_n))$ for $n \geq 2$, or at least determine whether it is almost unperforated.

As it turns out, it is not easy to find examples of stably finite C*-algebras whose Cuntz semigroups fail to be weakly cancellative. There exist commutative examples, but we do not know if simple ones exist as well.

Question 16.5. Does there exist a simple, unital, stably finite C*-algebra whose Cuntz semigroup does not have weak cancellation?

If a C*-algebra as above exists, then its stable rank will necessarily be greater than 1 by Theorem 10.3.

A Cu-semigroup is said to be *almost divisible* if, given $x, x' \in S$ with $x' \ll x$, then for all $n \in \mathbb{N}$ there is $y \in S$ such that $ny \leq x$ and $x' \leq (n+1)y$. If A is a $\mathcal{Z}$-stable C*-algebra, then $\mathrm{Cu}(A)$ is almost divisible, as was shown by Rørdam in [81]; this is proved very similar to Theorem 8.4. The question above is related to the following algebraic question, raised in [8]:

Question 16.6. Under what additional axioms (besides (O5) and (O6)) is a simple Cu-semigroup which is also almost unperforated and almost divisible necessarily weakly cancellative?

We now turn to connections to classification. As mentioned in the introduction, the Cuntz semigroup has been successfully used to classify interesting classes of nonsimple C*-algebras. The latest and most general result in this direction is due to Leonel Robert [73], and there is reason to believe that the class of C*-algebras that Robert considered is the largest that can be classified solely in terms of Cu.

Question 16.7. Are there variations of the Cuntz semigroup that can be used to classify larger classes of non-simple C*-algebras?

The question above is very vague, but it is motivated by the invariant $\mathrm{Cu}^\sim$ considered by Robert in [73] and by Robert-Santiago in [78], which is necessary to obtain classification in the non-unital setting. Another direction in which this question can be interpreted is by trying to incorporate K_1-information into the invariant. Some steps in this direction have been made by Cantier in [28] (see also [29]).

As we saw in Theorem 9.7, in the simple, separable, nuclear, finite $\mathcal{Z}$-stable setting, the Elliott invariant Ell contains the same information as the pair $(\mathrm{Cu}, \mathrm{K}_1)$. In particular, two simple, separable, unital, nuclear, finite $\mathcal{Z}$-stable C*-algebras A and B satisfying the UCT are isomorphic if and only if $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$ and $\mathrm{K}_1(A) \cong \mathrm{K}_1(B)$. The modern approach to classification focuses on the classification of homomorphisms, and it would thus be interesting to know to what extent the Cuntz semigroup can be used in this setting (for some results in this direction, see

[30]). Since the invariant needed to classify homomorphisms is larger than Ell, in particular including algebraic K_1-data, the following seems like a natural question:

Question 16.8. To what extent can the algebraic K_1-group be recovered from the Cuntz semigroup?

The Cuntz semigroup is expected to be useful for classification also beyond the $\mathcal{Z}$-stable setting. A positive answer to the following question would be a significant breakthrough in the area.

Question 16.9. Can one use $(\mathrm{Cu}(A), \mathrm{K}_1(A))$ to classify a class of simple, nuclear C^*-algebras bigger than the one in Theorem 9.3?

The study of group actions on C^*-algebras is a very fruitful one. Some of the most recent research in the area has shown that Cuntz semigroup techniques are extremely powerful in this setting (as proved in [44, 46, 45, 22, 21]), thus suggesting that the theory of group actions on Cuntz semigroups (as developed in [20, 23]) may lead to interesting constructions:

Problem 16.10. Develop the theory of crossed products and Rokhlin properties for group actions on Cu-semigroups.

For actions of compact groups, an equivariant version of the Cuntz semigroup, resembling equivariant K-theory, has been studied in [48]. For actions with the Rokhlin property, the induced dynamical system on Cu has been explored in [47, 42, 43].

KK-theory is a bivariant joint generalization of K-theory and K-homology: for two C^*-algebras A and B, the KK-group $\mathrm{KK}(A, B)$ is a natural homotopy equivalence class of (A, B)-Hilbert bimodules, and it behaves as K-homology in the first coordinate and as K-theory in the second. KK-theory provides a strong link between operator algebras, noncommutative geometry and index theory.

Problem 16.11. Use the bivariant version of Cu introduced in [9, 10] to establish more connections with noncommutative geometry.

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