Lecture 1: Crossed Products of C*-Algebras by Finite Groups

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- Lecture 1 (7 Dec. 2011): Crossed Products of C*-Algebras by Finite Groups.
- Lecture 2 (8 Dec. 2011): Crossed Products of AF Algebras by Actions with the Rokhlin Property.
- Lecture 3 (9 Dec. 2011): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.

A rough outline of all three lectures

- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
- Crossed products by actions with the tracial Rokhlin property.
- Applications of the tracial Rokhlin property.

General motivation

The material to be described is part of the structure and classification theory for simple nuclear C*-algebras (the Elliott program). More specifically, it is about proving that C*-algebras which appear in other parts of the theory (in these lectures, certain kinds of crossed product C*-algebras) satisfy the hypotheses of known classification theorems.

To keep things from being too complicated, we will consider crossed products by actions of finite groups. Nevertheless, even in this case, one can see some of the techniques which are important in more general cases.

Crossed product C*-algebras have long been important in operator algebras, for reasons having nothing to do with the Elliott program. It has generally been difficult to prove that crossed products are classifiable, and there are really only three cases in which there is a somewhat satisfactory theory: actions of finite groups on simple C*-algebras, minimal homeomorphisms of compact metric spaces, and actions of $\mathbb{Z}$ on simple C*-algebras.

Background

These lectures assume some familiarity with the basic theory of C*-algebras, as found, for example, in Murphy's book. K-theory will be occasionally used, but not in an essential way. A few other concepts will be important, such as tracial rank zero. They will be defined as needed, and some basic properties mentioned, usually without proof. Various side comments will assume more background, but these can be skipped.

There are extra slides at the ends of the files, containing extra material (such as additional examples). I do not expect to use them in the lectures. They are included so that they are present in the posted version of the slides.

Group actions on C*-algebras

Definition

Let G be a group and let A be a C*-algebra. An *action of G on A* is a homomorphism $g \mapsto \alpha_g$ from G to $\mathrm{Aut}(A)$.

That is, for each $g \in G$, we have an automorphism $\alpha_g \colon A \to A$, and $\alpha_1 = \mathrm{id}_A$ and $\alpha_g \circ \alpha_h = \alpha_{gh}$ for $g, h \in G$.

In these lectures, almost all groups will be discrete (usually finite). If the group has a topology, one requires that the function $g \mapsto \alpha_g(a)$, from G to A, be continuous for all $a \in A$.

We give examples of actions of finite groups, considering first actions on commutative C*-algebras. These come from actions on locally compact spaces, as described next.

Group actions on spaces

Definition

Let G be a group and let X be a set. Then an *action of G on X* is a map $(g, x) \mapsto gx$ from $G \times X$ to X such that:

- $1 \cdot x = x$ for all $x \in X$.
- $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

If G and X have topologies, then $(g, x) \mapsto gx$ is required to be (jointly) continuous.

When G is discrete, continuity means that $x \mapsto gx$ is continuous for all $g \in G$. Since the action of g^{-1} is also continuous, this map is in fact a homeomorphism.

Group actions on commutative C*-algebras

An action of G on X is a continuous map $(g, x) \mapsto gx$ from $G \times X$ to X such that:

- $1 \cdot x = x$ for all $x \in X$.
- $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$.

Lemma

Let G be a topological group and let X be a locally compact Hausdorff space. Suppose G acts continuously on X. Then there is an action $\alpha \colon G \to \mathrm{Aut}(C_0(X))$ such that $\alpha_g(f)(x) = f(g^{-1}x)$ for $g \in G$, $f \in C_0(X)$, and $x \in X$.

Every action of G on $C_0(X)$ comes this way from an action of G on X.

If G is discrete, this is obvious from the correspondence between maps of locally compact spaces and homomorphisms of commutative C*-algebras. (In the general case, one needs to check that the two continuity conditions correspond properly.)

Examples of group actions on spaces

Every action in this list of a group G on a compact space X gives an action of G on $C(X)$.

- Any group G has a trivial action on any space X, given by $gx = x$ for all $g \in G$ and $x \in X$.
- Any group G acts on itself by (left) translation: gh is the usual product of g and h.
- The finite cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ acts on the circle S^1 by rotation: the standard generator acts as multiplication by $e^{2\pi i/n}$.
- $\mathbb{Z}_2$ acts on S^1 via the order two homeomorphism $\zeta \mapsto \overline{\zeta}$.
- $\mathbb{Z}_2$ acts on S^n via the order two homeomorphism $x \mapsto -x$.

More examples of group actions on spaces

Every action in this list of a group G on a compact space X gives an action of G on $C(X)$.

- Let Z be a compact manifold, or a connected finite complex. (Much weaker conditions on Z suffice, but Z must be path connected.) Let $X = \widetilde{Z}$ be the universal cover of Z, and let $G = \pi_1(Z)$ be the fundamental group of Z. Then there is a standard action of G on X. Spaces with finite fundamental groups include real projective spaces (in which case this example was already on the previous slide) and lens spaces.
- The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{R}^2$ via the usual matrix multiplication. This action preserves $\mathbb{Z}^2$, and so is well defined on $\mathbb{R}^2/\mathbb{Z}^2 \cong S^1 \times S^1$. There are finite cyclic subgroups of orders 2, 3, 4, and 6, generated by
$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$
Restriction gives actions of these on $S^1 \times S^1$.

Group actions on noncommutative C*-algebras

Some elementary examples:

- For every group G and every C*-algebra A, there is a trivial action $\iota\colon G \to \mathrm{Aut}(A)$, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in A$.
- Suppose $g \mapsto z_g$ is a (continuous) homomorphism from G to the unitary group of a unital C*-algebra A. Then $\alpha_g(a) = z_g a z_g^*$ defines an action of G on A. (We write $\alpha_g = \mathrm{Ad}(z_g)$.) This is an *inner* action.
- As a special case, let G be a finite group, and let $g \mapsto z_g$ be a unitary representation of G on $\mathbb{C}^n$. Then $g \mapsto \mathrm{Ad}(z_g)$ defines an action of G on M_n.

Product type actions

We describe a particular "product type action". Let $A_n = (M_2)^{\otimes n}$, the tensor product of n copies of the algebra M_2 of 2×2 matrices. Thus $A_n \cong M_{2^n}$. Define
$$\varphi_n\colon A_n \to A_{n+1} = A_n \otimes M_2$$
by $\varphi_n(a) = a \otimes 1$. Let A be the (completed) direct limit $\varinjlim_n A_n$. (This is just the 2^∞ UHF algebra.) Define a unitary $v \in M_2$ by
$$v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define $z_n \in A_n$ by $z_n = v^{\otimes n}$. Define $\alpha_n \in \mathrm{Aut}(A_n)$ by $\alpha_n = \mathrm{Ad}(z_n)$. Then α_n is an inner automorphism of order 2. Using $z_{n+1} = z_n \otimes v$, one can easily check that $\varphi_n \circ \alpha_n = \alpha_{n+1} \circ \varphi_n$ for all n, and it follows that the α_n determine an order 2 automorphism α of A. Thus, we have an action of $\mathbb{Z}_2$ on A. This action is not inner, although it is "approximately inner".

General product type actions

We had $A = \varinjlim_n (M_2)^{\otimes n}$ with the action of $\mathbb{Z}_2$ generated by the direct limit automorphism

$$\varinjlim_n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{\otimes n}$$

We write this automorphism as

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

In general, one can use an arbitrary group, one need not choose the same unitary representation in each tensor factor, and the tensor factors need not all be the same size.

More examples of product type actions

We will later use the following two additional examples:

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_3,$$

and

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad}(\mathrm{diag}(-1, 1, 1, \ldots, 1)) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_{2^n+1}.$$

In the second one, there are supposed to be 2^n ones on the diagonal, giving a $(2^n + 1) \times (2^n + 1)$ matrix.

Irrational rotation algebras

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Recall the irrational rotation algebra A_θ, the (simple, and unique) C*-algebra generated by two unitaries u and v satisfying the commutation relation $vu = e^{2\pi i \theta} uv$.

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on A_θ by sending the matrix

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix}$$

to the automorphism determined by

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}}$$

and

$$\alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$

The algebra A_θ is often considered to be a noncommutative analog of the torus $S^1 \times S^1$, and this action is the analog of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$.

Irrational rotation algebras (continued)

Recall that $\mathrm{SL}_2(\mathbb{Z})$ has finite cyclic subgroups of orders 2, 3, 4, and 6, generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Restriction gives actions of these groups on the irrational rotation algebras.

In terms of generators of A_θ, and omitting the scalar factors (which are not necessary when one restricts to these subgroups), the action of $\mathbb{Z}_2$ is generated by

$$u \mapsto u^* \quad \text{and} \quad v \mapsto v^*,$$

and the action of $\mathbb{Z}_4$ is generated by

$$u \mapsto v \quad \text{and} \quad v \mapsto u^*.$$

Irrational rotation algebras (continued)

A_θ is generated by unitaries u and v such that $vu = e^{2\pi i\theta}uv$.

There is a "gauge action" on A_θ, which multiplies the generators by scalars of absolute value 1. In particular, the group $\mathbb{Z}_n$ acts on A_θ by sending a generator of the group to the automorphism determined by

$$u \mapsto e^{2\pi i/n}u \quad \text{and} \quad v \mapsto v.$$

One can also use

$$u \mapsto u \quad \text{and} \quad v \mapsto e^{2\pi i/n}v.$$

Tensor products

Assume (for convenience) that A is nuclear and unital. Then there is an action of $\mathbb{Z}_2$ on $A \otimes A$ generated by the "tensor flip" $a \otimes b \mapsto b \otimes a$.

Similarly, the symmetric group S_n acts on $A^{\otimes n}$.

The tensor flip on the 2^∞ UHF algebra A turns out to be conjugate to a product type action, namely

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{on} \quad \bigotimes_{n=1}^{\infty} M_4.$$

Another interesting example is gotten by taking A to be the Jiang-Su algebra Z. The algebra Z is simple, separable, unital, and nuclear. It has no nontrivial projections, its Elliott invariant is the same as for $\mathbb{C}$, and $Z \otimes Z \cong Z$.

Crossed products

Let G be a locally compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of G on a C*-algebra A. There is a *crossed product C*-algebra* $C^*(G, A, \alpha)$, which is a kind of generalization of the group C*-algebra $C^*(G)$. Crossed products are quite important in the theory of C*-algebras.

One motivation: Suppose G is a semidirect product $N \rtimes H$. The action of H on N gives an action $\alpha\colon H \to \mathrm{Aut}(C^*(N))$, and one has $C^*(G) \cong C^*(H, C^*(N), \alpha)$. Thus, crossed products appear even if one is only interested in group C*-algebras and unitary representations of groups.

Another motivation (not applicable to finite groups acting on spaces): The noncommutative version of X/G is the fixed point algebra A^G. In particular, for compact G, one can check that $C(X/G) \cong C(X)^G$. For noncompact groups, often X/G is very far from Hausdorff and A^G is far too small. The crossed product provides a much more generally useful algebra, which is the "right" substitute for the fixed point algebra when the action is free.

Crossed products by finite groups

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. As a vector space, $C^*(G, A, \alpha)$ is the group ring $A[G]$, consisting of all finite formal linear combinations of elements in G with coefficients in A. The multiplication and adjoint are given by

$$(a \cdot g)(b \cdot h) = (a[gbg^{-1}]) \cdot (gh) = (a\alpha_g(b)) \cdot (gh)$$

and

$$(a \cdot g)^* = \alpha_g^{-1}(a^*) \cdot g^{-1}$$

for $a, b \in A$ and $g, h \in G$, extended linearly. There is a unique norm which makes this a C*-algebra. (See below.)

If A is unital, the group elements $g = 1 \cdot g$ are in $A[G]$, and are unitary. We conventionally write u_g instead of g for the element of $A[G]$. Thus, a general element of $A[G]$ has the form $c = \sum_{g \in G} c_g u_g$ with $c_g \in A$ for $g \in G$. (This actually works even if A is not unital.)

If G is discrete but not finite, $C^*(G, A, \alpha)$ is the completion of $A[G]$ in a suitable norm. (Before completion, we have the *skew group ring*.)

Crossed products by finite groups (continued)

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. We construct a C* norm on the skew group ring $A[G]$.
Recall:

$$(a \cdot g)(b \cdot h) = (a\alpha_g(b)) \cdot (gh) \quad \text{and} \quad (a \cdot g)^* = \alpha_g^{-1}(a^*) \cdot g^{-1},$$

that is,

$$(au_g)(bu_h) = a\alpha_g(b)u_{gh} \quad \text{and} \quad (au_g)^* = \alpha_g^{-1}(a^*)u_{g^{-1}}.$$

Fix a faithful representation $\pi\colon A \to L(H_0)$ of A on a Hilbert space H_0. Set $H = l^2(G, H_0)$, the set of all $\xi = (\xi_g)_{g\in G}$ in $\bigoplus_{g\in G} H_0$, with the scalar product

$$\big\langle (\xi_g)_{g\in G},\, (\eta_g)_{g\in G} \big\rangle = \sum_{g\in G} \langle \xi_g, \eta_g \rangle.$$

Then define $\sigma\colon A[G] \to L(H)$ as follows. For $c = \sum_{g\in G} c_g u_g$,

$$(\sigma(c)\xi)_h = \sum_{g\in G} \pi(\alpha_h^{-1}(c_g))(\xi_{g^{-1}h})$$

for all $h \in G$.

Crossed products by finite groups (continued)

Recall:

$$(au_g)(bu_h) = a\alpha_g(b)u_{gh} \quad \text{and} \quad (au_g)^* = \alpha_g^{-1}(a^*)u_{g^{-1}}.$$

Also, for $c = \sum_{g\in G} c_g u_g$,

$$(\sigma(c)\xi)_h = \sum_{g\in G} \pi(\alpha_h^{-1}(c_g))(\xi_{g^{-1}h}).$$

If A is unital, then for $a \in A$ and $g \in G$, identify a with au_1 and get

$$(\sigma(a)\xi)_h = \pi(\alpha_{h^{-1}}(a))(\xi_h) \quad \text{and} \quad (\sigma(u_g)\xi)_h = \xi_{g^{-1}h}.$$

One can check that σ is a *-homomorphism. We will just check the most important part, which is that $\sigma(u_g)\sigma(b) = \sigma(\alpha_g(b))\sigma(u_g)$. We have

$$\big[\sigma(\alpha_g(b))\sigma(u_g)\xi\big]_h = \pi\big(\alpha_{h^{-1}}(\alpha_g(b))\big)(\sigma(u_g)\xi)_h = \pi(\alpha_{h^{-1}g}(b))(\xi_{g^{-1}h})$$

and

$$\big(\sigma(u_g)\sigma(b)\xi\big)_h = \big(\sigma(b)\xi\big)_{g^{-1}h} = \pi\big(\alpha_{h^{-1}g}(b)\big)\xi\big)_{g^{-1}h}.$$

Crossed products by finite groups (continued)

Recall: for $c = \sum_{g\in G} c_g u_g$,

$$(\sigma(c)\xi)_h = \sum_{g\in G} \pi(\alpha_h^{-1}(c_g))(\xi_{g^{-1}h}).$$

If A is unital, then for $a \in A$ and $g \in G$,

$$(\sigma(a)\xi)_h = \pi(\alpha_{h^{-1}}(a))(\xi_h) \quad \text{and} \quad (\sigma(u_g)\xi)_h = \xi_{g^{-1}h}.$$

For $c = \sum_{g\in G} c_g u_g$, it is easy to check that

$$\|\sigma(c)\| \leq \sum_{g\in G} \|c_g\|$$

and not much harder to check that

$$\|\sigma(c)\| \geq \max_{g\in G} \|c_g\|.$$

The norms on the right hand sides are equivalent, so $A[G]$ is complete in the norm $\|c\| = \|\sigma(c)\|$.

Crossed products by finite groups (continued)

We are still considering an action $\alpha\colon G \to \mathrm{Aut}(A)$ of a finite group G on a C*-algebra A.

We started with a faithful representation $\pi\colon A \to L(H_0)$ of A on a Hilbert space H_0. Then we constructed a representation $\sigma\colon A[G] \to L(l^2(G, H_0))$. We found that $A[G]$ is complete in the norm $\|c\| = \|\sigma(c)\|$. By standard theory, the norm $\|c\| = \|\sigma(c)\|$ is therefore the only norm in which $A[G]$ is a C*-algebra. In particular, it does not depend on the choice of π.

We return to the notation $C^*(G, A, \alpha)$ for the crossed product. The crossed product has a universal property, which we omit here.

Things are more complicated if G is discrete but not finite. (In particular, there may be more than one reasonable norm—since $A[G]$ isn't complete, this is not ruled out.) The situation is even more complicated if G is merely locally compact.

Examples of crossed products by finite groups

Let G be a finite group, and let $\iota\colon G \to \mathrm{Aut}(\mathbb{C})$ be the trivial action, defined by $\iota_g(a) = a$ for all $g \in G$ and $a \in \mathbb{C}$. Then $C^*(G, \mathbb{C}, \iota) = C^*(G)$, the group C*-algebra of G. (So far, G could be any locally compact group.)

Since we are assuming that G is finite, this is a finite dimensional C*-algebra, with $\dim(C^*(G)) = \mathrm{card}(G)$. If G is abelian, so is $C^*(G)$, so $C^*(G) \cong \mathbb{C}^{\mathrm{card}(G)}$.

In general, $C^*(G)$ turns out to be the direct sum of matrix algebras, one summand M_k for each unitary equivalence class of irreducible representations of G, with k being the dimension of the representation.

Now let A be any C*-algebra, and let $\iota_A\colon G \to \mathrm{Aut}(A)$ be the trivial action. It is not hard to see that $C^*(G, A, \iota_A) \cong C^*(G) \otimes A$. The elements of A "factor out", since $A[G]$ is just the ordinary group ring.

Examples of crossed products (continued)

Let G be a finite group, acting on $C(G)$ via the translation action on G. Set $n = \mathrm{card}(G)$. We describe how to prove that $C^*(G,\, C(G)) \cong M_n$.

Let $\alpha\colon G \to \mathrm{Aut}(C(G))$ denote the action. For $g \in G$, we let u_g be the standard unitary (as defined above), and we let $\delta_g \in C(G)$ be the function $\chi_{\{g\}}$. Then $\alpha_g(\delta_h) = \delta_{gh}$ for $g, h \in G$. For $g, h \in G$, set

$$v_{g,h} = \delta_g u_{gh^{-1}} \in C^*(G,\, C(G),\, \alpha).$$

These elements form a system of matrix units. We check:

$$\begin{aligned} v_{g_1,h_1} v_{g_2,h_2} &= \delta_{g_1} u_{g_1 h_1^{-1}} \delta_{g_2} u_{g_2 h_2^{-1}} \\ &= \delta_{g_1} \alpha_{g_1 h_1^{-1}}(\delta_{g_2}) u_{g_1 h_1^{-1}} u_{g_2 h_2^{-1}} = \delta_{g_1} \delta_{g_1 h_1^{-1} g_2} u_{g_1 h_1^{-1} g_2 h_2^{-1}}. \end{aligned}$$

Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is v_{g_1,h_2}. Similarly, $v_{g,h}^* = v_{h,g}$.

Since the elements δ_g span $C(G)$, the elements $v_{g,h}$ span $C^*(G,\, C(G),\, \alpha)$. So $C^*(G,\, C(G),\, \alpha) \cong M_n$ with $n = \mathrm{card}(G)$.

Examples of crossed products (continued)

Let G be a finite group, acting on $C(G)$ via the translation action on G. Set $n = \mathrm{card}(G)$. Then $C^*(G,\, C(G)) \cong M_n$.

Now consider G acting on $G \times X$, by translation on G and trivially on X. The same method gives $C^*(G,\, C_0(G \times X)) \cong C_0(X, M_n)$.

This result generalizes greatly: for any locally compact group G, one gets $C^*(G,\, C_0(G)) \cong K(L^2(G))$, etc.

Equivariant exact sequences

We will describe several more examples, mostly without proof. To understand what to expect, the following is helpful.

For $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$, we say that a homomorphism $\varphi\colon A \to B$ is *equivariant* if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

One can check (this is easy when G is finite) that an equivariant homomorphism $\varphi\colon A \to B$ induces a homomorphism

$$\overline{\varphi}\colon C^*(G, A, \alpha) \to C^*(G, B, \beta),$$

just by applying φ to the algebra elements. Thus, if the standard unitaries in $C^*(G, A, \alpha)$ are called u_g and the standard unitaries in $C^*(G, B, \beta)$ are called v_g, then

$$\overline{\varphi}\left(\sum_{g \in G} c_g u_g \right) = \sum_{g \in G} \varphi(c_g) v_g.$$

Equivariant exact sequences

The homomorphism φ is equivariant if $\varphi(\alpha_g(a)) = \beta_g(\varphi(a))$ for all $g \in G$ and $a \in A$.

Recall that equivariant homomorphisms induce homomorphisms of crossed products.

Theorem

Let G be a locally compact group. Let $0 \to J \to A \to B \to 0$ be an exact sequence of C*-algebras with actions γ of G on J, α of G on A, and β of G on B, and equivariant maps. Then the sequence

$$0 \longrightarrow C^*(G, J, \gamma) \longrightarrow C^*(G, A, \alpha) \longrightarrow C^*(G, B, \beta) \longrightarrow 0$$

is exact.

When G is finite, the proof is easy:

$$0 \longrightarrow J[G] \longrightarrow A[G] \longrightarrow B[G] \longrightarrow 0$$

is clearly exact.

Examples of crossed products (continued)

Recall the example from earlier: $\mathbb{Z}_n$ acts on the circle S^1 by rotation, with the standard generator acting by multiplication by $\omega = e^{2\pi i/n}$.

For any point $x \in S^1$, let

$$L_x = \{\omega^k x \colon k = 0, 1, \ldots, n-1\} \quad \text{and} \quad U_x = S^1 \setminus L_x.$$

Then L_x is equivariantly homeomorphic to $\mathbb{Z}_n$ with translation, and U_x is equivariantly homeomorphic to

$$\mathbb{Z}_n \times \left\{ e^{2\pi i t/n} x \colon 0 < t < 1 \right\} \cong \mathbb{Z}_n \times (0, 1).$$

The equivariant exact sequence

$$0 \longrightarrow C_0(U_x) \longrightarrow C(S^1) \longrightarrow C(L_x) \longrightarrow 0$$

gives the following exact sequence of crossed products:

$$0 \longrightarrow C_0((0,1), M_n) \longrightarrow C^*(\mathbb{Z}_n, C(S^1)) \longrightarrow M_n \longrightarrow 0.$$

With more work (details are in my crossed product notes), one can show that $C^*(\mathbb{Z}_n, C(S^1)) \cong C(S^1, M_n)$. The space S^1 on the right arises as the orbit space S^1/Z_n.

Examples of crossed products (continued)

We use the standard abbreviation $C^*(G, X) = C^*(G, C_0(X))$.

For the action of $\mathbb{Z}_n$ on the circle S^1 by rotation, we got $C^*(\mathbb{Z}_n, C(S^1)) \cong C(S^1, M_n)$.

Recall the example from earlier: $\mathbb{Z}_2$ acts on S^n via the order two homeomorphism $x \mapsto -x$.

Based on what happened with $\mathbb{Z}_n$ acting on the circle S^1 by rotation, one might hope that $C^*(\mathbb{Z}_2, S^n) \cong C(S^n/\mathbb{Z}_2, M_2)$. This is almost right, but not quite. In fact, $C^*(\mathbb{Z}_2, S^n)$ turns out to be the section algebra of a bundle over $S^n/\mathbb{Z}_2$ with fiber M_2, and the bundle is locally trivial—but not trivial.

Examples of crossed products (continued)

Recall the example from earlier: $\mathbb{Z}_2$ acts on S^1 via the order two homeomorphism $\zeta \mapsto \overline{\zeta}$.

Set

$$L = \{-1, 1\} \quad \text{and} \quad U = X \setminus L.$$

Then the action on L is trivial, and U is equivariantly homeomorphic to

$$\mathbb{Z}_2 \times \{x \in U \colon \mathrm{Im}(x) > 0\} \cong \mathbb{Z}_2 \times (-1, 1).$$

The equivariant exact sequence

$$0 \longrightarrow C_0(U) \longrightarrow C(S^1) \longrightarrow C(L) \longrightarrow 0$$

gives the following exact sequence of crossed products:

$$0 \longrightarrow C_0((-1,1), M_2) \longrightarrow C^*(\mathbb{Z}_2, C(S^1)) \longrightarrow C(L) \otimes C^*(\mathbb{Z}_2) \longrightarrow 0,$$

in which $C(L) \otimes C^*(\mathbb{Z}_2) \cong \mathbb{C}^4$. With more work (details are in my crossed product notes), one can show that $C^*(\mathbb{Z}_n, C(S^1))$ is isomorphic to

$$\{f \in C([-1, 1], M_2) \colon f(1) \text{ and } f(-1) \text{ are diagonal matrices}\}.$$

Crossed products by inner actions

Recall the inner action $\alpha_g = \mathrm{Ad}(z_g)$ for a continuous homomorphism $g \mapsto z_g$ from G to the unitary group of a C*-algebra A. The crossed product is the same as for the trivial action, in a canonical way.

Assume G is finite. Let $\iota\colon G \to \mathrm{Aut}(A)$ be the trivial action of G on A. Let $u_g \in C^*(G, A, \alpha)$ and $v_g \in C^*(G, A, \iota)$ be the unitaries corresponding to the group elements. The isomorphism φ sends $a \cdot u_g$ to $a z_g \cdot v_g$. This is clearly a linear bijection of the skew group rings.

We check the most important part of showing that φ is an algebra homomorphism. Recall that $u_g b = \alpha_g(b) u_g$ (and $v_g b = \iota_g(b) v_g = b v_g$). So we need $\varphi(u_g)\varphi(b) = \varphi(u_g b)$. We have

$$\varphi(u_g b) = \varphi(\alpha_g(b) u_g) = \alpha_g(b) z_g v_g$$

and, using $z_g b = \alpha_g(b) z_g$,

$$\varphi(u_g)\varphi(b) = z_g v_g v = z_g b v_g = \alpha_g(b) z_g v_g.$$

A detailed computation can be found in my crossed product notes.

Crossed products by product type actions

Recall the action of $\mathbb{Z}_2$ on the 2^∞ UHF algebra generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

Write it as $\alpha = \varinjlim_n \mathrm{Ad}(z_n)$ on $A = \varinjlim_n M_{2^n}$.

It is not hard to show that crossed products commute with direct limits. Since $\mathrm{Ad}(z_n)$ is inner, we get

$$C^*(\mathbb{Z}_2, M_{2^n}, \mathrm{Ad}(z_n)) \cong C^*(\mathbb{Z}_2) \otimes M_{2^n} \cong M_{2^n} \otimes M_{2^n}.$$

Now we have to use the explicit form of these isomorphisms to compute the maps in the direct system of crossed products, and then find the direct limit. In this particular case, the maps turn out to be

$$(a, b) \mapsto \big(\mathrm{diag}(a, b),\ \mathrm{diag}(a, b)\big),$$

and a computation with Bratteli diagrams shows that the direct limit is again the 2^∞ UHF algebra. (In general, the direct limit will be more complicated, and usually not a UHF algebra.)

Appendix 1: More examples of group actions on spaces

Every action in this list of a group G on a compact space X gives an action of G on $C(X)$.

- If G is a group and H is a (closed) subgroup (not necessarily normal), then G has a translation action on G/H.
- Let Y be a compact space, and set $X = Y \times Y$. Then $G = \mathbb{Z}_2$ acts on X via the order two homeomorphism $(y_1, y_2) \mapsto (y_2, y_1)$. Similarly, the symmetric group S_n acts on Y^n.

Appendix 2: More examples of group actions on C*-algebras

Let $A = M_2$, let $G = (\mathbb{Z}_2)^2$ with generators g_1 and g_2, and set

$$\alpha_1 = \mathrm{id}_A, \qquad \alpha_{g_1} = \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\alpha_{g_2} = \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \alpha_{g_1 g_2} = \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These define an action $\alpha\colon G \to \mathrm{Aut}(A)$ such that α_g is inner for all $g \in G$, but for which there is no homomorphism $g \mapsto z_g \in U(A)$ such that $\alpha_g = \mathrm{Ad}(z_g)$ for all $g \in G$. The point is that the implementing unitaries for α_{g_1} and α_{g_2} commute up to a scalar, but can't be appropriately modified to commute exactly.

Appendix 2: More examples (continued)

We take the generators of the Cuntz algebra $\mathcal{O}_d$ to be $s_1, s_2, \ldots, s_d$, satisfying

$$s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1 \quad \text{and} \quad s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1.$$

(In the second condition, if $d = \infty$, just the $s_j s_j^*$ are orthogonal.)

We give the general quasifree action here. Two special cases on the next slide have much simpler formulas.

Let $\rho\colon G \to L(\mathbb{C}^d)$ be a unitary representation of G. Write

$$\rho(g) = \begin{pmatrix} \rho_{1,1}(g) & \cdots & \rho_{1,d}(g) \\ \vdots & \ddots & \vdots \\ \rho_{d,1}(g) & \cdots & \rho_{d,d}(g) \end{pmatrix}$$

for $g \in G$. Then there exists a unique action $\beta^\rho\colon G \to \mathrm{Aut}(\mathcal{O}_d)$ such that

$$\beta_g^\rho(s_k) = \sum_{j=1}^{d} \rho_{j,k}(g) s_j$$

for $j = 1, 2, \ldots, d$. (This can be checked by a computation.)

Appendix 2: more examples (continued)

The Cuntz relations: $s_1^* s_1 = s_2^* s_2 = \cdots = s_d^* s_d = 1$ and $s_1 s_1^* + s_2 s_2^* + \cdots + s_d s_d^* = 1$.

Some special cases of quasifree actions, for which it is easy to see that they really are group actions:

- For $G = \mathbb{Z}_n$, choose n-th roots of unity $\zeta_1, \zeta_2, \ldots, \zeta_d$ and let a generator of the group multiply s_j by ζ_j.
- Let G be a finite group. Take $d = \mathrm{card}(G)$, and label the generators s_g for $g \in G$. Then define $\beta^G\colon G \to \mathrm{Aut}(\mathcal{O}_d)$ by $\beta_g^G(s_h) = s_{gh}$ for $g, h \in G$. (This is the quasifree action coming from regular representation of G.)
- Label the generators of $\mathcal{O}_\infty$ as $s_{g,j}$ for $g \in G$ and $j \in \mathbb{N}$. Define $\iota\colon G \to \mathrm{Aut}(\mathcal{O}_\infty)$ by $\iota_g(s_{h,j}) = s_{gh,j}$ for $g \in G$ and $j \in \mathbb{N}$. (This is the quasifree action coming from the direct sum of infinitely many copies of the regular representation of G.)

Appendix 2: More examples (continued)

Let A be a C*-algebra, and let $A \star A$ be the free product of two copies of A. Then there is an automorphism $\alpha \in \mathrm{Aut}(A \star A)$ which exchanges the two free factors. For $a \in A$, it sends the copy of a in the first free factor to the copy of the same element in the second free factor, and similarly the copy of a in the second free factor to the copy of the same element in the first free factor. This automorphism might be called the "free flip". It generates an action of $\mathbb{Z}_2$ on $A \star A$.

There are many generalizations. One can take the amalgamated free product $A \star_B A$ over a subalgebra $B \subset A$ (using the same inclusion in both copies of A), or the reduced free product $A \star_{\mathrm{r}} A$ (using the same state on both copies of A). There is a permutation action of S_n on the free product of n copies of A. And one can make any combination of these generalizations.

Appendix 3: The universal property of the crossed product

The crossed product $C^*(G, A, \alpha)$ (for G locally compact) is defined in such a way as to have a universal property which generalizes the universal property of the group C*-algebra $C^*(G)$.

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. A *covariant representation* of (G, A, α) on a Hilbert space H is a pair (v, π) consisting of a unitary representation $v\colon G \to U(H)$ (the unitary group of H) and a representation $\pi\colon A \to L(H)$ (the algebra of all bounded operators on H), satisfying the *covariance condition*

$$v(g)\pi(a)v(g)^* = \pi(\alpha_g(a))$$

for all $g \in G$ and $a \in A$. It is called *nondegenerate* if π is nondegenerate.

Appendix 3: The universal property (continued)

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A, and let (v, π) be a covariant representation of (G, A, α) on a Hilbert space H. Then the *integrated form* of (v, π) is the representation $\sigma\colon C_{\mathrm{c}}(G, A, \alpha) \to L(H)$ given by

$$\sigma(a)\xi = \int_G \pi(a(g))v(g)\xi \, dg.$$

$C^*(G, A, \alpha)$ is then a completion of $C_{\mathrm{c}}(G, A, \alpha)$, chosen to give:

Theorem

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a locally compact group G on a C*-algebra A. Then the integrated form construction defines a bijection from the set of nondegenerate covariant representations of (G, A, α) on a Hilbert space H to the set of nondegenerate representations of $C^*(G, A, \alpha)$ on the same Hilbert space.