Seoul National University short course: An introduction to the structure of crossed product C*-algebras.

Lecture 2: Explicit computations

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Recall (from the formula for the regular representation when G is discrete):

Corollary

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $\pi_0 \colon A \to L(H_0)$ be a representation, and let $\sigma \colon C^*_{\mathrm{r}}(G, A, \alpha) \to L(H) = L(L^2(G, H_0))$ be the associated regular representation. Let $a = \sum_{g \in G} a_g u_g \in C^*_{\mathrm{r}}(G, A, \alpha)$, with $a_g = 0$ for all but finitely many g.

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$$\xi(h) = \begin{cases} \eta & h = g \\ 0 & h \neq g. \end{cases}$$

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The last statement follows by continuity from "picking off coordinates" in the regular representation. □

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3. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $E_1(a^* a) = 0$, then $a = 0$.

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Proof of the properties of coefficients

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(2): Suppose $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $\sigma(a) = 0$. Fix $l \in G$. Taking $h = g^{-1}$ and $k = l^{-1} g^{-1}$ in the previous proposition, we get $(\pi_0 \circ \alpha_g)(E_l(a)) = 0$ for all $g \in G$. So $E_l(a) = 0$. This is true for all $l \in G$, so $a = 0$.

(3): As before, let $a = \sum_{g \in G} a_g u_g \in C_{\mathrm{c}}(G, A, \alpha)$. Then $a^* a = \sum_{g, h \in G} u_g^* a_g^* a_h u_h$, so

$$E_1(a^* a) = \sum_{g \in G} u_g^* a_g^* a_g u_g = \sum_{g \in G} \alpha_g^{-1}\big(E_g(a)^* E_g(a)\big).$$

In particular, for each fixed g, we have $E_1(a^* a) \geq \alpha_g^{-1}\big(E_g(a)^* E_g(a)\big)$. By continuity, this inequality holds for all $a \in C^*_{\mathrm{r}}(G, A, \alpha)$. Thus, if $E_1(a^* a) = 0$, then $E_g(a)^* E_g(a) = 0$ for all g, so $a = 0$ by Part (1). This completes the proof.

Injective representations of A always give injective regular representations of the reduced crossed product

It is true for general locally compact groups, not just discrete groups, that the regular representation of $C^*_{\mathrm{r}}(G, A, \alpha)$ associated to an injective representation of A is injective. See Theorem 7.7.5 of Pedersen's book.

The conditional expectation

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Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. Let $E = E_1 \colon C^*_{\mathrm{r}}(G, A, \alpha) \to A$ be as above. Prove that E has the following properties:

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1. $E(E(b)) = E(b)$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.

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3. $\|E(b)\| \leq \|b\|$ for all $b \in C^*_{\mathrm{r}}(G, A, \alpha)$.
4. If $a \in A$ and $b \in C^*_{\mathrm{r}}(G, A, \alpha)$, then $E(ab) = aE(b)$ and $E(ba) = E(b)a$.

The limits of coefficients

Unfortunately, in general $\sum_{g \in G} a_g u_g$ does not converge in $C^*_{\mathrm{r}}(G, A, \alpha)$, and it is very difficult to tell exactly which families of coefficients correspond to elements of $C^*_{\mathrm{r}}(G, A, \alpha)$.

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The limits of coefficients (continued)

Let's pursue this a little farther. The regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$ gives an injective map $\lambda \colon C^*(\mathbb{Z}) \to L(l^2(\mathbb{Z}))$.

The limits of coefficients (continued)

Let's pursue this a little farther. The regular representation of $\mathbb{Z}$ on $l^2(\mathbb{Z})$ gives an injective map $\lambda\colon C^*(\mathbb{Z}) \to L(l^2(\mathbb{Z}))$. Let $\delta_n \in l^2(\mathbb{Z})$ be the function

$$\delta_n(k) = \begin{cases} 1 & k = n \\ 0 & k \neq n. \end{cases}$$

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The limits of coefficients (continued)

Even if one understands completely what all the elements of $C^*_{\mathrm{r}}(G)$ are, and even if the action is trivial, understanding the elements of the reduced crossed product requires that one understand all the elements of the completed tensor product $C^*_{\mathrm{r}}(G) \otimes_{\min} A$.

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There is just one bright spot: although we will not prove it here, there is an analog for general crossed products by $\mathbb{Z}$ of the fact that the Cesaro means of the Fourier series of a continuous function always converge uniformly to the function. See Theorem 8.2.2 of Davidson's book.

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The discussion above is meant to point out the difficulties in dealing with crossed products by infinite groups. Despite all this, for some problems, finite groups are harder. Computing the K-theory of a crossed product by $\mathbb{Z}/2\mathbb{Z}$ is harder than computing the K-theory of a crossed product by any of $\mathbb{Z}$, $\mathbb{R}$, or even a (nonabelian) free group!

Preliminaries for computing crossed products

We will shortly do some explicit computations of examples. First, though, we give some useful preliminaries:

- Equivariant maps and functoriality.
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- Crossed products and direct limits.
- Notation for matrix units.

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In many of the cases we consider, the ideal structure of the crossed product can be derived from the Gootman-Rosenberg theorem. (See below for a little more about this theorem.) In some cases, one can then use this information to determine the entire structure of the crossed product.

Equivariant homomorphisms

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Definition

If (G, A, α) and (G, B, β) are G-algebras, then a homomorphism $\varphi \colon A \to B$ is said to be *equivariant* (or G-equivariant if the group must be specified) if for every $g \in G$, we have $\varphi \circ \alpha_g = \beta_g \circ \varphi$.

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For a fixed locally compact group G, the G-algebras and equivariant homomorphisms form a category.

The crossed product construction is functorial for equivariant homomorphisms

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$$C^*(G, A, \alpha) \to C^*(G, B, \beta) \quad \text{and} \quad C^*_{\mathrm{r}}(G, A, \alpha) \to C^*_{\mathrm{r}}(G, B, \beta).$$

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This construction makes the crossed product and reduced crossed product constructions functors from the category of G-algebras to the category of C*-algebras.

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This is straightforward. See the notes for details.

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$$0 \longrightarrow C^*(G, J, \gamma) \longrightarrow C^*(G, A, \alpha) \longrightarrow C^*(G, B, \beta) \longrightarrow 0$$

is exact.

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Let $0 \to J \to A \to B \to 0$ be an exact sequence of G-algebras, with actions γ on J, α on A, and β on B. Then the sequence

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Proofs can be found in the three places listed in the notes.

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The analog for reduced crossed products is in general false.

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The proof is done by combining the universal properties of direct limits and crossed products. See the notes.

Notation for matrix units

For any index set S, let $\delta_s \in l^2(S)$ be the standard basis vector, determined by

$$\delta_s(t) = \begin{cases} 1 & t = s \\ 0 & t \neq s. \end{cases}$$

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$$e_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad e_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

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Example: The trivial action (continued)

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Note how full and reduced crossed products parallel maximal and minimal tensor products.

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For preservation of adjoints:

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So φ_0 is an isometric isomorphism of *-algebras, and therefore extends to an isomorphism of the universal C*-algebras.

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Thus, if $g_2 \neq h_1$, the answer is zero, while if $g_2 = h_1$, the answer is v_{g_1,h_2}. Similarly, $v_{g,h}^* = v_{h,g}$. That is, the elements $v_{g,h}$ satisfy the relations for a system of matrix units indexed by G. Also, $\mathrm{span}\big(\{v_{g,h} \colon g, h \in G\}\big)$ is dense in $l^1(G, \, C_0(G), \, \alpha)$, and hence in $C^*(G, \, C_0(G), \, \alpha)$.

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If G acts on $G \times X$ by translation in the first factor and trivially in the second factor, we get the crossed product $C(X, \, K(l^2(G)))$. (In fact, this is true for an arbitrary action of G on X. See the notes.)

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Fix $n \in \mathbb{Z}_{>0}$, and let $G = \mathbb{Z}/n\mathbb{Z}$ act on S^1 via the rotation by $2\pi/n$, that is, with generator the homeomorphism $h(\zeta) = e^{2\pi i/n}\zeta$ for $\zeta \in S^1$.

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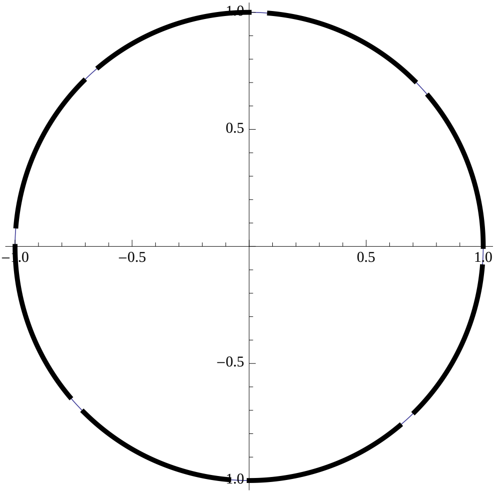
Example: $n = 8$



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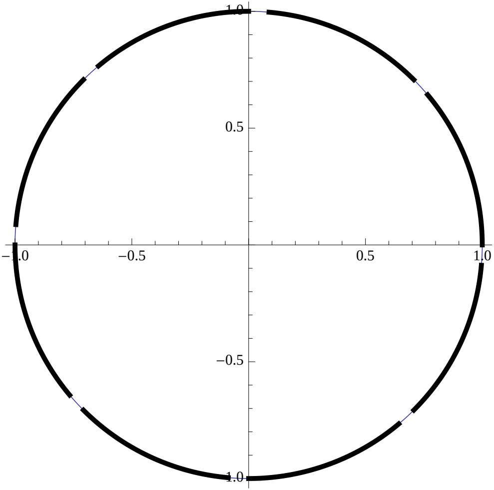

The heavy sections are the 8 translates of the closed arc Y from 1 to $\exp(2\pi i/9)$. They give a quotient of the crossed product isomorphic to $C(Y, M_8)$.

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

Here are the details. Let $\alpha \in \mathrm{Aut}(C(S^1))$ be the order n automorphism $\alpha(f) = f \circ h^{-1}$.

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Here are the details. Let $\alpha \in \mathrm{Aut}(C(S^1))$ be the order n automorphism $\alpha(f) = f \circ h^{-1}$. Thus, $\alpha(f)(\zeta) = f(e^{-2\pi i/n}\zeta)$ for $\zeta \in S^1$.

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$$s = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

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The key computation is

$$s \, \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \, s^* = \mathrm{diag}(\lambda_n, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}).$$

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

Set

$$B = \{f \in C([0,1], M_n) \colon f(0) = s f(1) s^*\}.$$

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Set

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Define $\varphi_0 \colon C(S^1) \to B$ by sending $f \in C(S^1)$ to the continuously varying diagonal matrix

$$\varphi_0(f)(t) = \mathrm{diag}\big(f\big(e^{2\pi i t/n}\big), \, f\big(e^{2\pi i (t+1)/n}\big), \, \ldots, \, f\big(e^{2\pi i (t+n-1)/n}\big) \big).$$

(For fixed t, the diagonal entries are obtained by evaluating f at the points in the orbit of $e^{2\pi i t/n}$.)

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(For fixed t, the diagonal entries are obtained by evaluating f at the points in the orbit of $e^{2\pi i t/n}$.) The diagonal entries of $f(0)$ are gotten from those of $f(1)$ by a forwards cyclic shift, so $\varphi_0(f)$ really is in B. For the same reason, we get

$$\begin{aligned} \varphi_0(\alpha(f))(t) &= \mathrm{diag}\big(f\big(e^{2\pi i(t-1)/n}\big), \, f\big(e^{2\pi i t/n}\big), \, \ldots, \, f\big(e^{2\pi i(t+n-2)/n}\big)\big) \\ &= s\varphi_0(f)(t)s^*. \end{aligned}$$

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

Now let $v \in C([0,1], M_n)$ be the constant function with value s.

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

Now let $v \in C([0,1], M_n)$ be the constant function with value s. Note that $v \in B$.

Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

Now let $v \in C([0,1], M_n)$ be the constant function with value s. Note that $v \in B$. The calculation just done implies that

$$\varphi_0(\alpha^k(f)) = v^k \varphi_0(f) v^{-k}$$

for $0 \leq k \leq n-1$.

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Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

Now let $v \in C([0,1], M_n)$ be the constant function with value s. Note that $v \in B$. The calculation just done implies that

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Example: $\mathbb{Z}/n\mathbb{Z}$ acting on S^1 by rotation by $e^{2\pi i/n}$ (continued)

For surjectivity, let $a \in B$, and write

$$a(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) & \cdots & a_{1,n}(t) \\ a_{2,1}(t) & a_{2,2}(t) & \cdots & a_{2,n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}(t) & a_{n,2}(t) & \cdots & a_{n,n}(t) \end{pmatrix}$$

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It remains to prove that $B \cong C(S^1, M_n)$.

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so $\psi(f)$ really is in B. It is easily checked that ψ is bijective.

Example: $x \mapsto -x$ on S^n

Example

Let $X = S^n = \{x \in \mathbb{R}^{n+1} \colon \|x\|_2 = 1\}$, and let $\mathbb{Z}/2\mathbb{Z}$ act by sending the nontrivial group element to the order 2 homeomorphism $x \mapsto -x$.

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Example: Complex conjugation on S^1

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Take $X = S^1 = \{\zeta \in \mathbb{C} \colon |\zeta| = 1\}$, and let $\mathbb{Z}/2\mathbb{Z}$ act by sending the nontrivial group element to the order 2 homeomorphism $\zeta \mapsto \overline{\zeta}$. Let $\alpha \in \mathrm{Aut}(C(S^1))$ be the corresponding automorphism.

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$$B = \{f \in C([-1, 1], M_2) \colon f(1) \text{ and } f(-1) \text{ are diagonal matrices}\}.$$

Example: Complex conjugation on S^1 (continued)

Here are the details. First, let $C_0 \subset M_2$ be the subalgebra consisting of all matrices of the form $\begin{pmatrix} \lambda & \mu \\ \mu & \lambda \end{pmatrix}$ with $\lambda, \mu \in \mathbb{C}$.

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for $f \in C(S^1)$ and $t \in [-1, 1]$.

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Picture for the definition of $\varphi_0(f)$)

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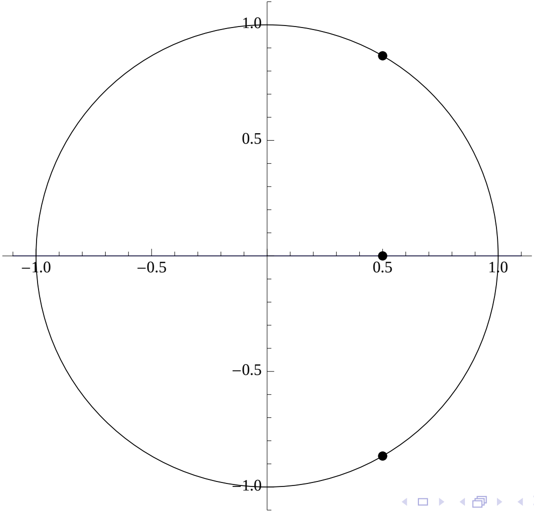

Example: Complex conjugation on S^1 (continued)

Therefore there is a homomorphism $\varphi \colon C^* (\mathbb{Z}/2\mathbb{Z}, X) \to C$ such that $\varphi|_{C(S^1)} = \varphi_0$ and φ sends the standard unitary u in $C^* (\mathbb{Z}/2\mathbb{Z}, X)$ to v.

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$$\varphi(f_0 + f_1 u)(t) = \begin{pmatrix} f_0\big(t + i\sqrt{1 - t^2}\big) & f_1\big(t + i\sqrt{1 - t^2}\big) \\ f_1\big(t - i\sqrt{1 - t^2}\big) & f_0\big(t - i\sqrt{1 - t^2}\big) \end{pmatrix}$$

for $f_1, f_2 \in C(S^1)$ and $t \in [-1, 1]$.

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We claim that φ is an isomorphism.

Example: Complex conjugation on S^1 (continued)

Therefore there is a homomorphism $\varphi\colon C^*(\mathbb{Z}/2\mathbb{Z}, X) \to C$ such that $\varphi|_{C(S^1)} = \varphi_0$ and φ sends the standard unitary u in $C^*(\mathbb{Z}/2\mathbb{Z}, X)$ to v. It is given by the formula

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We claim that φ is an isomorphism. Since

$$C^*(\mathbb{Z}/2\mathbb{Z}, X) = \{f_0 + f_1 u\colon f_1, f_2 \in C(S^1)\},$$

it is easy to check injectivity.

Example: Complex conjugation on S^1 (continued)

For surjectivity, let

$$a(t) = \begin{pmatrix} a_{1,1}(t) & a_{1,2}(t) \\ a_{2,1}(t) & a_{2,2}(t) \end{pmatrix}$$

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$$a_{1,1}(-1) = a_{2,2}(-1) \quad \text{and} \quad a_{2,1}(-1) = a_{1,2}(-1), \tag{1}$$

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$$f_0(\zeta) = \begin{cases} a_{1,1}(\mathrm{Re}(\zeta)) & \mathrm{Im}(\zeta) \geq 0 \\ a_{2,2}(\mathrm{Re}(\zeta)) & \mathrm{Im}(\zeta) \leq 0 \end{cases}$$

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for $\zeta \in S^1$. The relations (1) and (2) ensure that f_0 and f_1 are well defined at ± 1, and are continuous. One easily checks that $\varphi(f_0 + f_1 u) = a$. This proves surjectivity.

Example: Complex conjugation on S^1 (continued)

The algebra C is not quite what was promised. Set

$$w = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$

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In this example, one choice of matrix units in M_2 was convenient for the free orbits, while another choice was convenient for the fixed points. It seemed better to compute everything in terms of the choice convenient for the free orbits, and convert afterwards.

Example: $x \mapsto -x$ on $[-1, 1]$

Exercise

Let $\mathbb{Z}/2\mathbb{Z}$ act on $[-1, 1]$ via $x \mapsto -x$. Compute the crossed product.

Example: $(x_1, x_2, \ldots, x_n, x_{n+1}) \mapsto (x_1,\ x_2,\ \ldots,\ x_n,\ -x_{n+1})$ on S^n

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Let $\mathbb{Z}/2\mathbb{Z}$ act on

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \colon x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$

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This is a special case of G acting on G/H by translation. In the general case, it turns out that $C^*(G, \, G/H) \cong K(L^2(G/H)) \otimes C^*(H)$. Note that there is no twisting.

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Example: $\mathbb{Z}$ acting on $\mathbb{Z}/n\mathbb{Z}$ by translation (continued)

In $C(S^1)$ let z be the function $z(\zeta) = \zeta$ for all ζ.

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$$v = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 & 0 & z \\ 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

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(This unitary differs from the unitary s used before only in that here the upper right corner entry is z instead of 1.)

Example: $\mathbb{Z}$ acting on $\mathbb{Z}/n\mathbb{Z}$ by translation (continued)

Define $\varphi_0 \colon C(\mathbb{Z}/n\mathbb{Z}) \to M_n \otimes C(S^1)$ by $\varphi_0(\chi_{\{k\}}) = e_{k,k}$.

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Define $\varphi_0 \colon C(\mathbb{Z}/n\mathbb{Z}) \to M_n \otimes C(S^1)$ by $\varphi_0(\chi_{\{k\}}) = e_{k,k}$. Then one checks that $v\varphi_0(f)v^* = \varphi_0(\alpha(f))$ for all $f \in C(\mathbb{Z}/n\mathbb{Z})$.

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We use the following description of $M_n \otimes C(S^1)$: it is the universal unital C*-algebra generated by a system $(e_{j,k})_{1 \leq j,k \leq n}$ of matrix units such that $\sum_{j=1}^{n} e_{j,j} = 1$ and a central unitary y.

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The rest of the details are omitted; see the notes. The main point is to use the description of $M_n \otimes C(S^1)$ as the universal C*-algebra on the generators and relations above.

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Example: A product type action

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Let

$$\overline{\varphi}_n \colon C^*(\mathbb{Z}/2\mathbb{Z}, \, M_{2^n}, \, \operatorname{Ad}(z_n)) \to C^*(\mathbb{Z}/2\mathbb{Z}, \, M_{2^{n+1}}, \, \operatorname{Ad}(z_{n+1}))$$

be the corresponding map on the crossed products.

Example: A product type action (continued)

From what we did with inner actions, we get isomorphisms

$$\sigma_n \colon C^* (\mathbb{Z}/2\mathbb{Z}, \, M_{2^n}, \, \mathrm{Ad}(z_n)) \to M_{2^n} \oplus M_{2^n}$$

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$$\psi_n \colon M_{2^n} \oplus M_{2^n} \to M_{2^{n+1}} \oplus M_{2^{n+1}}$$

which makes the following diagram commute:

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$$\sigma_n(a + bu_n) = (a + bz_n, \, a - bz_n)$$

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We have enough information to write an explicit formula for this expression (see the notes), and it turns out that we can take

$$\psi_n(b, c) = \left(\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}, \, \begin{pmatrix} c & 0 \\ 0 & b \end{pmatrix} \right).$$

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Remarks on the product type example

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Exercises 4.23 and 4.24 in the notes combine direct limit methods with computations of the sort done above.

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We will see below that for $\theta \in \mathbb{R} \setminus \mathbb{Q}$, the algebra $C^*(\mathbb{Z}, S^1, h_\theta)$ is simple.