Seoul National University short course: An introduction to the structure of crossed product C*-algebras.

Lecture 3: Crossed products by minimal homeomorphisms

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If the action of G on X is not minimal, then there is a nontrivial invariant closed subset $T \subset X$, and $C^*_{\mathrm{r}}(G, X \setminus T)$ turns out to be a nontrivial ideal in $C^*_{\mathrm{r}}(G, X)$. Thus $C^*_{\mathrm{r}}(G, X)$ is not simple.

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Almost all work on minimal homeomorphisms has been on compact spaces. For these, we have the following equivalent conditions.

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4. Whenever $U \subset X$ is an open subset such that $h(U) \subset U$, then $U = \varnothing$ or $U = X$.
5. For every $x \in X$, the orbit $\{h^n(x) \colon n \in \mathbb{Z}\}$ is dense in X.

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5. For every $x \in X$, the orbit $\{h^n(x) \colon n \in \mathbb{Z}\}$ is dense in X.
6. For every $x \in X$, the forward orbit $\{h^n(x) \colon n \geq 0\}$ is dense in X.

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Exercise

Prove the lemma.

Examples of minimal actions

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Irrational rotations of the circle are minimal.

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There are other proofs of minimality.

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Other examples of minimal homeomorphisms include Furstenberg transformations, restrictions of Denjoy homeomorphisms of the circle to their minimal sets, and certain irrational time maps of suspension flows.

Minimal actions are plentiful

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Remarks on freeness

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An essentially free minimal action of an abelian group is free. This is because the fixed point set of any group element is invariant under the action.

Simplicity of the transformation group C*-algebra

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The usual proof for $G = \mathbb{Z}$ depends on Rokhlin type arguments. See the proof of Lemma VIII.3.7 of Davidson's book. We have avoided such arguments here, but will use them later. To obtain more information about simple transformation group C*-algebras, such arguments are necessary, at least with the current state of knowledge. Examples show that, in the absence of some form of the Rokhlin property, stronger structural properties of crossed products of noncommutative C*-algebras need not hold, even when they are simple.

First lemma

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Lemma

Let A be a C*-algebra, let $B \subset A$ be a subalgebra, and let ω be a state on A such that $\omega|_B$ is multiplicative. Then for all $a \in A$ and $b \in B$, we have $\omega(ab) = \omega(a)\omega(b)$ and $\omega(ba) = \omega(b)\omega(a)$.

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This is also a special case of a result of Choi.

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5. If $a \in C^*_{\mathrm{r}}(G, A, \alpha)$ and $E(a^* a) = 0$, then $a = 0$.

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We now have, using the second lemma at the fifth step,

$$\begin{aligned}\tfrac{1}{4}\|E(a^*a)\| &> \|b - a^*a\| \geq |\omega(b - a^*a)| = |\omega(b)| = \left|\sum\nolimits_{g \in S} \omega(b_g u_g)\right| \\ &= |\omega(b_1)| = |\omega_0(b_1)| = |b_1(x)| \geq E(a^*a)(x) - \|E(a^*a) - b_1\| \\ &\geq E(a^*a)(x) - \|a^*a - b\| > \tfrac{3}{4}\|E(a^*a)\| - \tfrac{1}{4}\|E(a^*a)\| = \tfrac{1}{2}\|E(a^*a)\|.\end{aligned}$$

Proof of the simplicity theorem (continued)

The set $C_0(X) + I$ is a C*-subalgebra of $C^*_{\mathrm{r}}(G, X)$. Let $\omega_0 \colon C_0(X) + I \to \mathbb{C}$ be the following composition:

$$C_0(X) + I \longrightarrow (C_0(X) + I)/I \xrightarrow{\cong} C_0(X)/(C_0(X) \cap I) = C_0(X) \xrightarrow{\mathrm{ev}_x} \mathbb{C}.$$

Then ω_0 is a homomorphism. Use the Hahn-Banach Theorem in the usual way to get a state $\omega \colon C^*_{\mathrm{r}}(G, X) \to \mathbb{C}$ which extends ω_0. Note that $\omega(a^* a) = 0$.

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This contradiction shows that $I \cap C_0(X) \neq 0$.

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The main theorem on minimal homeomorphisms

Definitions (with explanation) and the main corollary are below.

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Note: Some of the discussion here is not in the notes, and some of the results are in a slightly different order.

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The theorem also implies that $C^*(\mathbb{Z}, X, h)$ is classifiable in the sense of the Elliott program. (This is used in the proof of the corollary.)

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- $\dim\left([0,1]^{\mathbb{Z}}\right) = \infty$.

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There is machinery available to compute the range of ρ in the above theorem without computing $C^*(\mathbb{Z}, X, h)$. See, for example, Ruy Exel's Ph.D. thesis. (Reference to the published version in the notes.)

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(This is equivalent to the original definition for simple C*-algebras. See the notes.)

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Let A be a simple unital C*-algebra. Then A has tracial rank zero if for every finite subset $F \subset A$, every $\varepsilon > 0$, and every nonzero positive element $c \in A$, there exists a projection $p \in A$ and a unital finite dimensional subalgebra $D \subset pAp$ such that:

1. $\| [a, p] \| < \varepsilon$ for all $a \in F$.
2. $\mathrm{dist}(pap, D) < \varepsilon$ for all $a \in F$.
3. $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{cAc}$.

(This is equivalent to the original definition for simple C*-algebras. See the notes.)

The condition was originally called "tracially AF".

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One can get this result for the Cantor set by combining the main theorem with known classification results, so Putnam's argument doesn't give any more in the end.

Other known results: Minimal diffeomorphisms

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It should be possible to generalize to minimal homeomorphisms of finite dimensional compact metric spaces.

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There is also a collection of related results on crossed products of simple C*-algebras by actions of $\mathbb{Z}$ and of finite groups which have the tracial Rokhlin property, and generalizations.

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The proof is an exercise.

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$$C^*(\mathbb{Z}, X, h)_Y = C^*(C(X), \, uC_0(X \setminus Y)) \subset C^*(\mathbb{Z}, X, h).$$

$C^*(\mathbb{Z}, X, h)_Y$ is the C*-algebra of a groupoid

Although we will not use formally groupoids in these notes, it should be pointed out that $C^*(\mathbb{Z}, X, h)_Y$ is the C*-algebra of a subgroupoid of the transformation group groupoid $\mathbb{Z} \ltimes X$ made from the action of $\mathbb{Z}$ on X generated by h.

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For actions of $\mathbb{Z}^d$, it appears to be necessary to use subalgebras of the crossed product for which the only nice description is in terms of subgroupoids of the transformation group groupoid.

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Proof of Lemma 3

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We first claim that there is $N \in \mathbb{Z}_{>0}$ such that $\bigcup_{n=1}^{N} h^{-n}(Y) = X$. Set $U = \bigcup_{n=1}^{\infty} h^{-n}(Y)$, which is a nonempty open subset of X such that $U \subset h(U)$.

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It follows that for each fixed $y \in Y$, the sequence of iterates $h(y), h^2(y), \ldots$ of y under h must return to Y in at most N steps.

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We first claim that there is $N \in \mathbb{Z}_{>0}$ such that $\bigcup_{n=1}^{N} h^{-n}(Y) = X$. Set $U = \bigcup_{n=1}^{\infty} h^{-n}(Y)$, which is a nonempty open subset of X such that $U \subset h(U)$. Then $Z = X \setminus \bigcup_{n=1}^{\infty} h^{-n}(Y)$ is a closed subset of X such that $h(Z) \subset Z$, and $Z \neq X$. Therefore $Z = \varnothing$. So $U = X$, and the claim now follows from compactness of X.

It follows that for each fixed $y \in Y$, the sequence of iterates $h(y), \, h^2(y), \ldots$ of y under h must return to Y in at most N steps. Define the *first return time* $r(y)$ to be

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$$r(y) = \min\{n \geq 1 \colon h^n(y) \in Y\} \leq N.$$

Proof of Lemma 3 (continued)

Let $n(0) < n(1) < \cdots < n(l) \leq N$ be the values of r.

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$$Y = \coprod_{k=0}^{l} Y_k \quad \text{and} \quad X = \coprod_{k=0}^{l} \coprod_{j=1}^{n(k)} h^j(Y_k).$$

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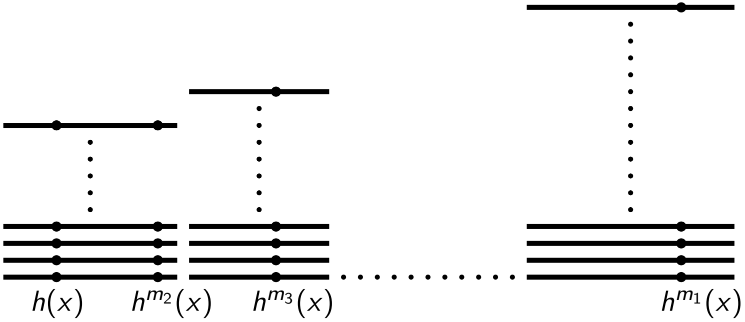
Rokhlin towers, with part of an orbit

We take $x \in Y$, in fact, $x \in Y_0$.

The bases of the towers are $h(Y_0), h(Y_1), \ldots, h(Y_l)$, and the heights are $n(0), n(1), \ldots, n(l)$. The tower over $h(Y_k)$ corresponds to a summand of $C^*(\mathbb{Z}, X, h)_Y$ isomorphic to $M_{n(k)} \otimes C(Y_k)$.



$$m_1 = n(0) + 1, \quad m_2 = n(0) + n(l) + 1, \quad m_3 = n(0) + n(l) + n(1) + 1$$

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$$p_k u f = u\big(p_k - \chi_{h^{n(k)}(Y_k)} + \chi_{Y_k}\big) f = u p_k f = u f p_k.$$

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It follows that p_k commutes with all elements of $C^*(\mathbb{Z}, X, h)_Y$. So it suffices to prove that $p_k C^*(\mathbb{Z}, X, h)_Y p_k$ is AF for each k.

Proof of Lemma 3 (continued)

Now $p_k C^*(\mathbb{Z}, X, h)_Y p_k$ is the C*-algebra generated by $C(X_k)$ and

$$u(\chi_{X \setminus Y})p_k = u(\chi_{X_k \setminus h^{n(k)}(Y_k)}) = \sum_{j=1}^{n(k)-1} u(\chi_{h^j(Y_k)})$$

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One can now check, although it is a bit tedious to write out the details, that there is an isomorphism $\psi_k \colon p_k C^*(\mathbb{Z}, X, h)_Y p_k \to M_{n(k)} \otimes C(Y_k)$

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Notation

Let A be a C*-algebra, and let $p, q \in A$ be projections. We write $p \sim q$ to mean that p and q are Murray-von Neumann equivalent in A,

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This shows that we can control the size of the leftover in the definition of tracial rank zero using projections in $C(X)$ instead of in the crossed product. This is a big advantage.

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(3) There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.

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The point is that $C^*(\mathbb{Z}, X, h)_Y$ is an AF algebra, and (3) says, in view of Lemma 4, that $1 - p$ is "small".

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$$h^{-N}(K), \, h^{-N+1}(K), \, \ldots, \, h^{N}(K)$$

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$$pbp = pb_0 p = pE(b)p.$$

Proof of Lemma 4 (continued)

Using this equation at the first step, we get

$$\|pcp-pE(c)p\| \leq \|pcp-pbp\|+\|pE(b)p-pE(c)p\| \leq 2\|c-b\| < 2\delta. \quad (1)$$

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$$v^*v = c^{1/2}pa^2pc^{1/2} \in \overline{cC^*(\mathbb{Z}, X, h)c}.$$

This completes the proof.

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An induction argument now shows that $\chi_{h(Y)} \sim \chi_{h^N(Y)}$ in $C^*(\mathbb{Z}, X, h)_Y$. Also, $\chi_{X \setminus Y} \sim \chi_{X \setminus h(Y)}$ in $C^*(\mathbb{Z}, X, h)_Y$.

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In this proof, we apply Berg's technique "over" a small subset of the space X. For some other related results on crossed products, the methods of proof known so far require an application "over" subsets which fill up most of X.

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The basic idea behind the proof of Lemma 6 is Berg's technique. It was originally used to prove that the direct sum of the bilateral shift and a finite cyclic shift is approximately unitarily equivalent, to within an error depending on the length of the finite cyclic shift, to the bilateral shift. The error goes to zero as the length of the finite cyclic shift goes to infinity.

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In this proof, we apply Berg's technique "over" a small subset of the space X. For some other related results on crossed products, the methods of proof known so far require an application "over" subsets which fill up most of X.

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Comment on the proof of Lemma 6: Berg's technique

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For actions of more general groups on compact metric spaces, to get the strongest expected results one needs a generalization or replacement, so far not found.

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Choose $\delta_0 > 0$ with $\delta_0 < \frac{1}{2}\varepsilon$ and so small that $d(x_1, x_2) < 4\delta_0$ implies $|f(x_1) - f(x_2)| < \frac{1}{4}\varepsilon$ for all $f \in F$.

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Choose $\delta > 0$ with $\delta \leq \delta_0$ and such that whenever $d(x_1, x_2) < \delta$ and $0 \leq k \leq N_0$, then $d(h^{-k}(x_1), \, h^{-k}(x_2)) < \delta_0$.

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Since h is minimal, there is $N > N_0 + 1$ such that $d(h^N(y),\, y) < \delta$.

Proof of Lemma 6 (continued)

Choose $N + N_0 + 1$ disjoint nonempty open subsets $U_{-N_0}, U_{-N_0+1}, \ldots, U_N \subset U$.

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$$h^{-N_0}(Y),\ h^{-N_0+1}(Y),\ \ldots,\ Y,\ h(Y),\ \ldots,\ h^N(Y)$$

are disjoint and all have diameter less than δ.

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Set $q_0 = \chi_Y$. For $-N_0 \leq n \leq N$ set

$$T_n = h^n(Y) \quad \text{and} \quad q_n = u^n q_0 u^{-n} = \chi_{h^n(Y)}.$$

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Choose $N + N_0 + 1$ disjoint nonempty open subsets $U_{-N_0}, U_{-N_0+1}, \ldots, U_N \subset U$. Using minimality again, choose $r_{-N_0}, r_{-N_0+1}, \ldots, r_N \in \mathbb{Z}$ such that $h^{r_l}(y) \in U_l$ for $-N_0 \leq l \leq N$. Since h is free, there is a compact open set $Y \subset X$ containing y such that

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Then the q_n are mutually orthogonal projections in $C(X)$.

Proof of Lemma 6 (continued)

We now have a sequence of projections, in principle going to infinity in both directions:

$$\ldots, \, q_{-N_0}, \, \ldots, \, q_{-1}, \, q_0, \, q_1, \, \ldots, \, q_{N-N_0}, \, \ldots, \, q_{N-1}, \, q_N, \, \ldots.$$

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The ones shown are orthogonal, and conjugation by u is the shift. The projections q_0 and q_N are the characteristic functions of compact open sets which are disjoint but close to each other, and similarly for the pairs q_{-1} and q_{N-1} down to q_{-N_0} and q_{N-N_0}.

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The ones shown are orthogonal, and conjugation by u is the shift. The projections q_0 and q_N are the characteristic functions of compact open sets which are disjoint but close to each other, and similarly for the pairs q_{-1} and q_{N-1} down to q_{-N_0} and q_{N-N_0}. We are now going to use Berg's technique to splice this sequence along the pairs of indices $(-N_0, \, N - N_0)$ through $(0, N)$, obtaining a loop of length N on which conjugation by u is approximately the cyclic shift.

Proof of Lemma 6 (continued)

Lemma 5 provides a partial isometry $w \in C^*(\mathbb{Z}, X, h)_Y$ such that $w^* w = q_0$ and $w w^* = q_N$.

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$$z_k = (a_k + b_k)^* v(k/N_0)(a_k + b_k) \in C^*(\mathbb{Z}, X, h)_Y.$$

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Therefore z_k is a unitary in the corner

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Now define $e_n = q_n$ for $0 \leq n \leq N - N_0$,

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Moreover, adding estimates on the differences of the matrix entries at the second step,

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Proof of Lemma 6 (continued)

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Proof of Lemma 6 (continued)

Moreover, adding estimates on the differences of the matrix entries at the second step,

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$$\|u e_{n-1} u^* - e_n\| \leq 2\|u z_{N-n+1} u^* - z_{N-n}\| < \varepsilon.$$

Proof of Lemma 6 (continued)

Moreover, adding estimates on the differences of the matrix entries at the second step,

$$\| u z_{k+1} u^* - z_k \| = \| v(k/N_0) - v((k-1)/N_0) \| \leq 2\pi / N_0 < \tfrac{1}{2}\varepsilon.$$

Now define $e_n = q_n$ for $0 \leq n \leq N - N_0$, and for $N - N_0 \leq n \leq N$ write $k = N - n$ and set $e_n = z_k q_{-k} z_k^*$. The two definitions for $n = N - N_0$ agree because $z_{N_0} q_{-N_0} z_{N_0}^* = q_{N-N_0}$, and moreover $e_N = e_0$. Therefore $u e_{n-1} u^* = e_n$ for $1 \leq n \leq N - N_0$, and also $u e_N u^* = e_1$, while for $N - N_0 < n \leq N$ we have

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Also, clearly $e_n \in C^*(\mathbb{Z}, X, h)_Y$ for all n.

Proof of Lemma 6 (continued)

Set $e = \sum_{n=1}^{N} e_n$ and $p = 1 - e$.

Proof of Lemma 6 (continued)

Set $e = \sum_{n=1}^{N} e_n$ and $p = 1 - e$. We verify that p satisfies (1) through (3):

1. $\|pa - ap\| < \varepsilon$ for all $a \in F \cup \{u\}$.
2. $pap \in pC^*(\mathbb{Z}, X, h)_Y p$ for all $a \in F \cup \{u\}$.
3. There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.

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First,

$$p - upu^* = ueu^* - e = \sum_{n=N_0+1}^{N} (ue_{n-1}u^* - e_n).$$

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The terms in the sum are orthogonal and have norm less than ε, so $\|upu^* - p\| < \varepsilon$. Furthermore, since $p \leq 1 - q_0 = 1 - \chi_Y$,

Proof of Lemma 6 (continued)

Set $e = \sum_{n=1}^{N} e_n$ and $p = 1 - e$. We verify that p satisfies (1) through (3):

1. $\| pa - ap \| < \varepsilon$ for all $a \in F \cup \{u\}$.
2. $pap \in pC^*(\mathbb{Z}, X, h)_Y p$ for all $a \in F \cup \{u\}$.
3. There is a compact open set $Z \subset U$ such that $1 - p \precsim \chi_Z$ in $C^*(\mathbb{Z}, X, h)$.

First,

$$p - upu^* = ueu^* - e = \sum_{n=N_0+1}^{N} (ue_{n-1}u^* - e_n).$$

The terms in the sum are orthogonal and have norm less than ε, so $\| upu^* - p \| < \varepsilon$. Furthermore, since $p \leq 1 - q_0 = 1 - \chi_Y$, we get $pup \in C^*(\mathbb{Z}, X, h)_Y$.

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Proof of Lemma 6 (continued)

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Next, let $f \in F$. The sets $T_0, T_1, \ldots, T_N$ all have diameter less than δ. We have $d(h^N(y), y) < \delta$, so the choice of δ implies that $d(h^n(y), h^{n-N}(y)) < \delta_0$ for $N - N_0 \leq n \leq N$.

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Next, let $f \in F$. The sets $T_0, T_1, \ldots, T_N$ all have diameter less than δ. We have $d(h^N(y), y) < \delta$, so the choice of δ implies that $d(h^n(y), h^{n-N}(y)) < \delta_0$ for $N - N_0 \leq n \leq N$. Also, $T_{n-N} = h^{n-N}(T_0)$ has diameter less than δ.

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are disjoint, there is $g \in C(X)$ which is constant on each of these sets and satisfies $\|f - g\| < \frac{1}{2}\varepsilon$.

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$$g e_n = g(q_{n-N} + q_n) e_n = \lambda_n (q_{n-N} + q_n) e_n = e_n (q_{n-N} + q_n) g = e_n g.$$

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This is (1) for f. That $pfp \in C^*(\mathbb{Z}, X, h)_Y$ follows from the fact that f and p are in this subalgebra.

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Let the values of g on these sets be λ_1 on S_1 through λ_N on S_N. Then $ge_n = e_n g = \lambda_n e_n$ for $0 \leq n \leq N - N_0$. For $N - N_0 < n \leq N$ we use $e_n \in (q_{n-N} + q_n) C^*(\mathbb{Z}, X, h)_Y (q_{n-N} + q_n)$ to get,

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This completes the proof.

Some comments on the general case

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for a suitable $y_0 \in X$, and such that $\operatorname{int}(Y_n) \neq \varnothing$ for all n.

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for a suitable $y_0 \in X$, and such that $\mathrm{int}(Y_n) \neq \varnothing$ for all n. Then

$$C^*(\mathbb{Z}, X, h)_{Y_0} \subset C^*(\mathbb{Z}, X, h)_{Y_1} \subset C^*(\mathbb{Z}, X, h)_{Y_2} \subset \cdots$$

and

$$\overline{\bigcup_{n=0}^{\infty} C^*(\mathbb{Z}, X, h)_{Y_n}} = C^*(\mathbb{Z}, X, h)_{\{y_0\}}.$$

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$C^*(\mathbb{Z}, X, h)_{\{y_0\}}$ is simple and has the same tracial states and K_0-group as $C^*(\mathbb{Z}, X, h)$. (These facts require proof. The one about K_0 is hard, but had already been done by Putnam.) Using this, and some results on dymanics (essentially "mean dimension zero"), the theory of direct limits of recursive subhomogeneous algebras can be used to prove that $C^*(\mathbb{Z}, X, h)_{\{y_0\}}$ has tracial rank zero. This turns out to be a sufficient substitute for $C^*(\mathbb{Z}, X, h)_Y$ being an AF algebra.

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We describe the proof of one of the pieces needed.

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Lemma

Let X be an infinite compact metric space, and let $h \colon X \to X$ be a minimal homeomorphism. Let $y \in X$. Then the restriction map $T(C^*(\mathbb{Z}, X, h)) \to T\big(C^*(\mathbb{Z}, X, h)_{\{y\}}\big)$ is a bijection.

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The proof follows from the following two results.

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The proof follows from the following two results. The first is well known, and holds in much greater generality.

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The proof follows from the following two results. The first is well known, and holds in much greater generality. The proofs are roughly the same, but the proof of the second is more complicated, since the algebra is smaller.

Tracial states and invariant measures

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Proof of the proposition

It is immediate that τ_μ is positive, and that $\tau_\mu(1) = 1$. So τ_μ is a state.

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We prove $\tau_\mu(ab) = \tau_\mu(ba)$ for all $a, b \in C^*(\mathbb{Z}, X, h)$. Since τ_μ is continuous, and since $C^*(\mathbb{Z}, X, h)$ is the closed linear span of all elements fu^m with $f \in C(X)$ and $m \in \mathbb{Z}$, it suffices to prove $\tau_\mu\big((fu^m)(gu^n)\big) = \tau_\mu\big((gu^n)(fu^m)\big)$ for all $f, g \in C(X)$ and $m, n \in \mathbb{Z}$.

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This completes the proof that τ_μ is a tracial state on $C^*(\mathbb{Z}, X, h)$.

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$$\tau(g_j f u^n) = \tau\big(g_j^{1/2} f u^n g_j^{1/2}\big) = \tau\big(g_j^{1/2} f \big(g_j^{1/2} \circ h^{-n}\big) u^n\big) = \tau(0) = 0.$$

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Summing over j gives $\tau(fu^n) = 0$. This completes the proof.

Tracial states on the subalgebra

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Proof of the lemma

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Now let τ be any tracial state on $C^*(\mathbb{Z}, X, h)_{\{y\}}$. Let μ be the Borel probability measure on X determined by $\tau(f) = \int_X f \, d\mu$ for $f \in C(X)$.

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Now let τ be any tracial state on $C^*(\mathbb{Z}, X, h)_{\{y\}}$. Let μ be the Borel probability measure on X determined by $\tau(f) = \int_X f \, d\mu$ for $f \in C(X)$. As in the proof of the previous proposition, we complete the proof by showing that μ is h-invariant, and that $\tau = \tau_\mu|_{C^*(\mathbb{Z}, X, h)_{\{y\}}}$.

Proof of the lemma (continued)

For the first, we again show that $\int_X (f \circ h^{-1}) \, d\mu = \int_X f \, d\mu$ for every $f \in C(X)$.

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We now use the trace property at the second step to get

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Thus μ is h-invariant.

Proof of the lemma (continued)

For the second, we first claim that $C^*(\mathbb{Z}, X, h)_{\{y\}}$ is the closed linear span of all elements of the form $f u^m$, with $f \in C(X)$ and $m \in \mathbb{Z}$, which actually happen to be in $C^*(\mathbb{Z}, X, h)_{\{y\}}$.

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